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Changes of Address.

BAKER, W. E., 255 *Shenly Fields Road, Selly Oak, Birmingham.*

BARTER, J. D., 2415-20 *Allen Avenue, Altadena, California, U.S.A.*

BURCHNALL, J. L., 1 *Field House Terrace, Durham.*

BURKILL, J. C., 20 *Chaucer Road, Cambridge.*

CAVE-BROWNE-CAVE, B. M., 36 *North Side, Streatham Common, S.W.*16.

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HAZLETT, O. C., *Box* 574, *Station A, Champaign, Illinois, U.S.A.*

HEATH, A. C., 80 *Bishop's Mansions, S.W.*6.

HUDSON, H. P , 44*a Ravenscroft Avenue, Golders Green, N.W.*11.

KUTTNER, B., *The University, Edgbaston, Birmingham.*

LAWRENCE, B. E., 267 *Belle Grove Road, Welling, Kent.*

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SOMMERFELD, A., *Dunantstrasse* 6, *München* 23, *Germany.*

WILSON, B. M., 3 *Dowsefield Lane, Calderstones, Liverpool.*

PAPERS

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EXPANSIONS OF ARITHMETICAL FUNCTIONS IN INFINITE SERIES

By R. D. Carmichael.

[Received and read 13 November, 1930.]

1. *Introduction.* The remarkable expansions of arithmetical functions obtained by Ramanujan in his notable memoir* on certain trigonometrical sums and their applications are contained as special cases of much more general expansions which have also other special cases of particular interest. It is the purpose of this paper to present these generalizations and to draw from the expansions some conclusions of importance obtained by means of a hitherto unnoticed fundamental property of the Ramanujan sums $c_q(n)$.

This property has led to the formulation (§ 2) of the notion of orthogonal arithmetical functions analogous to the notion of orthogonal functions in analysis. In the case of the Ramanujan functions the property in consideration is expressed by the relations

$$\sum_{n=1}^{q\rho} c_q(n)\,c_\rho(n) = 0 \text{ if } \rho \neq q, \qquad \sum_{n=1}^{q} c_q{}^2(n) = q\phi(q),$$

where $\phi(q)$ denotes Euler's ϕ-function of q. Orthogonal functions of a certain large class are constructed (§ 3) directly by means of roots of unity.

* *Trans. Camb. Phil. Soc.*, 13 (1918), 259–276; also No. 21 in Ramanujan's *Collected Papers* (1927), 179–199.

The Ramanujan functions $c_q(n)$ are generalized (§4) by aid of the L-series, the properties of the generalized functions are developed (§5) and they are employed (§6) in obtaining expansions of a general class of functions.

The principal direct applications made are those relating to the divisors of integers (§§7 and 8), to Euler's ϕ-function (§9), and to the representation of integers (§10) as sums of an even number of squares.

The most characteristic results of the paper are those (§11) relating to the asymptotic forms of certain functions defined by aid of the divisors of a given integer. The principal result in this connection is that stated by equation (11.10). The functions involved here have several applications which we hope to present later, and, in particular, an important application to the problem of the representation of integers as sums of squares.

2. *Definition of orthogonal arithmetical functions.* Let $f(n)$ and $g(n)$ be two functions whose values are defined for $n = 1, 2, 3, \ldots$. Such a pair of arithmetical functions will be called orthogonal when

(1) these functions have positive integral periods α and β respectively so that $f(n+\alpha) = f(n)$ and $g(n+\beta) = g(n)$ for all n;

(2) $\Sigma f(n)g(n) = 0$, the sum being formed for $n = 1, 2, \ldots, l$, where l is the least common multiple of α and β.

If two (finite or infinite) sets of functions

$$f_1(n),\ f_2(n),\ \ldots,\ \text{and}\ \ g_1(n),\ g_2(n),\ \ldots \tag{2.1}$$

are such that $f_i(n)$ and $g_j(n)$ are orthogonal when $i \neq j$, then the two sets are said to be biorthogonal. If the single set $f_1(n), f_2(n), \ldots$ is such that every function in it is orthogonal to every other function in it, then it is said to be a set of orthogonal arithmetical functions.

For the biorthogonal sets (2.1) we shall be interested mainly in the case in which the supplementary condition

$$\sum_{n=1}^{l_i} f_i(n)g_i(n) \neq 0 \tag{2.2}$$

is satisfied for each i, where l_i is the least common multiple of the periods of $f_i(n)$ and $g_i(n)$. For an orthogonal set $f_1(n), f_2(n), \ldots$, the supplementary condition takes the form

$$\sum_{n=1}^{a_i} f_i^2(n) \neq 0, \tag{2.3}$$

where a_i is the period of $f_i(n)$. It is certainly satisfied when $f_i(n)$, for every fixed i, is real for all n and is not identically zero.

If the biorthogonal functions (2.1) satisfy (2.2) and if an arithmetical function $h(n)$ has an expansion of the form

$$h(n) = a_1 f_1(n) + a_2 f_2(n) + a_3 f_3(n) + \ldots, \tag{2.4}$$

where the a's are independent of n, then, for a given ν, we have

$$\begin{aligned} \lim_{m=\infty} \frac{1}{m} \sum_{n=1}^{m} g_\nu(n) h(n) &= \lim_{m=\infty} \sum_{i=1}^{\infty} a_i \frac{1}{m} \sum_{n=1}^{m} f_i(n) g_\nu(n) \\ &= \sum_{i=1}^{\infty} a_i \lim_{m=\infty} \frac{1}{m} \sum_{n=1}^{m} f_i(n) g_\nu(n), \end{aligned} \tag{2.5}$$

provided that the limit in the first member exists and *provided that* the interchange of limiting operations effected in passing from the second to the third member is legitimate. If $i \neq \nu$ we have

$$\lim_{m=\infty} \frac{1}{m} \sum_{n=1}^{m} f_i(n) g_\nu(n) = 0,$$

since the sum indicated is bounded in consequence of the periodicity of $f_i(n)$ and $g_\nu(n)$ and their relation of orthogonality. Similarly, when $i = \nu$, we have

$$\begin{aligned} \lim_{m=\infty} \frac{1}{m} \sum_{n=1}^{m} f_\nu(n) g_\nu(n) &= \lim_{\mu=\infty} \frac{1}{\mu l_\nu} \sum_{n=1}^{\mu l_\nu} f_\nu(n) g_\nu(n) \\ &= \frac{1}{l_\nu} \sum_{n=1}^{l_\nu} f_\nu(n) g_\nu(n), \end{aligned}$$

where l_ν is the least common multiple of the periods of $f_\nu(n)$ and $g_\nu(n)$. Denoting the last member of this equation by λ_ν, we have uniquely

$$a_\nu = \frac{1}{\lambda_\nu} \lim_{m=\infty} \frac{1}{m} \sum_{n=1}^{m} g_\nu(n) h(n) \quad (\nu = 1, 2, \ldots). \tag{2.6}$$

This formal determination of the a_ν may be carried out equally well under broader hypotheses concerning the sets (2.1). We have not made essential use of the periodicity of the functions. All that has been necessary to the formal processes is covered by assuming that

$$\lim_{m=\infty} \frac{1}{m} \sum_{n=1}^{m} f_i(n) g_\nu(n)$$

exists and has a value $\lambda_{i\nu}$, where $\lambda_{i\nu} = 0$ when $i \neq \nu$ and $\lambda_{\nu\nu} = \lambda_\nu \neq 0$. The

notions of orthogonal and biorthogonal sets of functions may then be extended so that the sets (2.1) will be called biorthogonal when merely the conditions just named are satisfied; but such extension will not be employed in this paper. The later definition reduces to the earlier one in the case of periodic functions.

The processes employed in treating the expansion (2.4) are closely similar to like processes in the theory of Fourier series, and indeed in the more general theory of biorthogonal functions in analysis. These processes may be employed in constructing expansions like (2.4). Furthermore, if such an expansion is constructed by other means, as in some cases treated in this paper, then these processes may be employed in deriving veritable asymptotic relations.

3. *Direct construction of orthogonal arithmetical functions.* Roots of unity furnish effective means of constructing orthogonal and biorthogonal functions. Let p and q be two relatively prime integers, and likewise r and ρ, where q and ρ are positive and where p (or r) is zero only when q (or ρ) is unity. Then

$$(3.1)\qquad \sum_{n=1}^{q\rho} e^{2\pi i p n/q} e^{2\pi i r n/\rho} = 0 \text{ unless } \rho = q \text{ and } p+r \equiv 0 \pmod{q},$$

$$(3.2)\qquad \sum_{n=1}^{q} e^{2\pi i p n/q} e^{2\pi i r n/q} = q \text{ when } p+r \equiv 0 \pmod{q}.$$

For

$$\sum_{n=1}^{pq} e^{2\pi i (p/q+r/\rho) n}$$

is equal to $q\rho$ when $p/q+r/\rho$ is an integer and is otherwise equal to zero. Therefore the functions $e^{2\pi i p n/q}$ and $e^{2\pi i r n/\rho}$ are orthogonal except when $p/q+r/\rho$ is an integer. But this expression is a sum of two fractions in their lowest terms, and therefore can be integral in value only when $\rho = q$ (for if $p/q+r/\rho = \nu$, then $p\rho/q = \nu\rho - r$ and $rq/\rho = \nu q - p$, so that ρ is divisible by q and q is divisible by ρ, whence $\rho = q$). If $\rho = q$ the sum is integral when and only when $p+r \equiv 0 \bmod q$. Thence follow (3.1) and (3.2).

Let us now consider the functions

$$(3.3)\qquad F_q(n, \alpha) = \sum_{pRq} a_{pq} e^{2\pi i p n/q},$$

$$(3.4)\qquad F_\rho(n, \alpha) = \sum_{rR\rho} a_{r\rho} e^{2\pi i r n/\rho},$$

where the symbol pRq denotes that the sum is taken for any set of values

of p which form a reduced residue system modulo q and where a_{pq} ($a_{r\rho}$) is independent of n and is unaltered when p (r) is replaced by any number congruent to it modulo q (ρ). From (3.1) and (3.2) it follows readily that

$$(3.5) \qquad \sum_{n=1}^{q\rho} F_q(n, a) F_\rho(n, \alpha) = 0 \quad \text{if} \quad \rho \neq q,$$

$$(3.6) \qquad \sum_{n=1}^{q} F_q(n, a) F_q(n, \alpha) = q \sum_{pRq} a_{pq} \alpha_{q-p, q}.$$

Therefore the functions (3.3) and (3.4) are orthogonal.

If we form functions (3.3) and (3.4) for both q and ρ ranging over the positive integers, we have two sets of biorthogonal functions. The supplementary conditions are fulfilled if the last sum in (3.6) is different from zero for each q. The functions in either of these sets form an orthogonal set. For such an orthogonal set we have

$$(3.7) \qquad \sum_{n=1}^{q} F_q^2(n, a) = q \sum_{pRq} a_{pq} a_{q-p, q}.$$

If we write

$$(3.8) \quad C_q(n, b) = \sum_{pRq} b_{pq} \cos \frac{2\pi pn}{q}, \quad S_q(n, c) = \sum_{pRq} c_{pq} \sin \frac{2\pi pn}{q} \quad (q = 1, 2, \ldots),$$

where the b's and c's are defined like the a's, then

$$C_q(n, b) = F_q(n, a) \quad \text{if} \quad a_{pq} = \tfrac{1}{2}(b_{pq} + b_{q-p, q})$$

and

$$S_q(n, c) = F_q(n, a) \quad \text{if} \quad a_{pq} = \frac{1}{2i}(c_{pq} - c_{q-p, q}).$$

Therefore each of the sets in (3.8) is orthogonal and the two sets are biorthogonal.

For the special case $b_{pq} = 1 = a_{pq}$ we have the remarkable sums $c_q(n)$ of Ramanujan (*Collected Papers*, pp. 179-199), namely

$$(3.9) \qquad c_q(n) = \sum_{pRq} e^{2\pi i pn/q} = \sum_{pRq} \cos \frac{2\pi pn}{q} \quad (q = 1, 2, \ldots).$$

From the foregoing results it follows that

$$(3.10) \qquad \sum_{n=1}^{q\rho} c_q(n) c_\rho(n) = 0 \quad \text{if} \quad \rho \neq q, \qquad \sum_{n=1}^{q} c_q^2(n) = q\phi(q).$$

It was from an observation of these properties of $c_q(n)$ that I was led to

formulate the notion of orthogonal and biorthogonal arithmetical functions.

Ramanujan (*loc. cit.*, p. 188) also employed the following set, which likewise is orthogonal:

$$(3.11) \qquad S_q(n) = \sum_{\lambda} (-1)^{\frac{1}{2}(\lambda-1)} \sin \frac{2\pi n\lambda}{q} \quad (q = 1, 2, \ldots),$$

where λ ranges over the integers not exceeding q and prime to q.

Let us consider conditions on the quantities a_{pq} of (3.3) which are sufficient to secure the (often useful) condition that

$$(3.12) \qquad F_q(n, a) F_\rho(n, a) = F_{q\rho}(n, a) \quad \text{if} \quad (q, \rho) = 1,$$

where (q, ρ) denotes the greatest common divisor of q and ρ. Such conditions are

$$(3.13) \qquad a_{pq} a_{r\rho} = a_{p\rho+qr,\, q\rho}.$$

This follows readily from a comparison of the second members of the equations

$$F_q(n, a) F_\rho(n, a) = \sum_{pRq} \sum_{rR\rho} a_{pq} a_{r\rho} e^{2\pi i(p/q + r/\rho)n},$$

$$F_{q\rho}(n, a) = \sum_{\alpha Rq} \sum_{\beta R\rho} a_{\alpha\beta,\, q\rho} e^{2\pi i \alpha\beta n/(q\rho)},$$

and from the fact that $p\rho+qr$ $(\alpha\beta)$ runs over a reduced residue system modulo $q\rho$ when p and r (α and β) run over reduced residue systems with moduli q and ρ respectively.

When (3.12) is satisfied the functions $F_q(n, a)$ are determined by their values for the cases when q is a power of a prime. To avoid a trivial case we must have $F_1(n, a) \neq 0$, whence $F_1(n, a) = 1$ and $a_{01} = 1$. From (3.13) it follows that a_{pq} may be assigned at will for prime power values of q and that all the other a_{pq} are then uniquely determined by (3.13).

Since a_{pq} is employed only when $(p, q) = 1$, and since $a_{pq} = a_{p+q,\, q}$, it follows that a_{pq} may be treated as a function $f(p/q)$, where f is a function of a single variable and has the period 1. Then (3.13) becomes

$$(3.14) \qquad f\left(\frac{p}{q}\right) f\left(\frac{r}{\rho}\right) = f\left(\frac{p}{q} + \frac{r}{\rho}\right).$$

One solution of this equation is obviously $a_{pq} = f(p/q) = e^{2\pi i \alpha p/q}$, where α is an integer independent of p and q. Then, if we write

$$(3.15) \qquad c_q(n+\alpha) = \sum_{pRq} e^{2\pi i(n+\alpha)p/q} \quad (q = 1, 2, \ldots),$$

we have a set of orthogonal functions such that

$$c_q(n+a)\,c_\rho(n+a) = c_{q\rho}(n+a) \quad \text{if} \quad (q, \rho) = 1. \tag{3.16}$$

For $a = 0$ we have a property of Ramanujan's functions first set forth by G. H. Hardy, *Proc. Camb. Phil. Soc.*, 20 (1921), 264.

4. *Arithmetical functions defined by aid of L-series.* By

$$\chi(a), \quad \chi_1(a), \quad \chi_2(a), \quad \ldots,$$

we denote any characters with moduli k, k_1, k_2, ... respectively [Landau, *Vorlesungen über Zahlentheorie*, 1 (1927), 83]. By $\chi_0(a)$, $\chi_{10}(a)$, $\chi_{20}(a)$, ..., we denote the principal characters with moduli $k, k_1, k_2, \ldots$ respectively. If any symbol for a character is written with an argument which is not an integer, it is to be understood that the value is to be any conveniently assigned one; such symbols with non-integral arguments are employed only for convenience, and they are always introduced in terms which contain a factor equal to zero. The character which is equal to unity for all arguments will often be replaced by 1.

We use the L-series (Landau, *loc. cit.*, 88, 90)

$$L(s, \chi) = \sum_{a=1}^{\infty} \frac{\chi(a)}{a^s}, \tag{4.1}$$

$$\frac{1}{L(s, \chi)} = \sum_{a=1}^{\infty} \frac{\mu(a)\chi(a)}{a^s}, \tag{4.2}$$

where $\mu(a)$ is the Möbius function (Landau, *loc. cit.*, 20). By s we denote the complex variable $s = \sigma + it$, where σ and t are real. Our most interesting results arise by confining s to real values.

Let $\eta_q(n)$ denote the function

$$\eta_q(n) = \sum_{\nu=1}^{q} \cos \frac{2\pi\nu n}{q} = \sum_{\nu=1}^{q} e^{2\pi i\nu n/q}, \tag{4.3}$$

so that $\eta_q(n)$ has the value q or 0 according as q is or is not a factor of n.

Let λ be any positive quantity. Then we have

$$L(r+s-\lambda, \chi_1)\,L(s, \chi_2) = \sum_{\nu=1}^{\infty} \frac{\chi_1(\nu)\,\nu^{\lambda-r}}{\nu^s} \sum_{\nu=1}^{\infty} \frac{\chi_2(\nu)}{\nu^s} \tag{4.4}$$

$$= \sum_{n=1}^{\infty} n^{-s} \sum_{d \mid n} \chi_1(d)\,\chi_2\left(\frac{n}{d}\right) d^{\lambda-r},$$

where the last summation sign indicates that d ranges over the divisors of n. Now we have

$$(4.5) \qquad \sum_{d|n} \chi_1(d)\chi_2\left(\frac{n}{d}\right) d^{\lambda-r} = \sum_{\nu=1}^{\infty} \frac{\chi_1(\nu)\chi_2(n/\nu)\eta_\nu{}^\lambda(n)}{\nu^r}$$

and

$$(4.6) \qquad \frac{1}{L(r,\chi)} \sum_{\nu=1}^{\infty} \frac{\chi_1(\nu)\chi_2(n/\nu)\eta_\nu{}^\lambda(n)}{\nu^r} = \sum_{q=1}^{\infty} \frac{c_q^{(\lambda)}(n,\chi,\chi_1,\chi_2)}{q^r},$$

where

$$(4.7) \qquad c_q^{(\lambda)}(n,\chi,\chi_1,\chi_2) = \sum_{d|q} \mu\left(\frac{q}{d}\right)\chi\left(\frac{q}{d}\right)\chi_1(d)\chi_2\left(\frac{n}{d}\right)\eta_d{}^\lambda(n).$$

If desired, we may further restrict d to be a divisor of n, since λ is positive and $\eta_d(n)$ is zero when d is not a divisor of n.

From equations (4.4), (4.5), and (4.6), we have

$$(4.8) \qquad \frac{L(r+s-\lambda,\chi_1)\,L(s,\chi_2)}{L(r,\chi)} = \sum_{n=1}^{\infty}\sum_{q=1}^{\infty} \frac{c_q^{(\lambda)}(n,\chi,\chi_1,\chi_2)}{q^r n^s}.$$

One might employ this identity for the purpose of defining these functions c_q.

Multiplying both members of (4.6) by $L(r,\chi)$ and forming the product of the two Dirichlet series in the resulting second member, we have an identity from which we find that

$$(4.9) \qquad \chi_1(q)\chi_2\left(\frac{n}{q}\right)\eta_q{}^\lambda(n) = \sum_{d|q} \chi\left(\frac{q}{d}\right) c_d^{(\lambda)}(n,\chi,\chi_1,\chi_2).$$

We shall employ the functions $c_q^{(\lambda)}(n,\chi,\chi_1,\chi_2)$ in expanding several important arithmetical functions and in deriving asymptotic relations involving them. It is convenient to employ simpler notations for the more important cases, in accordance with the following definitions:

$$(4.10) \qquad c_q(n,\chi,\chi_1,\chi_2) = c_q^{(\lambda)}(n,\chi,\chi_1,\chi_2),$$

$$(4.11) \qquad c_q^{(\lambda)}(n,\chi,\chi_1) = c_q^{(\lambda)}(n,\chi,\chi_1,1), \quad c_q(n,\chi,\chi_1) = c_q^{(1)}(n,\chi,\chi_1),$$

$$(4.12) \qquad c_q^{(\lambda)}(n) = c_q^{(\lambda)}(n,1,1), \quad c_q(n) = c_q^{(1)}(n).$$

The function $c_q(n)$ is the Ramanujan function already denoted by this symbol. From (4.9) and (4.7) we have the following relations stated by Ramanujan (*loc. cit.*, 180):

$$(4.13) \qquad \eta_q(n) = \sum_{d|q} c_d(n), \quad c_q(n) = \sum_{d|q} \mu\left(\frac{q}{d}\right)\eta_d(n).$$

Furthermore from (4.7) it follows that

$$c_q^{(\lambda)}(n, \chi, \chi) = \chi(q)\, c_q^{(\lambda)}(n). \tag{4.14}$$

5. *Properties of the functions* $c_q^{(\lambda)}(n, \chi, \chi_1, \chi_2)$. From (4.7) it follows that

$$|c_q^{(\lambda)}(n, \chi, \chi_1, \chi_2)| \leqslant \sigma_\lambda(\nu), \quad [\nu = (q, n)], \tag{5.1}$$

where $\sigma_\lambda(\nu)$ is the sum of the λ-th powers of the divisors of ν and (q, n) denotes the greatest common divisor of q and n. Hence

$$c_q^{(\lambda)}(n, \chi, \chi_1, \chi_2) = O(1) \tag{5.2}$$

if either of q or n is fixed and the other becomes infinite. In both cases it is assumed that λ, χ, χ_1, χ_2 remain fixed.

If $(m, q) = 1$, then $\eta_q(mn) = \eta_q(n)$. Hence from (4.7) we see that

$$c_q^{(\lambda)}(mn, \chi, \chi_1, \chi_2) = \chi_2(m)\, c_q^{(\lambda)}(n, \chi, \chi_1, \chi_2) \text{ if } (m, q) = 1. \tag{5.3}$$

Now, if $(q, \rho) = 1$, we have from (4.7) the relation

$$c_q^{(\lambda)}(n, \chi, \chi_1, \chi_2)\, c_\rho^{(\lambda)}(n, \chi, \chi_1, \chi_2)$$

$$= \sum_{d \mid q} \sum_{\delta \mid \rho} \mu\left(\frac{q\rho}{d\delta}\right) \chi\left(\frac{q\rho}{d\delta}\right) \chi_1(d\delta)\, \chi_2\left(\frac{n}{d}\right) \chi_2\left(\frac{n}{\delta}\right) \eta_d^\lambda(n)\, \eta_\delta^\lambda(n).$$

Since d and δ are relatively prime, we have $\eta_d(n)\, \eta_\delta(n) = \eta_{d\delta}(n)$. If χ_2 is the principal character χ_{20} and if $(k_2, n) = 1$, then $\chi_2(n/d)\, \chi_2(n/\delta) = \chi_2(n/d\delta)$ whenever n is a multiple of both d and δ, since χ_{20} has the value 1 for each of the indicated arguments. Therefore we have

$$c_q^{(\lambda)}(n, \chi, \chi_1, \chi_{20})\, c_\rho^{(\lambda)}(n, \chi, \chi_1, \chi_{20}) = c_{q\rho}^{(\lambda)}(n, \chi, \chi_1, \chi_{20}) \tag{5.4}$$

$$\text{if } (q, \rho) = 1 = (k_2, n).$$

Replacing χ_{20} by 1 we have

$$c_q^{(\lambda)}(n, \chi, \chi_1)\, c_\rho^{(\lambda)}(n, \chi, \chi_1) = c_{q\rho}^{(\lambda)}(n, \chi, \chi_1) \text{ if } (q, \rho) = 1. \tag{5.5}$$

For the case when $\lambda = 1$ and $\chi(a) \equiv 1 \equiv \chi_1(a)$, that is, for the case of Ramanujan's functions $c_q(n)$, we have $c_q(n)\, c_\rho(n) = c_{q\rho}(n)$ if $(q, \rho) = 1$, a result which is due to G. H. Hardy, *Proc. Camb. Phil. Soc.*, 20 (1921), 264.

If $(k_2, n) = 1$, then we have $c_1^{(\lambda)}(n, \chi, \chi_1, \chi_{20}) = \chi_{20}(n) = 1$, and

$$|c_q^{(\lambda)}(n, \chi, \chi_1, \chi_{20})| \leqslant q^\lambda \Pi(1 + \omega^{-\lambda}) \quad (q > 1), \tag{5.6}$$

where the product is formed for the prime factors ω of q. When q is a prime power this may be proved directly from (4.7), since there are at most

two terms different from zero in the sum indicated. The proof is then completed by aid of (5.5). The inequality may be compared with (5.1).

For $c_q^{(\lambda)}(n)$ we have a somewhat more precise inequality, namely

$$|c_q^{(\lambda)}(n)| \leqslant q^\lambda, \tag{5.7}$$

for all positive integers q and n, the sign of equality holding only when $q = 1$. From (4.7) it follows that when q is a prime power the value of $c_q^{(\lambda)}(n)$ is numerically equal to the difference of two integers neither of which is greater than q^λ. Therefore $|c_q^{(\lambda)}(n)| < q^\lambda$ when q is a prime power. From this result and the multiplicative property (5.5) we conclude that (5.7) is true. Indeed a somewhat more precise inequality could be obtained by refining the argument.

From the property of orthogonality it follows that

$$\sum_{n=1}^{\alpha l} c_q(n)\, c_\rho(n) = 0 \quad \text{if} \quad \rho \neq q,$$

where $l = (q, \rho)$ and α is any positive integer. Therefore

$$\left| \sum_{n=1}^{m} c_q(n)\, c_\rho(n) \right| \leqslant (l-1)\, q\rho < q^2 \rho^2 \quad \text{if} \quad \rho \neq q \quad \text{and} \quad l = (q, \rho). \tag{5.8}$$

We have seen that the second equation in (4.13) readily yields the value of $c_q(n)$ when q is a prime power, and that the other values of $c_q(n)$ are then readily computed by means of the relation $c_q(n)\, c_\rho(n) = c_{q\rho}(n)$ if $(q, \rho) = 1$. But $c_q(n)$ is the sum of the n-th powers of the primitive q-th roots of unity. By means of Newton's formulae (Bôcher's *Higher algebra*, p. 244) the elementary symmetric functions of these primitive roots are readily computed for any particular value of q, whence one obtains the polynomial having these primitive roots of unity for its roots.

6. *Expansions in terms of* $c_q^{(\lambda)}(n, \chi, \chi_1, \chi_2)$. Let $F(u, v)$ denote any function of u and v and form the function $D(n, \chi_1, \chi_2)$, namely,

$$D(n, \chi_1, \chi_2) = \sum_{d \mid n} \chi_1(d)\, \chi_2\left(\frac{n}{d}\right) F\left(d, \frac{n}{d}\right) = \sum_{d \mid n} \chi_1\left(\frac{n}{d}\right) \chi_2(d)\, F\left(\frac{n}{d}, d\right). \tag{6.1}$$

Using the first form of D in (6.1) we have

$$\begin{aligned} D(n, \chi_1, \chi_2) &= \sum_{d \mid n} d^{-\lambda}\, \eta_d^\lambda(n)\, \chi_1(d)\, \chi_2\left(\frac{n}{d}\right) F\left(d, \frac{n}{d}\right) \\ &= \sum_{\nu=1}^{t} \nu^{-\lambda}\, \eta_\nu^\lambda(n)\, \chi_1(\nu)\, \chi_2\left(\frac{n}{\nu}\right) F\left(\nu, \frac{n}{\nu}\right) \quad (t \geqslant n), \end{aligned}$$

where the last sum is taken for $\nu = 1, 2, \ldots, [t]$, $[t]$ denoting the greatest integer not exceeding t. Substituting in the last member of the foregoing equation the value of $\chi_1(\nu)\chi_2(n/\nu)\eta_\nu^\lambda(n)$ from (4.9) and collecting the coefficients of $c_1, c_2, c_3, \ldots$, we have

$$(6.2)\quad D(n, \chi_1, \chi_2) = c_1^{(\lambda)}(n, \chi, \chi_1, \chi_2) \sum_{\nu=1}^{t} \frac{1}{\nu^\lambda} \chi(\nu) F\left(\nu, \frac{n}{\nu}\right)$$

$$+ c_2^{(\lambda)}(n, \chi, \chi_1, \chi_2) \sum_{\nu=1}^{\frac{1}{2}t} \frac{1}{(2\nu)^\lambda} \chi(\nu) F\left(2\nu, \frac{n}{2\nu}\right)$$

$$+ c_3^{(\lambda)}(n, \chi, \chi_1, \chi_2) \sum_{\nu=1}^{\frac{1}{3}t} \frac{1}{(3\nu)^\lambda} \chi(\nu) F\left(3\nu, \frac{n}{3\nu}\right) + \ldots \quad (t \geqslant n),$$

where a sum with upper limit less than unity is to be replaced by zero.

Using the second form of D in (6.1) and proceeding in a similar manner, we find that

$$(6.3)\quad D(n, \chi_1, \chi_2) = c_1^{(\lambda)}(n, \chi, \chi_2, \chi_1) \sum_{\nu=1}^{t} \frac{1}{\nu^\lambda} \chi(\nu) F\left(\frac{n}{\nu}, \nu\right)$$

$$+ c_2^{(\lambda)}(n, \chi, \chi_2, \chi_1) \sum_{\nu=1}^{\frac{1}{2}t} \frac{1}{(2\nu)^\lambda} \chi(\nu) F\left(\frac{n}{2\nu}, 2\nu\right)$$

$$+ c_3^{(\lambda)}(n, \chi, \chi_2, \chi_1) \sum_{\nu=1}^{\frac{1}{3}t} \frac{1}{(3\nu)^\lambda} \chi(\nu) F\left(\frac{n}{3\nu}, 3\nu\right) + \ldots \quad (t \geqslant n).$$

In (6.2) and (6.3) we have generalizations of formulae due to Ramanujan [(3.1) and (3.2) on p. 181 of his *Collected Papers*], his formulae being the special cases when $\lambda = 1$ and $\chi(a) = \chi_1(a) \equiv \chi_2(a) \equiv 1$. A similar generalization of the remaining formulae in the third and fourth sections of his memoir is possible. We require one of them, namely (6.4) below, a generalization of his (3.3). It may be derived in the following manner.

Let

$$F_1(u, v) = F(u, v) \log u, \quad F_2(u, v) = F(u, v) \log v.$$

Then

$$D(n, \chi_1, \chi_2) \log n = \sum_{d \mid n} \chi_1(d) \chi_2\left(\frac{n}{d}\right) F\left(d, \frac{n}{d}\right) \left(\log d + \log \frac{n}{d}\right)$$

$$= \sum_{d \mid n} \chi_1(d) \chi_2\left(\frac{n}{d}\right) F_1\left(d, \frac{n}{d}\right) + \sum_{d \mid n} \chi_1(d) \chi_2\left(\frac{n}{d}\right) F_2\left(d, \frac{n}{d}\right).$$

Expand the two sums in the last member of this equation in accordance

with equations (6.2) and (6.3) respectively. Then we have

$$(6.4)\quad D(n, \chi_1, \chi_2)\log n$$

$$= c_1^{(\lambda)}(n, \chi, \chi_1, \chi_2)\sum_{\nu=1}^{t}\frac{\log\nu}{\nu^\lambda}\chi(\nu)F\left(\nu, \frac{n}{\nu}\right)$$

$$+c_2^{(\lambda)}(n, \chi, \chi_1, \chi_2)\sum_{\nu=1}^{\frac{1}{2}t}\frac{\log 2\nu}{(2\nu)^\lambda}\chi(\nu)F\left(2\nu, \frac{n}{2\nu}\right)$$

$$+c_3^{(\lambda)}(n, \chi, \chi_1, \chi_2)\sum_{\nu=1}^{\frac{1}{3}t}\frac{\log 3\nu}{(3\nu)^\lambda}\chi(\nu)F\left(3\nu, \frac{n}{3\nu}\right)+\cdots$$

$$+c_1^{(\lambda)}(n, \chi, \chi_2, \chi_1)\sum_{\nu=1}^{r}\frac{\log\nu}{\nu^\lambda}\chi(\nu)F\left(\frac{n}{\nu}, \nu\right)$$

$$+c_2^{(\lambda)}(n, \chi, \chi_2, \chi_1)\sum_{\nu=1}^{\frac{1}{2}r}\frac{\log 2\nu}{(2\nu)^\lambda}\chi(\nu)F\left(\frac{n}{2\nu}, 2\nu\right)$$

$$+c_3^{(\lambda)}(n, \chi, \chi_2, \chi_1)\sum_{\nu=1}^{\frac{1}{3}r}\frac{\log 3\nu}{(3\nu)^\lambda}\chi(\nu)F\left(\frac{n}{3\nu}, 3\nu\right)+\cdots\quad (t\geqq n,\ r\geqq n).$$

It should be observed that it is not necessary to use the same function χ in (6.2) and (6.3), or in the two parts of the second member of (6.4).

The results in this section show that there is a very large class of arithmetical functions such that each function of the class is expansible in terms of the functions $c_q^{(\lambda)}(n, \chi, \chi_1, \chi_2)$. Some important special cases will be treated later.

7. *On the divisors of a given integer.* Let us consider the function

$$(7.1)\quad \sigma_s(n, \chi_1, \chi_2) = \sum_{d\mid n}\chi_1(d)\chi_2\left(\frac{n}{d}\right)d^s = \sum_{d\mid n}\chi_1\left(\frac{n}{d}\right)\chi_2(d)\left(\frac{n}{d}\right)^s$$

$$= n^s\sigma_{-s}(n, \chi_2, \chi_1).$$

The function $\sigma_s(n, \chi_1, \chi_2)$ is the special case of $D(n, \chi_1, \chi_2)$ in which $F(u, v) = u^s$. Expansions of this function are therefore afforded by the results in § 6, but we prefer to begin by obtaining them more directly. It is convenient to use more compact symbols for some of the special cases. Hence we introduce the definitions

$$(7.2)\quad \sigma_s(n, \chi_1) = \sigma_s(n, \chi_1, 1),\quad \sigma_s(n) = \sigma_s(n, 1) = \sigma_s(n, 1, 1).$$

The function $\sigma_s(n)$ is the sum of the s-th powers of the divisors of n.

We have

$$(7.3)\quad \sigma_{-s}(n, \chi_1, \chi_2) = \sum_{d|n} \frac{\chi_1(d)\chi_2(n/d)\eta_d^\lambda(n)}{d^{s+\lambda}} = \sum_{\nu=1}^{\infty} \frac{\chi_1(\nu)\chi_2(n/\nu)\eta_\nu^\lambda(n)}{\nu^{s+\lambda}}$$

$$= L(s+\lambda, \chi) \frac{1}{L(s+\lambda, \chi)} \sum_{\nu=1}^{\infty} \frac{\chi_1(\nu)\chi_2(n/\nu)\eta_\nu^\lambda(n)}{\nu^{s+\lambda}}.$$

Comparing with (4.6) we see that

$$(7.4)\quad \sigma_{-s}(n, \chi_1, \chi_2) = L(s+\lambda, \chi) \sum_{\nu=1}^{\infty} \frac{c_\nu^{(\lambda)}(n, \chi, \chi_1, \chi_2)}{\nu^{s+\lambda}}.$$

From (7.1) and (7.4) we see that

$$(7.5)\quad \sigma_s(n, \chi_1, \chi_2) = n^s L(s+\lambda, \chi) \sum_{\nu=1}^{\infty} \frac{c_\nu^{(\lambda)}(n, \chi, \chi_2, \chi_1)}{\nu^{s+\lambda}}.$$

To investigate the region of convergence of the series in the second member of (7.4) we observe that this series is the product of two Dirichlet series, one of which is the third member of equation (7.3), while the other is the Dirichlet series for the function $1/L(s+\lambda, \chi)$, and that the former of these two factor series converges absolutely for all values of s, since it contains only a finite number of terms different from zero; therefore the convergence of the series in (7.4) depends on that of the series in the expansion

$$(7.6)\quad \frac{1}{L(s+\lambda, \chi)} = \sum_{a=1}^{\infty} \frac{\chi(a)\mu(a)}{a^{s+\lambda}}.$$

The series in (7.4) converges [converges absolutely] when the series in (7.6) has this property. The latter series converges absolutely when $\sigma+\lambda > 1$. The abscissa of convergence of the series in (7.6) is not known. [When $\chi(a) \equiv 1$ it is known (Littlewood, Paris *Comptes rendus*, Jan. 29, 1912) that the series converges when $\sigma+\lambda > \frac{1}{2}$, provided that the hitherto unproved Riemann hypothesis concerning the zeros of $\zeta(s)$ is true; and some consequences of this fact are drawn by Ramanujan (*loc. cit.*, pp. 185-6).] But it is known (Landau, *Handbuch der Primzahlen*, pp. 631-7, 687-96) that the series in (7.6) converges when $\sigma+\lambda \geqslant 1$, and also that all the series obtained from it by repeated term-by-term differentiation converge when $\sigma+\lambda \geqslant 1$. From this it follows that the series in (7.6) converges when $\sigma+\lambda \geqslant 1$; the convergence is absolute when $\sigma+\lambda > 1$. Likewise, the series obtained by repeated term-by-term differentiation of the series in (7.4) are convergent when $\sigma+\lambda \geqslant 1$.

From these considerations it follows in particular that the relation

$$(7.7)\quad \sigma_{-s}(n, \chi_1, \chi_2) \frac{1}{L(s+\lambda, \chi)} = \sum_{\nu=1}^{\infty} \frac{c_\nu^{(\lambda)}(n, \chi, \chi_1, \chi_2)}{\nu^{s+\lambda}}$$

is valid when $\sigma+\lambda \geqslant 1$.

When $\chi \equiv \chi_0$, a principal character, then the function $1/L(s+\lambda, \chi)$ approaches 0 as $s+\lambda$ approaches 1 through real values greater than 1; otherwise it approaches a finite value $1/L(1, \chi)$ which is different from zero.

We distinguish two cases. If χ is the principal character $\chi_0 \pmod k$ (including the case when $k = 1$), and if in (7.7) we set $s+\lambda = 1$, we have

$$\sum_{q=1}^{\infty} \frac{1}{q} c_q^{(\lambda)}(n, \chi_0, \chi_1, \chi_2) = 0. \tag{7.8}$$

If, furthermore, we take $\chi_2(a) \equiv 1$ and $\chi_1(a) = \chi_0(a)$ and employ (4.14), we find that

$$\Sigma_q' \frac{1}{q} c_q^{(\lambda)}(n) = 0, \tag{7.9}$$

where the sum is taken for all positive integers q which are prime to the given positive integer k. When $k = 1$ and $\lambda = 1$, we have the following remarkable special case due to Ramanujan (*loc. cit.*, 185):

$$c_1(n) + \tfrac{1}{2} c_2(n) + \tfrac{1}{3} c_3(n) + \ldots = 0.$$

Similarly, from (7.7) we have

$$\sigma_{\lambda-1}(n, \chi_1, \chi_2) = L(1, \chi) \sum_{q=1}^{\infty} \frac{1}{q} c_q^{(\lambda)}(n, \chi, \chi_1, \chi_2), \tag{7.10}$$

where $\chi(a)$ is any character modulo k other than the principal character. Here the factor $L(1, \chi)$ has the (non-zero) value (Landau, *Handbuch*, p. 436)

$$L(1, \chi) = \int_0^1 \frac{\chi(1) + \chi(2)\, u + \chi(3)\, u^2 + \ldots + \chi(k)\, u^{k-1}}{1 - u^k}\, du. \tag{7.11}$$

When $(k_2, n) = 1$ we see readily from (5.4) that

$$\sum_{q=1}^{\infty} \frac{c_q^{(\lambda)}(n, \chi, \chi_1, \chi_{20})}{q^s} = \prod_a \left(1 + \frac{c_a^{(\lambda)}(n, \chi, \chi_1, \chi_{20})}{a^s} + \frac{c_{a^2}^{(\lambda)}(n, \chi, \chi_1, \chi_{20})}{a^{2s}} + \ldots\right) \tag{7.12}$$

$$= \prod_a \left(1 - \frac{c_a^{(\lambda)}(n, \chi, \chi_1, \chi_{20})}{a^s}\right)^{-1} \quad (\sigma > 1),$$

where the products are taken for a running over the primes in order of magnitude. Then from (7.4) we have

$$\sigma_{-s}(n, \chi_1, \chi_{20}) = L(s+\lambda, \chi) \prod_a \left(1 - \frac{c_a^{(\lambda)}(n, \chi, \chi_1, \chi_{20})}{a^{s+\lambda}}\right)^{-1} \quad (\sigma + \lambda > 1). \tag{7.13}$$

Similarly when $(k_1, n) = 1$, we have from (7.5) the relation

$$(7.14)\quad \sigma_s(n, \chi_{10}, \chi_2) = n^s L(s+\lambda, \chi) \prod_a \left(1 - \frac{c_\nu^{(\lambda)}(n, \chi, \chi_{10}, \chi_2)}{a^{s+\lambda}}\right)^{-1} \quad (\sigma+\lambda > 1).$$

The following special cases, obtained by putting $\chi_1(a) \equiv \chi(a)$ and $\chi_2(a) \equiv 1$, are worthy of notice:

$$(7.15)\quad \sigma_{-s}(n, \chi) = L(s+\lambda, \chi) \sum_{q=1}^{\infty} \chi(q)\, c_q^{(\lambda)}(n)\, q^{-s-\lambda} \qquad (\sigma+\lambda \geqslant 1),$$

$$(7.16)\quad \sigma_{-s}(n, \chi) = L(s+\lambda, \chi) \prod_a \{1 - \chi(q)\, c_q^{(\lambda)}(n)\, a^{-s-\lambda}\}^{-1} \quad (\sigma+\lambda > 1),$$

$$(7.17)\quad \sigma_{\lambda-1}(n, \chi) = L(1, \chi) \sum_{q=1}^{\infty} \frac{1}{q} \chi(q)\, c_q^{(\lambda)}(n),$$

where in the last equation $\chi(a)$ is not a principal character. When $\lambda = 1$ and $\chi(a) \equiv 1$, formula (7.15) reduces to a result due to Ramanujan [equation (6.1), *loc. cit.*, p. 184].

8. *Further properties of divisors.* Now we have

$$(8.1)\quad \sigma_s(n, \chi_1, \chi_2) = D(n, \chi_1, \chi_2) \text{ if } F(u, v) = u^s.$$

Then from formulae (6.2), (6.3), and (6.4) we have the following results:

$$(8.2)\quad \sigma_{-s}(n, \chi_1, \chi_2) = c_1^{(\lambda)}(n, \chi, \chi_1, \chi_2) \sum_{\nu=1}^{t} \frac{\chi(\nu)}{\nu^{s+\lambda}} + c_2^{(\lambda)}(n, \chi, \chi_1, \chi_2) \sum_{\nu=1}^{\frac{1}{2}t} \frac{\chi(\nu)}{(2\nu)^{s+\lambda}} + c_3^{(\lambda)}(n, \chi, \chi_1, \chi_2) \sum_{\nu=1}^{\frac{1}{3}t} \frac{\chi(\nu)}{(3\nu)^{s+\lambda}} + \ldots,$$

$$(8.3)\quad n^s \sigma_{-s}(n, \chi_1, \chi_2) = c_1^{(\lambda)}(n, \chi, \chi_2, \chi_1) \sum_{\nu=1} \chi(\nu)\, \nu^{s-\lambda} + c_2^{(\lambda)}(n, \chi, \chi_2, \chi_1) \sum_{\nu=1}^{\frac{1}{2}t} \chi(\nu)(2\nu)^{s-\lambda} + \ldots,$$

$$(8.4)\quad \sigma_{-s}(n, \chi_1, \chi_2) \log n = c_1^{(\lambda)}(n, \chi, \chi_1, \chi_2) \sum_{\nu=1} \frac{\chi(\nu) \log \nu}{\nu^{s+\lambda}} + c_2^{(\lambda)}(n, \chi, \chi_1, \chi_2) \sum_{\nu=1}^{\frac{1}{2}t} \frac{\chi(\nu) \log 2\nu}{(2\nu)^{s+\lambda}} + \ldots + n^{-s}\{c_1^{(\lambda)}(n, \chi, \chi_2, \chi_1) \sum_{\nu=1}^{r} \chi(\nu) \log \nu \,.\, \nu^{s-\lambda} + c_2^{(\lambda)}(n, \chi, \chi_2, \chi_1) \sum_{\nu=1}^{\frac{1}{2}r} \chi(\nu) \log 2\nu \,.\, (2\nu)^{s-\lambda} + \ldots\},$$

where $t \geqslant n$ and $r \geqslant n$. When $\lambda = 1$ and $\chi(a) \equiv \chi_1(a) \equiv \chi_2(a) \equiv 1$, the last three equations become respectively equations (5.1), (5.2), and (5.3) of Ramanujan's memoir (*loc. cit.*, p. 183).

Let us take $\sigma+\lambda > 1$ and consider equation (8.2) as t increases indefinitely. Then we have

$$\sum_{\nu=1}^{t/m} \frac{\chi(\nu)}{(m\nu)^{s+\lambda}} = \sum_{\nu=1}^{\infty} \frac{\chi(\nu)}{(m\nu)^{s+\lambda}} + O\left(\frac{1}{mt^{\sigma+\lambda-1}}\right) = \frac{1}{m^{s+\lambda}} L(s+\lambda, \chi) + O\left(\frac{1}{mt^{\sigma+\lambda-1}}\right).$$

On the right-hand side of (8.2) the number of sums is $[t]$. Then, if we employ (5.2), we see that

$$\sigma_{-s}(n, \chi, \chi_1, \chi_2) - L(s+\lambda, \chi) \sum_{\nu=1}^{t} \frac{c_\nu^{(\lambda)}(n, \chi, \chi_1, \chi_2)}{\nu^{s+\lambda}} = O\left(\frac{1}{t^{\sigma+\lambda-1}} \sum_{\nu=1}^{t} \frac{1}{\nu}\right).$$

Letting t become infinite, we are led again to formula (7.4) and thence to (7.5), so that we have a second proof of these two relations; this proof is based on the same considerations as those by aid of which Ramanujan proved the special case indicated at the end of § 7.

A similar generalization may be found for Ramanujan's formula (6.2) (*loc. cit.*, p. 184), by letting t become infinite in (8.4) and keeping r fixed. We merely state the result

$$(8.5) \quad \sigma_{-s}(n, \chi_1, \chi_2) \left\{\frac{L'(s+\lambda, \chi)}{L(s+\lambda, \chi)} + \log n\right\}$$

$$= L(s+\lambda, \chi) \sum_{q=1}^{\infty} \frac{c_q^{(\lambda)}(n, \chi, \chi_1, \chi_2) \log q}{q^{s+\lambda}}$$

$$+ n^{-s} \{c_1^{(\lambda)}(n, \chi, \chi_2, \chi_1) \sum_{\nu=1}^{r} \chi(\nu) \log \nu \,.\, \nu^{s-\lambda} + \ldots\},$$

where $r \geqslant n$ and $\sigma+\lambda > 1$.

If both the conditions $\lambda+\sigma > 1$ and $\lambda-\sigma > 1$ are satisfied, then we may allow both t and r to become infinite in (8.4), or we may allow r to become infinite in (8.5); thus we find that

$$(8.6) \quad \sigma_{-s}(n, \chi_1, \chi_2) \left\{\frac{L'(s+\lambda, \chi)}{L(s+\lambda, \chi)} + \frac{L'(\lambda-s, \chi)}{L(\lambda-s, \chi)} + \log n\right\}$$

$$= L(s+\lambda, \chi) \sum_{q=1}^{\infty} \frac{c_q^{(\lambda)}(n, \chi, \chi_1, \chi_2) \log q}{q^{s+\lambda}}$$

$$+ n^{-s} L(\lambda-s, \chi) \sum_{q=1}^{\infty} \frac{c_q^{(\lambda)}(n, \chi, \chi_2, \chi_1) \log q}{q^{\lambda-s}}.$$

From the discussion of the series in (7.6) it follows that both members of equation (7.7) are expansible in powers of $s+\lambda-1$. For the case

when $\chi(a) \equiv \chi_0(a)$, the principal character modulo k, one obtains relations of interest by equating coefficients of the first and second powers of $s+\lambda-1$ in these expansions. We have

$$\frac{1}{L(s+\lambda, \chi_0)} = c_{0k}(s+\lambda-1)+(c_{1k}-\gamma c_{0k})(s+\lambda-1)^2+\ldots,$$

where γ is Euler's constant and where $c_{01}=1$, $c_{11}=0$, and

$$c_{0k} = \prod_{p \mid k} \frac{p}{p-1}, \quad c_{1k} = -c_{0k} \sum_{p \mid k} \frac{\log p}{p-1} \quad (k>1),$$

p in each case ranging over the prime factors of k. Also we have

$$\sigma_{-s}(n, \chi_1, \chi_2) = \sum_{d \mid n} \chi_1(d)\chi_2\left(\frac{n}{d}\right) d^{\lambda-1} d^{-(s+\lambda-1)}$$

$$= \sum_{d \mid n} \chi_1(d)\chi_2\left(\frac{n}{d}\right) d^{\lambda-1}\{1-(s+\lambda-1)\log d+\ldots\}.$$

Equating coefficients as indicated, we have

$$(8.7) \qquad \sum_{q=1}^{\infty} \frac{1}{q} c_q^{(\lambda)}(n, \chi_0, \chi_1, \chi_2) \log q = -c_{0k}\sigma_{\lambda-1}(n, \chi_1, \chi_2),$$

$$(8.8) \quad \sum_{q=1}^{\infty} \frac{1}{q} c_q^{(\lambda)}(n, \chi_0, \chi_1, \chi_2)(\log q)^2$$

$$= -2c_{0k} \sum_{d \mid n} \chi_1(d)\chi_2\left(\frac{n}{d}\right) d^{\lambda-1} \log d + 2(c_{1k}-\gamma c_{0k})\sigma_{\lambda-1}(n, \chi_1, \chi_2).$$

[Special cases of these formulae are given by Ramanujan in (8.3) and (8.4) (*loc. cit.*, p. 186).] It was by equating constant terms in the two expansions that we obtained equation (7.8). The results obtained by this process are obviously deep-lying, owing to the character of their dependence on the convergence of the series in (7.6).

9. *On Euler's function* $\phi(n)$. This section contains a generalization and extension of § 9 of Ramanujan's memoir (*loc. cit.*, pp. 186-7). If $\phi(n)$ is the number of integers prime to and not exceeding n, then

$$\phi(n) = \sum_{d \mid n} \mu(d) \frac{n}{d}.$$

We generalize $\phi(n)$ by introducing the function

$$(9.1) \qquad \phi_s(n, \chi_1, \chi_2) = \sum_{d \mid n} \mu(d)\chi_1(d)\chi_2\left(\frac{n}{d}\right)\left(\frac{n}{d}\right)^s$$

$$= D(n, \chi_1, \chi_2) \quad \text{for} \quad F(u, v) = \mu(u) v^s.$$

Then from (6.2) we have

$$(9.2)\quad n^{-s}\phi_s(n, \chi_1, \chi_2) = c_1^{(\lambda)}(n, \chi, \chi_1, \chi_2) \sum_{\nu=1}^{t} \frac{\chi(\nu)\mu(\nu)}{\nu^{s+\lambda}}$$
$$+c_2^{(\lambda)}(n, \chi, \chi_1, \chi_2) \sum_{\nu=1}^{\frac{1}{2}t} \frac{\chi(\nu)\mu(2\nu)}{(2\nu)^{s+\lambda}}$$
$$+c_3^{(\lambda)}(n, \chi, \chi_1, \chi_2) \sum_{\nu=1}^{\frac{1}{3}t} \frac{\chi(\nu)\mu(3\nu)}{(3\nu)^{s+\lambda}} + \ldots \quad (t \geqq n).$$

If $\sigma+\lambda > 1$ we may allow t to become infinite as in § 8; thus we find that

$$(9.3)\quad \frac{\phi_s(n, \chi_1, \chi_2)}{n^s} = c_1^{(\lambda)}(n, \chi, \chi_1, \chi_2) \sum_{\nu=1}^{\infty} \frac{\chi(\nu)\mu(\nu)}{\nu^{s+\lambda}}$$
$$+c_2^{(\lambda)}(n, \chi, \chi_1, \chi_2) \sum_{\nu=1}^{\infty} \frac{\chi(\nu)\mu(2\nu)}{(2\nu)^{s+\lambda}}$$
$$+c_3^{(\lambda)}(n, \chi, \chi_1, \chi_2) \sum_{\nu=1}^{\infty} \frac{\chi(\nu)\mu(3\nu)}{(3\nu)^{s+\lambda}} + \ldots \quad (\sigma+\lambda > 1).$$

Now we have

$$(9.4)\quad \frac{1}{L(s, \chi)} = \sum_{a=1}^{\infty} \frac{\chi(a)\mu(a)}{a^s} = \prod_{\omega} \left(1 - \frac{\chi(\omega)}{\omega^s}\right),$$

where ω ranges over the primes in order of magnitude. If m is a given positive integer and we omit from $\mu(m)$ multiplied by the product in the last member of (9.4) those factors for which ω is a divisor of m, we obtain the series

$$\mu(m) \sum_{a}{}' \frac{\chi(a)\mu(a)}{a^s} = \sum_{\nu=1}^{\infty} \frac{\chi(\nu)\mu(m\nu)}{\nu^s},$$

where, in the first sum, a ranges over the integers prime to m. Therefore

$$\sum_{\nu=1}^{\infty} \frac{\chi(\nu)\mu(m\nu)}{(m\nu)^{s+\lambda}} = \frac{\mu(m)}{L(s+\lambda, \chi)\phi_{s+\lambda}(m, \chi, 1)},$$

since

$$\phi_s(m, \chi, 1) = m^s \prod_{p|m} \left(1 - \frac{\chi(p)}{p^s}\right),$$

where the product is taken for the prime factors p of m. Then from (9.3) we have the relation

$$(9.5)\quad n^{-s}\phi_s(n, \chi_1, \chi_2) L(s+\lambda, \chi) = \sum_{\nu=1}^{\infty} \frac{\mu(\nu) c_\nu^{(\lambda)}(n, \chi, \chi_1, \chi_2)}{\phi_{s+\lambda}(\nu, \chi, 1)} \quad (\sigma+\lambda > 1).$$

If we introduce the special notation

$$(9.6)\quad \phi_s(n, \chi) = \phi_s(n, \chi, 1), \quad \phi_s(n) = \phi_s(n, 1, 1),$$

then it is obvious that

$$\phi_s(n, \chi_0) = \phi_s(n) \quad \text{if} \quad (n, k) = 1. \tag{9.7}$$

Then from (9.5) we have

$$\frac{\phi_s(n)}{n^s} = \frac{1}{L(s+\lambda, \chi_0)} \sum_{\nu}{}' \frac{\mu(\nu)\, c_\nu^{(\lambda)}(n)}{\phi_{s+\lambda}(\nu)} \quad \text{if} \quad (n, k) = 1, \tag{9.8}$$

the summation being for ν prime to k. When $\lambda = 1$ and $k = 1$, this becomes Ramanujan's formula (9.6) (*loc. cit.*, p. 187).

It is easy to show that

$$\frac{L(r+s-\lambda, \chi_2)}{L(r, \chi_1)} = \sum_{n=1}^{\infty} \frac{\phi_{\lambda-s}(n, \chi_1, \chi_2)}{n^r}. \tag{9.9}$$

This relation might well serve as the definition of the function ϕ. If in (4.8) we replace χ, χ_1, χ_2 by χ_1, χ_2, χ respectively and compare the result with (9.9), we see that

$$L(s, \chi)\, \phi_{\lambda-s}(n, \chi_1, \chi_2) = \sum_{\nu=1}^{\infty} \frac{c_n^{(\lambda)}(\nu, \chi_1, \chi_2, \chi)}{\nu^s}. \tag{9.10}$$

A special case of this is due to Ramanujan (*loc. cit.*, p. 199). Equation (9.10) may also be proved directly by writing the left member as a product of two Dirichlet series

$$\sum_{\nu=1}^{\infty} \frac{\chi(\nu)}{\nu^s} \sum_{\nu=1}^{\infty} \frac{\mu(n/\nu)\chi_1(n/\nu)\chi_2(\nu)\eta_\nu^\lambda(n)}{\nu^s}$$

and then multiplying in the usual way. Thence it follows readily that the series in the second member of (9.10) converges when $\sigma > 1$, and furthermore that it converges when $\sigma > 0$, provided that χ is not a principal character.

But the series in (9.10) also converges for all positive σ in some other cases, and, in particular, in the case when $\lambda = 1$, $\chi_1(a) \equiv 1 \equiv \chi_2(a)$, $\chi(a) \equiv \chi_0(a)$, and $n > 1$, since for every positive integer a we have

$$\begin{aligned} \sum_{\nu=1}^{akn} c_n(\nu, 1, 1, \chi_0) &= \sum_{d \mid n} \mu\left(\frac{n}{d}\right) \sum_{\nu=1}^{akn} \chi_0\left(\frac{\nu}{d}\right) \eta_d(\nu) \\ &= \sum_{d\ n} \mu\left(\frac{n}{d}\right) \sum_{\nu=1}^{akn/d} \chi_0(\nu)\, d \\ &= akn\phi(k) \sum_{d \mid n} \mu\left(\frac{n}{d}\right) = 0 \quad \text{if} \quad n > 1. \end{aligned}$$

Therefore we have

$$(9.11) \qquad \sum_{\nu=1}^{\infty} \frac{c_n(\nu, 1, 1, \chi_0)}{\nu^s} = L(s, \chi_0)\, n^{1-s} \sum_{d|n} \mu(d)\, d^{s-1} \quad (n>1,\ \sigma>0),$$

where it is understood that the second member for $s=1$ is to be replaced by the limit approached by it as s approaches 1, so that in this case we have

$$(9.12) \qquad \sum_{\nu=1}^{\infty} \frac{1}{\nu} c_n(\nu, 1, 1, \chi_0) = -a_k \Lambda(n) \quad (n>1),$$

where $\Lambda(n)$ is $\log \omega$ if n is a power of a prime ω and is otherwise zero, and where

$$a_1 = 1, \quad a_k = \prod_{p|k} \left(1 - \frac{1}{p}\right) \quad (k>1),$$

the product being taken for p ranging over the prime factors of k. The case $k=1$ of these results is due to Ràmanujan (*loc. cit.*, p. 199).

10. *Representations as sums of squares.* We denote by

$$R_i(a) \quad (i=1, 2, 3, 4),$$

the four characters modulo 8, where

$$R_1(a) = 1, 1, 1, 1; \qquad R_2(a) = 1, -1, 1, -1;$$

$$R_3(a) = 1, 1, -1, -1; \quad R_4(a) = 1, -1, -1, 1,$$

according as $a \equiv 1, 3, 5, 7 \pmod 8$. Let $d_i(n)$ be the number of factors of n of the form $8m+i$. We let $N[n = x^2+y^2]$ denote the number of representations of n as a sum of two squares and employ other similar notations.

It is well known that $N[n = x^2+y^2] = 4\{d_1(n)+d_5(n)-d_3(n)-d_7(n)\}$. But the latter sum is equal to $4\sigma_0(n, R_2)$. Therefore, from (7.17) and (7.11), it follows that

$$(10.1) \quad N[n = x^2+y^2] = \pi\{c_1(n) - \tfrac{1}{3}c_3(n) + \tfrac{1}{5}c_5(n) - \tfrac{1}{7}c_7(n) + \ldots\}.$$

Simplifying by aid of (7.9) for $\lambda = 1$ and $k = 2$ we have

$$(10.2) \qquad N[n = x^2+y^2] = 2\pi\{c_1(n) + \tfrac{1}{5}c_5(n) + \tfrac{1}{9}c_9(n) + \ldots\}.$$

In a similar way, by employing the characters $R_i(a)$ and known relations involving divisors of n, we have the following results:

$$(10.3)\quad N\,[n=x^2+2y^2]=\pi\sqrt{2}\sum_{\nu=0}^{\infty}\left\{\frac{1}{8\nu+1}c_{8\nu+1}(n)+\frac{1}{8\nu+3}c_{8\nu+3}(n)\right\},$$

$$(10.4)\quad N\,[n=x^2-2y^2,\ 2x>3y\geqslant 0]$$

$$=\sqrt{2}\log(\sqrt{2}+1)\sum_{\nu=0}^{\infty}\left\{\frac{1}{8\nu+1}c_{8\nu+1}(n)+\frac{1}{8\nu+7}c_{8\nu+7}(n)\right\},$$

$$(10.5)\quad N\,[n=x^2+y^2+z^2+t^2,\ n\text{ odd}]=\pi^2 n\left\{c_1(n)+\frac{c_3(n)}{3^2}+\frac{c_5(n)}{5^2}+\ldots\right\},$$

$$(10.6)\quad N\,[2^a m=x^2+y^2+z^2+t^2,\ a>0,\ m\text{ odd}]$$

$$=3\pi^2 m\left\{c_1(m)+\frac{c_3(m)}{3^2}+\frac{c_5(m)}{5^2}+\ldots\right\},$$

$$(10.7)\quad N\,[4m=x^2+y^2+z^2+t^2,\ mxyzt\text{ odd}]$$

$$=\frac{\pi^2 m}{8}\left\{c_1(m)+\frac{c_3(m)}{3^2}+\frac{c_5(m)}{5^2}+\ldots\right\}.$$

Ramanujan (*loc. cit.*, p. 189) gives the equation

$$(10.8)\quad N\,[n=x^2+y^2+z^2+t^2]$$

$$=\pi^2 n\left\{c_1(n)-\frac{c_4(n)}{2^2}+\frac{c_3(n)}{3^2}-\frac{c_8(n)}{4^2}+\frac{c_5(n)}{5^2}-\frac{c_{12}(n)}{6^2}+\ldots\right\}.$$

Comparing this with (10.5) we see that

$$(10.9)\qquad \frac{c_4(n)}{1^2}+\frac{c_8(n)}{2^2}+\frac{c_{12}(n)}{3^2}+\ldots=0\ \text{ if } n \text{ is odd.}$$

Using $D_i(n)$, $i=1, 2, 3, 4$, for the functions $D(n)$ employed by Ramanujan (*loc. cit.*, pp. 187-189) in the expansions of his functions X_i, $i=1, 2, 3, 4$, respectively, we may without difficulty verify the relations

$$(10.10)\quad D_1(n)=(-1)^{n-1}\{2\sigma_{s-1}(n, R_1)-\sigma_{s-1}(n)\},$$

$$(10.11)\quad D_2(n)=\sigma_{s-1}(n, R_1)+2n^{s-1}\sigma_{-s+1}(\tfrac{1}{2}n, R_1)-n^{s-1}\sigma_{-s+1}(\tfrac{1}{2}n),$$

$$(10.12)\quad D_3(n)=n^{s-1}\sigma_{-s+1}(n, R_2),$$

$$(10.13)\quad D_4(n)=\sigma_{s-1}(n, R_2),$$

where a σ with first argument not equal to an integer is to be replaced by zero. [There are also other ways in which to express these functions in terms of the σ-functions (see Ramanujan, *loc. cit.*, p. 158).] Thence one may show readily that the important functions $\delta_{2s}(n)$, discussed by Ramanujan, have the following values:

$$(10.14)\quad \delta_{2s}(n) = \frac{2s}{(2^s-1)\,B_{\frac{1}{2}s}}(-1)^{n-1}\{2\sigma_{s-1}(n,\ R_1)-\sigma_{s-1}(n)\} \quad \text{when} \quad s = 4l > 0,$$

$$(10.15)\quad \delta_{2s}(n) = \frac{2s}{(2^s-1)\,B_{\frac{1}{2}s}}\{\sigma_{s-1}(n,\ R_1)+2n^{s-1}\sigma_{-s+1}(\tfrac{1}{2}n,\ R_1) - n^{s-1}\sigma_{-s+1}(\tfrac{1}{2}n)\} \quad \text{when} \quad s = 4l+2 > 0,$$

$$(10.16)\quad \delta_{2s}(n) = \frac{2^{s+1}}{E_{\frac{1}{2}(s-1)}}\{n^{s-1}\sigma_{-s+1}(n, R_2)+2^{1-s}\sigma_{s-1}(n,\ R_2)\} \quad \text{when} \quad s = 4l+1 > 1,$$

$$(10.17)\quad \delta_{2s}(n) = \frac{2^{s+1}}{E_{\frac{1}{2}(s-1)}}\{n^{s-1}\sigma_{-s+1}(n,\ R_2)-2^{1-s}\sigma_{s-1}(n,\ R_2)\} \quad \text{when} \quad s = 4l+3 > 0,$$

the B's and E's being respectively the Bernoullian and Eulerian numbers (see the next section).

The expressions containing the argument $\frac{1}{2}n$ may be replaced by simpler ones. Let $n = 2^\alpha m$, where m is odd. Then it may be shown without difficulty that

$$(10.18)\quad \sigma_{s-1}(n,\ R_1)+2n^{s-1}\sigma_{-s+1}(\tfrac{1}{2}n,\ R_1)-n^{s-1}\sigma_{-s+1}(\tfrac{1}{2}n) = g_\alpha\,\sigma_{s-1}(m),$$

where $$g_0 = 1, \quad g_\alpha = 2^{\alpha(s-1)}+2-\frac{2^{\alpha(s-1)}-1}{2^{s-1}-1} \quad \text{when} \quad \alpha > 0.$$

The properties of the σ-functions are thus available for determining properties of Ramanujan's functions D and δ, which he associates with the problem of the representation of integers as sums of an even number of squares. One may also obtain similar relations involving his functions $S_{2s}(n)$, which he (*loc. cit.*, pp. 190-192) associates with the problem of the representation of integers as sums of an even number of triangular numbers.

11. *Asymptotic relations involving divisors.* Putting $\lambda = 1$ in (7.15), we have

$$(11.1)\qquad \sigma_{-s}(n,\ \chi) = L(s+1,\ \chi)\sum_{q=1}^{\infty}\frac{\chi(q)\,c_q(n)}{q^{s+1}} \quad (\sigma > 0).$$

We employ the well-known formulae:

$$1+\frac{1}{2^{2\lambda}}+\frac{1}{3^{2\lambda}}+\frac{1}{4^{2\lambda}}+\ldots=\pi^{2\lambda}\frac{2^{2\lambda-1}B_\lambda}{(2\lambda)!} \quad (\lambda>0), \tag{11.2}$$

$$1+\frac{1}{3^{2\lambda}}+\frac{1}{5^{2\lambda}}+\frac{1}{7^{2\lambda}}+\ldots=\pi^{2\lambda}\frac{(2^{2\lambda}-1)B_\lambda}{2.(2\lambda)!} \quad (\lambda>0), \tag{11.3}$$

$$1-\frac{1}{3^{2\lambda+1}}+\frac{1}{5^{2\lambda+1}}-\frac{1}{7^{2\lambda+1}}+\ldots=\frac{\pi^{2\lambda+1}E_\lambda}{2^{2\lambda+2}.(2\lambda)!} \quad (\lambda\geqslant 0), \tag{11.4}$$

where λ is an integer, the B's are the Bernoullian numbers

$$B_1=\tfrac{1}{6},\quad B_2=\tfrac{1}{30},\quad B_3=\tfrac{1}{42},\quad B_4=\tfrac{1}{30},\quad B_5=\tfrac{5}{66},\quad \ldots,$$

and the E's are the Eulerian numbers

$$E_0=1,\quad E_1=1,\quad E_2=5,\quad E_3=61,\quad E_4=1385,\quad \ldots,$$

defined by the expansion

$$\sec x=\sum_{\nu=0}^{\infty}\frac{x^{2\nu}}{(2\nu)!}E_\nu.$$

Putting $\chi(a)\equiv 1$ and $s=2\lambda-1$ in (11.1), where λ is a positive integer, we have

$$\sigma_{2\lambda-1}(n)=n^{2\lambda-1}\sigma_{-(2\lambda-1)}(n)=n^{2\lambda-1}\pi^{2\lambda}\frac{2^{2\lambda-1}B_\lambda}{(2\lambda)!}\sum_{q=1}^{\infty}\frac{c_q(n)}{q^{2\lambda}}. \tag{11.5}$$

With the same substitution for s and with $\chi(a)$ equal to 1 or 0 according as a is odd or even, we have for odd n the relation

$$\sigma_{2\lambda-1}(n)=n^{2\lambda-1}\pi^{2\lambda}\frac{(2^{2\lambda}-1)B_\lambda}{2.(2\lambda)!}\sum_{q=1}^{\infty}\frac{c_{2q-1}(n)}{(2q-1)^{2\lambda}}. \tag{11.6}$$

By using the principal character modulo k the results in the last two equations may be generalized to give a formula for $\sigma_{2\lambda-1}(n)$ when n is prime to k. Other (but somewhat less simple) results hold also for general n.

If we multiply both sides of equation (11.1) by $c_\rho(n)$, sum for n from 1 to m and divide by m, we obtain the relation

$$\frac{1}{m}\sum_{n=1}^{m}c_\rho(n)\sigma_{-s}(n,\chi)=L(s+1,\chi)\frac{\chi(\rho)}{\rho^{s+1}}\frac{1}{m}\sum_{n=1}^{m}c_\rho^2(n)+S, \tag{11.7}$$

where

$$S=L(s+1,\chi)\sum_{q}{}'\frac{\chi(q)}{q^{s+1}}\frac{1}{m}\sum_{n=1}^{m}c_q(n)c_\rho(n),$$

the prime in Σ' denoting that in forming the sum the value $q=\rho$ is to be omitted.

By aid of (3.10) we see that

$$(11.8)\quad L(s+1,\chi)\frac{\chi(\rho)}{\rho^{s+1}}\frac{1}{m}\sum_{n=1}^{m}c_\rho^2(n)=L(s+1,\chi)\frac{\chi(\rho)\phi(\rho)}{\rho^s}+O\left(\frac{1}{m}\right).$$

Using (5.8), we show that

$$|S|\leqslant|L(s+1,\chi)|\sum_q{}'\frac{1}{q^{\tau+1}}\frac{1}{m}\left|\sum_{n=1}^{m}c_q(n)c_\rho(n)\right|$$

$$\leqslant\frac{1}{m}|L(s+1,\chi)|\sum_q{}'q^2\rho^2/q^{\tau+1}.$$

Therefore

$$(11.9)\qquad S=O\left(\frac{1}{m}\right)\quad\text{if}\quad\sigma>2.$$

From (11.7), (11.8), and (11.9), we conclude that

$$(11.10)\quad\frac{1}{m}\sum_{n=1}^{m}c_\rho(n)\sigma_{-s}(n,\chi)=L(s+1,\chi)\rho^{-s}\chi(\rho)\phi(\rho)+O\left(\frac{1}{m}\right)\quad(\sigma>2).$$

This general asymptotic relation is of considerable interest. Special cases of it have appeared in several papers; the reader will find references and analyses in Chapter 10 of volume 1 of Dickson's *History of the theory of numbers*, under the names Cesàro (pp. 291-294), Gronwall (p. 322), Gegenbauer (p. 301), and others.

Putting $\chi(a)\equiv 1$ in (11.10) we have

$$(11.11)\quad\frac{1}{m}\sum_{n=1}^{m}c_\rho(n)\sigma_{-s}(n)=\rho^{-s}\phi(\rho)\left(1+\frac{1}{2^{s+1}}+\frac{1}{3^{s+1}}+\ldots\right)+O\left(\frac{1}{m}\right)\ (\sigma>2).$$

Replacing s by the positive odd integer $2\lambda-1$ and employing (11.2), we have

$$(11.12)\qquad\frac{1}{m}\sum_{n=1}^{m}c_\rho(n)\sigma_{-(2\lambda-1)}(n)=\frac{\phi(\rho)}{\rho^{2\lambda-1}}\pi^{2\lambda}\frac{2^{2\lambda-1}B_\lambda}{(2\lambda)!}+O\left(\frac{1}{m}\right)\quad(\lambda>1),$$

λ being an integer.

Taking $\rho=1$ in (11.12) and remembering that $c_1(n)\equiv 1$, we have the theorem:

When λ is an integer greater than unity, the sum of the $(2\lambda-1)$-th powers of the reciprocals of the divisors of n is in the mean (on the average) equal to

$$\pi^{2\lambda}\frac{2^{2\lambda-1}B_\lambda}{(2\lambda)!},$$

the error term of the average for $n=1, 2, \ldots, m$ being $O(1/m)$.

Now $c_2(n) = (-1)^n$. Taking $\rho = 1$ and $\rho = 2$ in (11.12), we have two relations which may be combined so as to lead to the following theorem:

When λ is an integer greater than unity, the sum of the $(2\lambda-1)$-th powers of the reciprocals of the divisors of the odd [even] numbers

$$1,\ 3,\ \ldots,\ 2m-1 \quad [2,\ 4,\ \ldots,\ 2m]$$

is in the mean equal to

$$\pi^{2\lambda}\,\frac{(2^{2\lambda-1}+1)\,B_\lambda}{(2\lambda)!} \quad \left[\pi^{2\lambda}\,\frac{(2^{2\lambda-1}-1)\,B_\lambda}{(2\lambda)!}\right],$$

the error term in each case being of the form $O(1/m)$.

From the fact that no $c_q(n)$ is equal to zero for every n, and from the orthogonality of the functions $c_1(n)$, $c_2(n)$, ..., it follows that there is no linear relation among any finite number of these functions. Therefore, by considering (11.11) for the values 1, 2, ..., α of ρ, the mean of the sum of the s-th powers ($\sigma > 2$) of the reciprocals of the divisors of the m numbers $\tau\alpha+\beta$, $\tau = 0, 1, \ldots, m-1$, may be found for any given positive integral values of α and β. When s is an odd integer greater than unity we may obtain compact expressions for these asymptotic values in terms of classic constants.

When $\chi(a)$ is the principal character $\chi_0(a)$, we have $\sigma_{-s}(n, \chi_0)$ equal to the sum $\gamma_{-s}(n, k)$ of the s-th powers of the reciprocals of those divisors of n which are prime to the modulus k. Then we have

$$(11.13) \qquad \frac{1}{m}\sum_{n=1}^{m} c_\rho(n)\,\gamma_{-s}(n, k) = \frac{\chi_0(\rho)\,\phi(\rho)}{\rho^s}\,\sum_q{}' \frac{1}{q^{s+1}} + O\left(\frac{1}{m}\right) \quad (\sigma > 2),$$

where the prime in Σ' indicates that q ranges over the integers which are prime to k.

Putting $s = 2\lambda-1$, taking $k = 2$, and employing (11.3), we have

$$(11.14) \qquad \frac{1}{m}\sum_{n=1}^{m} c_\rho(n)\,\gamma_{-(2\lambda-1)}(n, 2) = \frac{\chi_0(\rho)\,\phi(\rho)}{\rho^{2\lambda-1}}\,\frac{(2^{2\lambda}-1)\,B_\lambda}{2\,.\,(2\lambda)!}\,\pi^{2\lambda} + O\left(\frac{1}{m}\right),$$

where $\chi_0(\rho)$ is 0 or 1 according as ρ is even or odd and where λ is an integer greater than unity.

For $\rho = 1$ and $\lambda = 2$ and 3 we have results due to Gegenbauer (Dickson, *loc. cit.*, p. 301), namely, the mean value of the cubes [fifth powers] of the reciprocals of the odd divisors of n is $\pi^4/96$ [$\pi^6/960$].

If $\rho = dv$, where d is any divisor of k greater than unity, then $\chi_0(dv) = 0$, whence we have from (11.13) the following remarkable

relation:

$$\lim_{m=\infty} \frac{1}{m} \sum_{n=1}^{m} c_{dv}(n)\gamma_{-s}(n, k) = 0 \quad [(d, k) > 1, \ \sigma > 2]. \tag{11.15}$$

This may obviously be generalized by means of (11.10).

Let us now replace $\chi(a)$ in (11.1) by the character $R_2(a)$ defined in § 10. Then we have

$$\sigma_{-s}(n, R_2) = \left(1 - \frac{1}{3^{s+1}} + \frac{1}{5^{s+1}} - \frac{1}{7^{s+1}} + \ldots\right) \times \left(c_1(n) - \frac{c_3(n)}{3^{s+1}} + \frac{c_5(n)}{5^{s+1}} - \frac{c_7(n)}{7^{s+1}} + \ldots\right) \quad (\sigma \geqslant 0). \tag{11.16}$$

The validity of this result when $\sigma = 0$ is implied by the discussion of convergence in § 7. It is easy to see that $\sigma_{-s}(n, R_2)$ is equal to the sum of the reciprocals of the s-th powers of those divisors of n which are of the form $4x+1$ minus the sum of the reciprocals of the s-th powers of those divisors of n which are of the form $4x+3$.

From (11.10) we have

$$\frac{1}{m} \sum_{n=1}^{m} c_\rho(n)\sigma_{-s}(n, R_2) = \frac{R_2(\rho)\phi(\rho)}{\rho^s}\left(1 - \frac{1}{3^{s+1}} + \frac{1}{5^{s+1}} - \ldots\right) + O\left(\frac{1}{m}\right) \quad (\sigma > 2). \tag{11.17}$$

Using (11.4) we obtain from the last two equations the following two:

$$\sigma_{-2\lambda}(n, R_2) = \frac{\pi^{2\lambda+1} E_\lambda}{2^{2\lambda+2} \cdot (2\lambda)!}\left\{c_1(n) - \frac{c_3(n)}{3^{2\lambda+1}} + \frac{c_5(n)}{5^{2\lambda+1}} - \ldots\right\}, \tag{11.18}$$

where λ is a non-negative integer;

$$\frac{1}{m} \sum_{n=1}^{m} c_\rho(n)\sigma_{-2\lambda}(n, R_2) = \frac{\pi^{2\lambda+1} E_\lambda}{2^{2\lambda+2} \cdot (2\lambda)!}\rho^{-2\lambda} R_2(\rho)\phi(\rho) + O\left(\frac{1}{m}\right), \tag{11.19}$$

where λ is an integer greater than unity.

Other special cases of interest arise from (11.1) and (11.10), and in particular those which are obtained when $\chi(a)$ is replaced by $R_3(a)$ or $R_4(a)$ as defined in § 10.

From equations (10.14) to (10.17) it is obvious that the results in this section imply important asymptotic properties of functions arising in the problem of the representation of integers as sums of an even number of squares; but we shall not develop such applications at the present.

SUB-SPACES OF A SPACE WITH TORSION

By H. A. HAYDEN.

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1. *Introduction.*

1.1. *Summary**. The curvature tensor $\Omega_{ij}^{\ a}$ of a sub-space of a Riemannian space is symmetric in i, j, and several theorems of Riemannian geometry which appear to involve $\Omega_{ij}^{\ a}$ actually involve only the symmetric part $\Omega_{(ij)}^{\ a}$. The object of the first part of this paper §§ (2–4) is to point out some of the consequences of the asymmetry of the tensor $\Omega_{ij}^{\ a}$ for spaces with torsion. In the later part of the paper (§§ 5, 6) we consider the geometry of a sub-space of a sub-space (V_l in V_m in V_n) and are led to develop a theory of *asymptotic sub-spaces*, these bearing the same relation to geodesic sub-spaces (in the Riemannian case†) as asymptotic lines to geodesic lines.

1.2. *Induced tensors and induced connection.* If V_n is a metrical space with metric tensor $g_{\alpha\beta}$ and torsion tensor $S_{\beta\gamma}^{\ \ \alpha}$, the affine connection $\Gamma_{\beta\gamma}^{\alpha}$ is defined by the equations

$$\nabla_\alpha g_{\beta\gamma} \equiv \frac{\partial g_{\beta\gamma}}{\partial x^\alpha} - \Gamma_{\beta\alpha}^{\lambda} g_{\lambda\gamma} - \Gamma_{\gamma\alpha}^{\lambda} g_{\beta\lambda} = 0, \tag{1}$$

$$\Gamma_{\beta\gamma}^{\alpha} - \Gamma_{\gamma\beta}^{\alpha} = 2S_{\beta\gamma}^{\ \ \alpha}, \tag{2}$$

there being $\frac{1}{2}n^2(n+1)$ equations (1) and $\frac{1}{2}n^2(n-1)$ equations (2), making n^3 in all, from which to determine the n^3 quantities $\Gamma_{\beta\gamma}^{\alpha}$. We find, in fact, on solving for the Γ's, that‡

$$\Gamma_{\beta\gamma}^{\alpha} = \left\{ \begin{matrix} \alpha \\ \beta\gamma \end{matrix} \right\} + S_{\beta\gamma}^{\ \ \alpha} - 2S_{\cdot(\beta\gamma)}^{\alpha}, \tag{3}$$

where $\left\{ \begin{matrix} \alpha \\ \beta\gamma \end{matrix} \right\}$ are the Christoffel symbols.

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* For the notation employed, *cf.* H. A. Hayden, *Proc. London Math. Soc.* (2), 33 (1931), 22–31, except that V_n there denotes a Riemannian space and here a general metrical space. Greek indices take values 1, 2, ..., n; Latin indices values 1, ..., m, except where otherwise stated. Indices written vertically above or below a letter have a purely ordinal significance.

† Or to *hyperplanes* (§ 4.1 *post*) in the non-Riemannian case.

‡ J. A. Schouten, *Der Ricci-Kalkül* (1924), 73.

If the torsion tensor vanishes identically (*i.e.* if the connection is symmetric) the space is Riemannian; in that case we shall denote it* by $\overset{0}{V}_n$.

If X_m is an m-dimensional manifold contained in V_n and defined by equations

$$x^\alpha = x^\alpha(x'^1, \ldots, x'^m) \quad (\alpha = 1, \ldots, n),$$

and if $T_{\alpha\beta\gamma\ldots}$ is any covariant tensor in V_n at a point of X_m, the numbers

$$T'_{ijk\ldots} \equiv T_{\alpha\beta\gamma\ldots}\theta_i^{\ \alpha}\theta_j^{\ \beta}\theta_k^{\ \gamma}\ldots,$$

where

$$\theta_i^{\ \alpha} \equiv \frac{\partial x^\alpha}{\partial x'^i},$$

constitute a covariant tensor in X_m (*i.e.* $T'_{ijk\ldots}$ behaves like a covariant tensor for transformations of the X_m-coordinates x'^i). We call $T'_{ijk\ldots}$ the *induced X_m-tensor* derived from the V_n-tensor $T_{\alpha\beta\gamma\ldots}$.

In particular, from $g_{\alpha\beta}$ we derive the tensor

$$g'_{ij} \equiv g_{\alpha\beta}\theta_i^{\ \alpha}\theta_j^{\ \beta}, \tag{4}$$

the *induced metric tensor*.

If g'^{ij} is the reciprocal element to g'_{ij} in the determinant $|g'_{ij}|$, we define

$$\theta^i_{\ .\alpha} \equiv g'^{ij}g_{\alpha\beta}\theta_j^{\ \beta}, \tag{5}$$

and we can then define induction for V_n-tensors of mixed ranks on X_m. For, if $T^{\alpha\beta\ldots}_{\lambda\mu\ldots}$ is any V_n-tensor at a point of X_m, it is easily verified that

$$T'^{ij\ldots}_{kl\ldots} \equiv T^{\alpha\beta\ldots}_{\lambda\mu\ldots}\theta^i_{\ .\alpha}\theta^j_{\ .\beta}\ldots\theta_k^{\ \lambda}\theta_l^{\ \mu}\ldots$$

is an X_m-tensor. This is defined as the *induced tensor* derived from $T^{\alpha\beta\ldots}_{\lambda\mu\ldots}$.

In particular, we obtain the *induced torsion tensor*

$$S'^{\ \ i}_{jk} \equiv S^{\ \ \alpha}_{\beta\gamma}\theta^i_{\ .\alpha}\theta_j^{\ \beta}\theta_k^{\ \gamma}. \tag{6}$$

The tensors g'_{ij}, $S'^{\ \ i}_{jk}$ determine a connection $\Gamma'^{\ i}_{jk}$, defined by the equations [analogous to (1) and (2)]

$$\frac{\partial g'_{ij}}{\partial x'^k} - \Gamma'^{\ l}_{ik}g'_{lj} - \Gamma'^{\ l}_{jk}g'_{il} = 0, \tag{7}$$

$$\Gamma'^{\ i}_{jk} - \Gamma'^{\ i}_{kj} = 2S'^{\ \ i}_{jk}; \tag{8}$$

this we call the *induced connection*.

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* *Cf.* Enea Bortolotti, *Rend. di Palermo*, 51 (1927), 98-105 (100).

If now we endow X_m with a metric and a torsion tensor it becomes a V_m (metrical affine space), and if its metric and torsion tensors are those induced by V_n we say that it is *immersed* in V_n. In that case its connection is the induced connection Γ'^{i}_{jk}.

Solving (7) and (8), we have [*cf.* (3)]

$$\Gamma'^{i}_{jk} = \left\{ \begin{matrix} i \\ jk \end{matrix} \right\}' + S'^{\cdot\cdot i}_{jk} - 2S'^{i}_{\cdot(jk)}, \tag{9}$$

where $\left\{ \begin{matrix} i \\ jk \end{matrix} \right\}'$ are the Christoffel symbols formed from g'_{ij}, and from (3) and (9) we have

$$\Gamma'^{i}_{jk} = \Gamma^{\alpha}_{\beta\gamma}\theta^{i}_{\cdot\alpha}\theta^{\beta}_{j}\theta^{\gamma}_{k} + \theta^{i}_{\cdot\alpha}\partial\theta^{\alpha}_{j}/\partial x'^{k}. \tag{10}$$

The importance of choosing the induced metric and induced torsion tensor as metric and torsion tensor for V_m lies in the fact that with such a choice of fundamental tensors the following theorems hold*:

(i) A vector tangential to V_m has the same length whether measured in V_n or in V_m;

(ii) If a field of vectors tangential to V_m is parallel in V_n along a curve in V_m, it is also a parallel field in V_m along the same curve (though not *vice versa* in general).

1.3. *Absolute differentiation of tensors in a sub-space.* The absolute differential ($DT^{\cdots}_{\cdots}$) and the covariant derivative ($\nabla_k T^{\cdots}_{\cdots}$) of tensors having double tensor character, with respect to V_n and to V_m, have been defined for Riemannian spaces in a previous paper†. They are defined for spaces with torsion in a similar manner, the Γ's in this case being coefficients of connection and not Christoffel symbols.

The curvature tensor of V_m in V_n is defined as

$$\Omega^{\cdot\alpha}_{ij} \equiv \nabla_i \theta^{\alpha}_{j} = \frac{\partial\theta^{\alpha}_{j}}{\partial x'^{i}} + \Gamma^{\alpha}_{\gamma\beta}\theta^{\beta}_{i}\theta^{\gamma}_{j} - \Gamma'^{k}_{ji}\theta^{\alpha}_{k}. \tag{11}$$

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* Most authors take these two theorems (or their equivalents) as *defining* the induced metric and induced connection. *Cf.* Schouten, *loc. cit.*, 137, 158, 175; V. Hlavatý, *Věstník Král. Čes. Spol. Nauk*, 2 (1926), 11; E. Bortolotti, *Boll. Unione Matem. Italiana*, 7 (1928), 87–94. A more general form of the second theorem above is: If v^{α} is any tangential vector field over V_m, and v'^{i} is the induced vector field,

$$\nabla'_k v'^{i} = \theta^{\lambda}_{k}\theta^{i}_{\cdot\alpha}\nabla_{\lambda}v^{\alpha},$$

where ∇'_k denotes covariant differentiation with respect to the induced connection.

† Hayden, *loc. cit.*, 23–24 (§ 3); *vide* also Duschek und Mayer, *Lehrbuch der Differentialgeometrie* 2 (1930), 157 ("verallgemeinerte Riccidifferential").

1.4. *Analytical preliminaries.*

1.41. From (11), (10), and the fact that

$$\theta^i_{\cdot\alpha}\,\theta_j^{\cdot\alpha} = \delta^i_j, \tag{12}$$

which follows immediately from (5), we have

$$\Omega_{ij}^{\cdot\cdot\alpha}\,\theta^k_{\cdot\alpha} = 0\,; \tag{13}$$

whence each of the m^2 vectors $\Omega_{ij}^{\cdot\cdot\alpha}$ $(i, j = 1, \ldots, m)$ is perpendicular to each of the vectors $\theta_i^{\cdot\alpha}$ $(i = 1, \ldots, m)$ and is therefore normal to V_m.

Thus, if $\underset{r}{\chi^\alpha}$ $(r = 1, \ldots, q$, where $q \equiv n-m)$ are q mutually orthogonal unit vector fields over V_m, each orthogonal to V_m,

$$\Omega_{ij}^{\cdot\cdot\alpha} = -\sum_{r=1}^{q} \overset{r}{\omega}_{ij}\,\underset{r}{\chi^\alpha}, \tag{14}$$

where the ω's are so defined.

If $\Omega_{[ij]}^{\cdot\cdot\alpha}$ is the skew-symmetric part of the curvature tensor, viz. $\frac{1}{2}(\Omega_{ij}^{\cdot\cdot\alpha} - \Omega_{ji}^{\cdot\cdot\alpha})$, we have, from (11),

$$\Omega_{[ij]}^{\cdot\cdot\alpha} = S'^{\cdot\cdot k}_{ij}\,\theta_k^{\cdot\alpha} - S_{\beta\gamma}^{\cdot\cdot\alpha}\,\theta_i^{\cdot\beta}\,\theta_j^{\cdot\gamma}. \tag{15}$$

Hence

$$\overset{r}{\omega}_{[ij]} = S_{\beta\gamma}^{\cdot\cdot\alpha}\,\underset{r}{\chi_\alpha}\,\theta_i^{\cdot\beta}\,\theta_j^{\cdot\gamma}, \tag{16}$$

and

$$\Omega_{[ij]}^{\cdot\cdot\alpha} = S_{\gamma\beta}^{\cdot\cdot\lambda}\,C^\alpha_\lambda\,\theta_i^{\cdot\beta}\,\theta_j^{\cdot\gamma}, \tag{17}$$

where

$$C^\alpha_\beta \equiv \sum_{r=1}^{q} \underset{r}{\chi^\alpha}\,\underset{r}{\chi_\beta}.$$

The content of equation (15) is shown in the following geometrical interpretation. Let dx'^i, $\delta x'^i$ be two infinitesimal vectors in V_m, at the same point, and suppose dx'^i displaced parallelly in V_m along $\delta x'^i$, and $\delta x'^i$ displaced parallelly in V_m along dx'^i. If V_m is Riemannian we obtain a closed parallelogram, but if V_m has torsion the vector joining the free ends of the displaced vectors is given (to the second order of infinitesimals) by $2S'^{\cdot\cdot k}_{ij}\,dx'^i\,\delta x'^j$. This is the *torsion vector* of V_m associated with the bivector $dx'^{[i}\,\delta x'^{j]}$. If now we regard the infinitesimal vectors as V_n-vectors, their components being $dx^\alpha \equiv \theta_i^{\cdot\alpha}\,dx'^i$ and $\delta x^\alpha \equiv \theta_i^{\cdot\alpha}\,\delta x'^i$, and suppose them displaced parallelly *in* V_n, we obtain the torsion vector of V_n associated with the same bivector. It is $2S_{\beta\gamma}^{\cdot\cdot\alpha}\,dx^\beta\,\delta x^\gamma$, *i.e.*

$$2S_{\beta\gamma}^{\cdot\cdot\alpha}\,\theta_i^{\cdot\beta}\,\theta_j^{\cdot\gamma}\,dx'^i\,\delta x'^j.$$

Also the torsion vector of V_m, when regarded as a V_n-vector, has

components $2\theta_k^\alpha S'^{\cdot\cdot k}_{ij}\, dx'^i \delta x'^j$. Hence (15) expresses that the V_n-torsion vector is the sum of the two components:

(i) the V_m-torsion vector, tangential to V_m,

(ii) the vector $2\Omega^{\cdot\cdot\alpha}_{[ji]}\, dx'^i \delta x'^j$, normal to V_m.

The tangential component vanishes for every bivector if V_m is Riemannian. The normal component vanishes for every bivector and for every V_m if V_n is *semi-symmetric*, *i.e.* if

$$S^{\cdot\cdot\alpha}_{\beta\gamma} = S_{[\beta}\delta^\alpha_{\gamma]}.$$

Both components vanish if V_n is Riemannian.

1.42. Since the m-space tangential to V_m is determined by the m vectors θ_i^α $(i = 1, \ldots, m)$ and the q-space normal to V_m by the q vectors $\underset{r}{\chi^\alpha}$ $(r = 1, \ldots, q)$, every V_n-vector at a point of V_m is expressible linearly in terms of those n vectors.

From the relations

$$g_{\alpha\beta}\underset{r}{\chi^\alpha}\theta_j^\beta = 0,$$

we have
$$g_{\alpha\beta}\theta_j^\beta \nabla_i \underset{r}{\chi^\alpha} = -g_{\alpha\beta}\underset{r}{\chi^\alpha}\Omega^{\cdot\cdot\beta}_{ij} = \overset{r}{\omega}_{ij}.$$

Defining
$$\overset{rs}{v}_i \equiv \underset{r}{\chi_\alpha}\nabla_i \underset{s}{\chi^\alpha}, \tag{18}$$

we have, therefore,
$$\nabla_i \underset{r}{\chi^\alpha} = \overset{r}{\omega}{}^j_i\, \theta_j^\alpha + \sum_{s=1}^{q} \overset{sr}{v}_i \underset{s}{\chi^\alpha}. \tag{19}$$

(This is the generalization for a V_m in V_n of Weingarten's equations.)

Further, on differentiating the relations

$$g_{\alpha\beta}\underset{r}{\chi^\alpha}\underset{s}{\chi^\beta} = \delta^s_r,$$

we have
$$\overset{rs}{v}_i + \overset{sr}{v}_i = 0.$$

Hence, if $m = n-1$, equation (19) becomes

$$\nabla_i \chi^\alpha = \omega^j_i\, \theta_j^\alpha, \tag{20}$$

where χ^α is the unit normal to V_{n-1}.

2. *Lines of curvature.*

2.1. V_{n-1} *in* V_n: *Two properties which cease to be equivalent for spaces with torsion.* A line of curvature of a $\overset{0}{V}_{n-1}$ in a $\overset{0}{V}_n$ is characterized by

either of the following properties:

(i) Its tangent at every point is in one of the principal directions of $\overset{0}{V}_{n-1}$;

(ii) $D\chi^a/Ds$ is tangential to the curve at every point (χ^a being the unit normal to $\overset{0}{V}_{n-1}$ and D/Ds denoting absolute differentiation along the curve).

In the case of a space with torsion these two properties are not equivalent, in general. For, if ξ'^i is any unit vector in V_{n-1},

$$\frac{D\chi^a}{Ds} = \nabla_i \chi^a \cdot \xi'^i = \omega_i^{\cdot k}\, \theta_k^{\cdot a}\, \xi'^i,$$

and the condition that this should be in the direction ξ'^i [property (ii)] is

$$\omega_i^{\cdot k}\, \theta_k^{\cdot a}\, \xi'^i = \sigma \xi^a,$$

where σ is a scalar; that is

$$(\omega_i^{\cdot k} - \sigma \delta_i^k)\, \xi'^i\, \theta_k^{\cdot a} = 0.$$

Multiplying by $g_{a\beta}\, \theta_j^{\cdot \beta}$, we have

$$(\omega_{ij} - \sigma g'_{ij})\, \xi'^i = 0, \tag{21}$$

and σ must clearly be a root of the equation

$$|\omega_{ij} - \sigma g'_{ij}| = 0. \tag{22}$$

We now turn to property (i); the normal curvature (ρ) of V_{n-1} at a point P in a given direction is defined as being equal to $\omega_{ij}\, \xi'^i \xi'^j$, where ξ'^i is the unit vector at P in that direction. It is the first curvature (in V_n) of the geodesic of V_m through P in that direction.

The principal directions of V_{n-1} at P are the directions for which the normal curvature has a stationary value. They correspond to the principal axes of the hyperquadric (indicatrix)

$$\omega_{ij}\, x'^i x'^j = 1, \tag{23}$$

i.e.

$$\omega_{(ij)}\, x'^i x'^j = 1, \tag{23'}$$

and are given by

$$(\omega_{(ij)} - \rho g'_{ij})\, \xi'^i = 0, \tag{24}$$

where ρ is a root of the equation

$$|\omega_{(ij)} - \rho g'_{ij}| = 0. \tag{25}$$

Hence the directions given by (21) are not, in general, principal directions, and the two properties (i) and (ii) are not equivalent.

Assuming that the metric is positive definite, the roots of (25) are all real, and property (i) defines real curves (*lines of curvature*), but some of the roots of (22) may be complex, and it may happen that there are no *real* curves defined by property (ii).

We shall denote the roots of (22) and (25) by $\underset{u}{\sigma}$, $\underset{u}{\rho}$ $(u = 1, \ldots, m)$ and the unit vectors in the corresponding directions, given by (21) and (24), by $\underset{u}{\Lambda'^i}$, $\underset{u}{\lambda'^i}$ respectively.

Two directions ξ'^i, η'^i at a point of V_{n-1} are defined as *conjugate* if

$$\omega_{(ij)}\,\xi'^i\eta'^j = 0,$$

two such directions corresponding to conjugate diameters of the hyperquadric (23). This relation is reciprocal (*i.e.*, involutory), just as for Riemannian spaces.

The principal directions $\underset{u}{\lambda'^i}$ at a point are mutually orthogonal and mutually conjugate, but the same is not true in general of the directions $\underset{u}{\Lambda'^i}$. For, if $\underset{u}{\rho}$, $\underset{v}{\rho}$ are two distinct roots of (25) and $\underset{u}{\sigma}$, $\underset{v}{\sigma}$ two distinct roots of (22),

$$(\omega_{(ij)} - \underset{u}{\rho} g'_{ij})\,\underset{u}{\lambda'^i} = 0$$
$$(\omega_{(ij)} - \underset{v}{\rho} g'_{ij})\,\underset{v}{\lambda'^i} = 0$$

Multiplying these by $\underset{v}{\lambda'^j}$, $\underset{u}{\lambda'^j}$ respectively and subtracting,

$$(\underset{u}{\rho} - \underset{v}{\rho})\, g'_{ij}\,\underset{u}{\lambda'^i}\,\underset{v}{\lambda'^j} = 0,$$

whence

$$g'_{ij}\,\underset{u}{\lambda'^i}\,\underset{v}{\lambda'^j} = 0. \qquad (26_A)$$

Also, on adding,

$$\omega_{(ij)}\,\underset{u}{\lambda'^i}\,\underset{v}{\lambda'^j} = 0 \qquad (u \neq v), \quad (27_A)$$

while

$$\omega_{(ij)}\,\underset{u}{\lambda'^i}\,\underset{u}{\lambda'^j} = \underset{u}{\rho}. \qquad (28_A)$$

$$(\omega_{ij} - \underset{u}{\sigma} g'_{ij})\,\underset{u}{\Lambda'^i} = 0$$
$$(\omega_{ij} - \underset{v}{\sigma} g'_{ij})\,\underset{v}{\Lambda'^i} = 0$$

Multiplying these by $\underset{v}{\Lambda'^j}$, $\underset{u}{\Lambda'^j}$ respectively and subtracting,

$$(\underset{u}{\sigma} - \underset{v}{\sigma})\, g'_{ij}\,\underset{u}{\Lambda'^i}\,\underset{v}{\Lambda'^j} = 2\omega_{[ij]}\,\underset{u}{\Lambda'^i}\,\underset{v}{\Lambda'^j},$$

whence

$$g'_{ij}\,\underset{u}{\Lambda'^i}\,\underset{v}{\Lambda'^j} = 2S_{\beta\gamma}{}^{\cdot\cdot\alpha}\,\underset{u}{\Lambda^\beta}\,\underset{v}{\Lambda^\gamma}\,\chi_\alpha/(\underset{u}{\sigma} - \underset{v}{\sigma}) \qquad (26_B)$$

Also, on adding,

$$\omega_{(ij)}\,\underset{u}{\Lambda'^i}\,\underset{v}{\Lambda'^j} = \tfrac{1}{2}(\underset{u}{\sigma} + \underset{v}{\sigma})\, g'_{ij}\,\underset{u}{\Lambda'^i}\,\underset{v}{\Lambda'^j}$$
$$= \frac{\underset{u}{\sigma} + \underset{v}{\sigma}}{\underset{u}{\sigma} - \underset{v}{\sigma}}\, S_{\beta\gamma}{}^{\cdot\cdot\alpha}\,\underset{u}{\Lambda^\beta}\,\underset{v}{\Lambda^\gamma}\,\chi_\alpha \qquad (u \neq v), \quad (27_B)$$

while

$$\omega_{(ij)}\,\underset{u}{\Lambda'^i}\,\underset{u}{\Lambda'^j} = \underset{u}{\sigma}. \qquad (28_B)$$

Equation (28_B) shows that $\underset{u}{\sigma}$ is the normal curvature of V_{n-1} in the

direction $\underset{u}{\Lambda'^i}$. It follows from (27B) that two Λ directions are conjugate if, and only if, either (i) the sum of the normal curvatures in those directions is zero, or (ii) the torsion vector of V_n associated with those directions is tangential to V_{n-1}.

In the latter case they are also orthogonal.

The Λ directions are identical with the principal directions at every point if V_n is semi-symmetric (and in particular, of course, if it is Riemannian).

2.2. *Rodrigues' formula and an analogue.* For displacement in a principal direction $\underset{u}{\lambda'^i}$ we have

$$\frac{D\chi^a}{Ds} = \nabla_i \chi^a \underset{u}{\lambda'^i} = \omega_i^{\cdot k}\, \theta_k^{\cdot a} \underset{u}{\lambda'^i}$$

$$= \underset{u\,u}{\rho} \lambda'^k \theta_k^{\cdot a} + g'^{jk}\, \omega_{[ij]} \underset{u}{\lambda'^i} \theta_k^{\cdot a}.$$

Since the principal directions at a point are mutually orthogonal,

$$g'^{jk} = \sum_{v=1}^{m} \underset{v}{\lambda'^j}\, \underset{v}{\lambda'^k},$$

and hence

$$\frac{D\chi^a}{Ds} = \underset{u\,u}{\rho} \lambda^a + \sum_v \omega_{[ij]} \underset{u}{\lambda'^i}\, \underset{v}{\lambda'^j}\, \underset{v}{\lambda^a},$$

i.e.

$$\boxed{\frac{D\chi^a}{Ds} = \underset{u\,u}{\rho} \lambda^a + \sum_v S_{\beta\gamma}^{\cdot\cdot\mu} \underset{u}{\lambda^\beta}\, \underset{v}{\lambda^\gamma}\, \chi_\mu\, \underset{v}{\lambda^a}.} \tag{29}$$

(This is the generalization for a space with torsion of Rodrigues' formula.)

Since $S_{\beta\gamma}^{\cdot\cdot\mu}$ is skew-symmetric in β, γ, the term under the summation sign corresponding to $v = u$ is zero, so that the component of $D\chi^a/Ds$ along $\underset{u}{\lambda^a}$ itself is $\underset{u}{\rho}$, as in the Riemannian case.

For displacement in a Λ direction, say $\underset{u}{\Lambda'^i}$, we have the simpler formula

$$\frac{D\chi^a}{Ds} = \omega_i^{\cdot k}\, \theta_k^{\cdot a} \underset{u}{\Lambda'^i} = \underset{u\,u}{\sigma} \Lambda'^i \theta_i^{\cdot a},$$

i.e.

$$\boxed{\frac{D\chi^a}{Ds} = \underset{u\,u}{\sigma} \Lambda^a.} \tag{30}$$

2.3. *Case of V_m in V_n ($m < n-1$).* If $\underset{r}{\chi^a}$ is any unit vector field normal to V_m, the corresponding tensor $\overset{r}{\omega}_{ij}$ determines m principal directions

at each point of V_m, namely the directions for which $\overset{r}{\rho}$ has stationary values, where

$$\overset{r}{\rho} = \overset{r}{\omega}_{ij}\,\xi'^i\,\xi'^j$$

with

$$g'_{ij}\,\xi'^i\,\xi'^j = 1;$$

$\overset{r}{\rho}$ is the length of the component along the given normal $\underset{r}{\chi}{}^\alpha$ of the normal curvature vector $\Omega_{ij}{}^\alpha\,\xi'^i\,\xi'^j$ in the direction ξ'^i.

We shall denote the principal normal curvatures corresponding to $\underset{r}{\chi}{}^\alpha$ by $\underset{u}{\overset{r}{\rho}}$ $(u = 1, \ldots, m)$ and the unit vectors in the principal directions by $\underset{u}{\overset{r}{\lambda}}{}'^i$. The tensor $\overset{r}{\omega}_{ij}$ also determines m directions $\underset{u}{\overset{r}{\Lambda}}{}'^i$ and m scalars $\underset{u}{\overset{r}{\sigma}}$, analogous to those for a V_{n-1}.

2.31. If ξ'^i is any unit vector in V_m, the absolute derivative of the normal $\underset{r}{\chi}{}^\alpha$ in the direction of ξ'^i is given by

$$D\underset{r}{\chi}{}^\alpha/Ds = \left(\overset{r}{\omega}_i{}^k\,\theta_k{}^\alpha + \sum_{s=1}^{q} \overset{sr}{v}_i\,\underset{s}{\chi}{}^\alpha\right)\xi'^i,$$

and the condition that this should be in the direction of ξ'^i is

$$D\underset{r}{\chi}{}^\alpha/Ds = \overset{r}{\sigma}\xi^\alpha = \overset{r}{\sigma}\xi'^i\,\theta_i{}^\alpha,$$

where $\overset{r}{\sigma}$ is a scalar;

i.e.

$$(\overset{r}{\omega}_i{}^k - \overset{r}{\sigma}\delta_i^k)\,\xi'^i\,\theta_k{}^\alpha + \sum_s \overset{sr}{v}_i\,\xi'^i\,\underset{s}{\chi}{}^\alpha = 0,$$

which is equivalent to the two sets of conditions

$$(\overset{r}{\omega}_{ij} - \overset{r}{\sigma}g'_{ij})\,\xi'^i = 0 \quad (j = 1, \ldots, m), \tag{31}$$

$$\overset{sr}{v}_i\,\xi'^i = 0 \quad (s = 1, \ldots, q). \tag{32}$$

It is therefore necessary, but by no means sufficient, that ξ'^i should be in one of the directions $\underset{u}{\overset{r}{\Lambda}}{}'^i$.

For the $q-1$ conditions (32)—there are only $q-1$, since $\overset{rr}{v}_i$ is identically zero—express that ξ'^i must be perpendicular to each of the $q-1$ vectors $\overset{rs}{v}_i$ $(s = 1, \ldots, r-1, r+1, \ldots, q)$ in V_m at the point. If more than $m-1$ of those vectors are linearly independent*, equations (32) will not possess a solution at all, still less one which will satisfy (31).

* Not more than m of them can be linearly independent since they all lie in V_m.

2.32. Corresponding to (29) and (30), we have for a V_m in V_n the following relations, which are easily verified.

(i) For displacement in a principal direction $\overset{r}{\underset{u}{\lambda'^i}}$,

$$\boxed{\frac{D\chi^\alpha}{\underset{r}{Ds}} = \underset{u\,u}{\overset{r\ r}{\rho\lambda^\alpha}} + \sum_v S_{\beta\gamma}{}^{\cdot\cdot\mu}\, \overset{r}{\underset{u}{\lambda^\beta}}\, \overset{r}{\underset{v}{\lambda^\gamma}}\, \chi_\mu\, \overset{r}{\underset{v}{\lambda^\alpha}} + \sum_s \overset{sr\ r}{\underset{u\ s}{v_i \lambda'^i}} \chi^\alpha} \tag{33}$$

(Rodrigues' formula for V_m in V_n.)

(ii) For displacement in a direction $\overset{r}{\underset{u}{\Lambda'^i}}$,

$$\boxed{\frac{D\chi^\alpha}{\underset{r}{Ds}} = \underset{u\,u}{\overset{r\ r}{\sigma\Lambda^\alpha}} + \sum_s \overset{sr\ r}{\underset{s\ u}{v_i \Lambda'^i}} \chi^\alpha.} \tag{34}$$

In the latter case the component of $\underset{r}{D\chi^\alpha/Ds}$ tangential to V_m is in the direction of displacement and is of length $\overset{r}{\underset{u}{\sigma}}$.

2.4. *The relative curvature* ($\bar{K}$) *in terms of the* ρ*'s and* σ*'s.* The relative curvature tensor (L_{ijkl}) of V_m in V_n is the difference of the absolute curvature tensor of $V_m(R'_{ijkl})$ and the induced curvature tensor of $V_n(R_{\alpha\beta\gamma\delta}\theta_i^{\cdot\alpha}\theta_j^{\cdot\beta}\theta_k^{\cdot\gamma}\theta_l^{\cdot\delta})$, and is related to the tensor $\Omega_{ij}^{\cdot\cdot\alpha}$ by Gauss's equation, viz.

$$L_{ijkl} = \Omega_{ik}^{\cdot\cdot\alpha}\,\Omega_{jl\alpha} - \Omega_{jk}^{\cdot\cdot\alpha}\,\Omega_{il\alpha}.$$

If ξ'^i, η'^i are unit vectors in two orthogonal directions in V_m at a point P, the relative curvature of V_m, at P, for the orientation determined by those directions is, by definition, $L_{ijkl}\,\xi'^i\eta'^j\,\xi'^k\eta'^l$.

The relative curvature ($\bar{K}$) of V_m in V_n at P is defined as the mean of the relative curvatures for the $\frac{1}{2}m(m-1)$ different orientations determined by any m mutually orthogonal directions in V_m at P when taken in pairs. It is independent of the particular directions chosen; for, if $\underset{u}{\eta'^i}$ $(u = 1, \ldots, m)$ are any such directions,

$$\bar{K} = \frac{2}{m(m-1)} \sum'_{(uv)} L_{ijkl}\, \underset{u}{\eta'^i}\, \underset{v}{\eta'^j}\, \underset{u}{\eta'^k}\, \underset{v}{\eta'^l},$$

where $\sum'_{(uv)}$ denotes summation for all *pairs* of values u, v (u, v ranging from

1 to m) for which $u \neq v$, and, since L_{ijkl} is skew-symmetric in i, j and in k, l,

$$\bar{K} = -\frac{1}{m(m-1)} \sum_{u=1}^{m} \sum_{v=1}^{m} L_{ijki} \eta'^{i}_{u} \eta'^{l}_{u} \eta'^{j}_{v} \eta'^{k}_{v}$$
$$= -\frac{1}{m(m-1)} g'^{jk} g'^{il} L_{ijkl}.$$

For a $\overset{0}{V}_m$ in $\overset{0}{V}_n$ there is the well known relation*

$$\bar{K} = \frac{2}{m(m-1)} \sum_{r=1}^{q} \Sigma'_{(uv)} \overset{r}{\rho}_{u} \overset{r}{\rho}_{v}. \tag{35}$$

We can obtain the corresponding formula for spaces with torsion by following the proof usually given* for (35). Thus, choosing as our m directions the principal directions at P, we have, using (27_A) and (28_A),

$$\Sigma'_{(uv)} L_{ijkl} \lambda'^{i}_{u} \lambda'^{j}_{v} \lambda'^{k}_{u} \lambda'^{l}_{v} = \sum_{r} \Sigma'_{(uv)} (\overset{r}{\omega}_{ik} \overset{r}{\omega}_{jl} - \overset{r}{\omega}_{jk} \overset{r}{\omega}_{il}) \lambda'^{i}_{u} \lambda'^{j}_{v} \lambda'^{k}_{u} \lambda'^{l}_{v}$$
$$= \sum_{r} \Sigma'_{(uv)} \overset{r}{\rho}_{u} \overset{r}{\rho}_{v} - \sum_{r} \Sigma'_{(uv)} \overset{r}{\omega}_{[jk]} \overset{r}{\omega}_{[il]} \lambda'^{i}_{u} \lambda'^{j}_{v} \lambda'^{k}_{u} \lambda'^{l}_{v}$$
$$= \sum_{r} \Sigma'_{(uv)} \overset{r}{\rho}_{u} \overset{r}{\rho}_{v} - \tfrac{1}{2} \sum_{u} \sum_{v} S^{\cdot\cdot\alpha}_{\beta\gamma} S_{\mu\nu\delta} C^{\delta}_{\alpha} \lambda^{\gamma}_{u} \lambda^{\mu}_{u} \lambda^{\beta}_{v} \lambda^{\nu}_{v},$$

and, since

$$\sum_{u} \lambda^{\alpha}_{u} \lambda_{\beta u} = g^{\alpha}_{\beta} - C^{\alpha}_{\beta} = \theta^{\alpha}_{i} \theta^{i}_{\cdot\beta},$$

we obtain finally

$$\boxed{\sum_{r} \Sigma'_{(uv)} \overset{r}{\rho}_{u} \overset{r}{\rho}_{v} = \tfrac{1}{2} m(m-1) \bar{K} + \tfrac{1}{2} S^{\cdot\cdot\alpha}_{\beta\gamma} S^{\mu\nu}_{\cdot\cdot\delta} \theta^{\beta}_{i} \theta^{i}_{\cdot\mu} \theta^{\gamma}_{j} \theta^{j}_{\cdot\nu} C^{\delta}_{\alpha}.} \tag{36}$$

The expression for $\bar{K}$ in terms of the σ's is more simple, being directly analogous to (35), to which it reduces of course for the Riemannian case. To obtain it we cannot follow the method above, since that involves the mutual orthogonality of the λ's, while the Λ's are not in general mutually orthogonal. We can obtain it directly, however, by expanding the determinant in the equation for the σ's, viz.

$$|\overset{r}{\omega}_{ij} - \overset{r}{\sigma} g'_{ij}| = 0.$$

Putting $|g'_{ij}| \equiv \Delta$, we have

$$\overset{r}{\sigma}{}^{m} \Delta - \overset{r}{\sigma}{}^{m-1} \overset{r}{\omega}_{ij} \frac{\partial \Delta}{\partial g'_{ij}} + \frac{1}{(2!)^2} \overset{r}{\sigma}{}^{m-2} (\overset{r}{\omega}_{ik} \overset{r}{\omega}_{jl} - \overset{r}{\omega}_{jk} \overset{r}{\omega}_{il}) \frac{\partial^2 \Delta}{\partial g'_{ik} \partial g'_{jl}} - \ldots = 0,$$

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* D. J. Struik, *Grundzüge der mehrdimensionalen Differentialgeometrie* (1922), 128; Schouten, *loc. cit.*, 200.

whence
$$\sum_{r\,(uv)\,u\,v}\Sigma'\overset{r}{\sigma}\overset{r}{\sigma}=\frac{1}{4\Delta}L_{ijkl}\frac{\partial^2\Delta}{\partial g'_{ik}\partial g'_{jl}}$$
$$=\frac{1}{4\Delta^2}L_{ijkl}\left(\frac{\partial\Delta}{\partial g'_{ik}}\frac{\partial\Delta}{\partial g'_{jl}}-\frac{\partial\Delta}{\partial g'_{jk}}\frac{\partial\Delta}{\partial g'_{il}}\right)$$
$$=\tfrac{1}{4}L_{ijkl}(g'^{ik}g'^{jl}-g'^{jk}g'^{il})$$
$$=-\tfrac{1}{2}g'^{jk}g'^{il}L_{ijkl}=\tfrac{1}{2}m(m-1)\bar{K}.$$

Thus
$$\boxed{\sum_{r\,(uv)\,u\,v}\Sigma'\overset{r}{\sigma}\overset{r}{\sigma}=\tfrac{1}{2}m(m-1)\bar{K}.} \qquad (37)$$

3. *Asymptotic lines.*

An asymptotic line of order p will be defined in the same way as for Riemannian spaces, viz. by the property that its first p normals* relative to V_n are all tangential to V_m. It is easily seen that the asymptotic directions are self-conjugate, and that the characteristic property of asymptotic lines of the first order of a $\overset{0}{V}_m$ in $\overset{0}{V}_n$, viz. that the absolute derivative along such a curve of any unit vector field normal to $\overset{0}{V}_m$ is perpendicular to the curve, remains true for spaces with torsion.

In fact, all the results of my previous paper† on asymptotic lines of a $\overset{0}{V}_m$ in $\overset{0}{V}_n$ remain true for spaces with torsion, except equation (18) which becomes‡
$$\underset{m}{\kappa^2}=-(m-1)H+2S_{\beta\gamma}^{\ \ .\alpha}\Omega_{i\ .\alpha}^{\ .j}\theta_j^{\ .\beta}\theta_k^{\ .\gamma}\underset{1}{\xi'}^i\underset{1}{\xi'}^k. \qquad (38)$$

4. *Geodesic sub-spaces and hyperplanes.*

4.1. *Definition of a hyperplane.* A $\overset{0}{V}_m$ immersed in a $\overset{0}{V}_n$ is defined as a geodesic sub-space§ of $\overset{0}{V}_n$ if every geodesic of $\overset{0}{V}_m$ is also a geodesic of $\overset{0}{V}_n$ (property A), and it is well known that this definition is equivalent to either of the following: every vector $\begin{cases}\text{tangential}\\ \text{normal}\end{cases}$ to $\overset{0}{V}_m$ remains

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* The normals to a curve are derived by the same process as for Riemannian spaces, and the Frenet formulae hold without alteration.

† Already cited (foot-note *, p. 27).

‡ For the case of a V_{n-1} in V_n, *cf.* Hlavatý, *Comptes rendus*, 186 (1928), 1258.

§ Ricci, *Rend. Accad. Lincei*, 5, 12^1 (1903), 409–420; Schouten, *loc. cit.*, 181–183; E. Cartan, "La géometrie des éspaces de Riemann", *Mém. Sci. Math.*, fasc. 9 (1925), 39–43.

$\left\{\begin{matrix}\text{tangential}\\ \text{normal}\end{matrix}\right.$ to $\overset{0}{V}_m$ when it is displaced parallelly in $\overset{0}{V}_n$ along any curve in $\overset{0}{V}_m$ (property B). This may also be stated in the following form: the vector difference of any two vectors at neighbouring points of $\overset{0}{V}_m$, each $\left\{\begin{matrix}\text{tangential}\\ \text{normal}\end{matrix}\right.$ to $\overset{0}{V}_m$, is itself $\left\{\begin{matrix}\text{tangential}\\ \text{normal}\end{matrix}\right.$ to $\overset{0}{V}_m$.

We shall see that in the case of a space with torsion the two properties (A and B) above are not equivalent in general.

A sub-space V_m of V_n defined by property A* we shall continue to call a *geodesic sub-space* of V_n; one defined by property B we shall call a *hyperplane* of V_n. It is the latter which are the complete analogues of the planes of ordinary Euclidean space†.

If $\underset{1}{\xi}^a$, $\underset{1}{\xi}'^i$ are the components in the x-coordinates and x'-coordinates, respectively, of the unit tangent to a curve C in V_m,

$$\underset{1}{\xi}^a = \underset{1}{\xi}'^i \theta_i^{\ a},$$

and hence

$$\frac{D\underset{1}{\xi}^a}{Ds} = \theta_i^{\ a} \frac{D\underset{1}{\xi}'^i}{Ds} + \Omega_{ij}^{\ \ a} \underset{1}{\xi}'^i \underset{1}{\xi}'^j, \tag{39}$$

where the differentiation is along the curve itself.

Property A is therefore equivalent to the following:

$$\Omega_{ij}^{\ \ a} \underset{1}{\xi}'^i \underset{1}{\xi}'^j = 0$$

all along every geodesic of V_m.

Since we can draw a geodesic through every point of V_m in every direction, it follows that *necessary and sufficient conditions for a geodesic sub-space are*

$$\Omega_{(ij)}^{\ \ a} = 0. \tag{40}$$

Now let η^a be any vector tangential to V_m and η'^i its components in V_m, so that

$$\eta^a = \eta'^i \theta_i^{\ a}.$$

For parallel displacement in V_n along the curve C,

$$\frac{D\eta^a}{Ds} \equiv \theta_i^{\ a} \frac{D\eta'^i}{Ds} + \Omega_{ij}^{\ \ a} \underset{1}{\xi}'^i \eta'^j = 0,$$

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* In the case of a non-Riemannian space we use "geodesics" in the sense of "auto-parallel curves" (what American writers call "paths"), not in the sense of minimal curves.

† The most natural definition of a plane in ordinary space is provided by the alternative form of property B given above.

and hence property B, in its first form, is equivalent to the following:

$$\Omega_{ij}^{\ \ a}\,\underset{1}{\xi}^{i}\eta'^{j}=0$$

for arbitrary η'^{i} and $\underset{1}{\xi}^{i}$. Thus *necessary and sufficient conditions for a hyperplane are*

$$\Omega_{ij}^{\ \ a}=0. \tag{41}$$

(It is easily verified that the second form of property B, involving normal vectors, leads to the same conditions.)

The conditions for a hyperplane are therefore more stringent than those for a geodesic sub-space.

4.2. *Some distinguishing properties.* In order that a geodesic sub-space should be a hyperplane it must satisfy the additional conditions

$$\Omega_{[ij]}^{\ \ a}=0,$$

i.e.

$$S_{\beta\gamma}^{\ \ \lambda}\,C_{\lambda}^{\alpha}\theta_{i}^{\beta}\theta_{j}^{\gamma}=0.$$

The distinction is more easily discernible in the following theorems.

THEOREM I. *A necessary and sufficient condition for a geodesic sub-space to be a hyperplane is that the relative curvature tensor vanishes at every point.*

That it is necessary is obvious from (41). To see that it is sufficient, suppose that

$$L_{ijkl}=0 \quad (i,j,k,l=1,\ldots,m),$$

so that, in particular,

$$L_{ijij}=0 \quad (i,j=1,\ldots,m;\ \textit{not summed for } i,j),$$

i.e.

$$\Omega_{ij}^{\ \ a}\Omega_{ji\,a}-\Omega_{jj}^{\ \ a}\Omega_{ii\,a}=0$$

$$(i,j=1,\ldots,m;\ \textit{not summed for } i,j).$$

Since V_m is a geodesic sub-space, the tensor $\Omega_{ij}^{\ \ a}$ is skew-symmetric in i, j, and the preceding equation reduces to

$$\Omega_{ij}^{\ \ a}\,\Omega_{ij\,a}=0.$$

Thus each of the vectors $\Omega_{ij}^{\ \ a}$ (for fixed i, j) is of zero length, whence

$$\Omega_{ij}^{\ \ a}=0,$$

and V_m is a hyperplane.

THEOREM II. *A necessary and sufficient condition for V_m to be a* {*geodesic sub-space* / *hyperplane* *of V_n is that every curve in V_m is an asymptotic line* {*of the first order.* / *of highest order (viz. $m-1$) with $\underset{m}{\kappa}$ equal to zero.*

The first part of the theorem, relating to geodesic sub-spaces, is evident. To prove the second part, we observe that the conditions for an asymptotic line of order $m-1$ are*

$$\Omega_{ij}^{\ a}\ \underset{1}{\xi'^i}\ \underset{r}{\xi'^j} = 0 \quad (r = 1, \ldots, m-1),$$

and that along such a curve†

$$\underset{m}{\kappa}\ \underset{m+1}{\xi^a} = \Omega_{ij}^{\ a}\ \underset{1}{\xi'^i}\ \underset{m}{\xi'^j}.$$

Hence the condition stated in the theorem is equivalent to

$$\Omega_{ij}^{\ a}\ \underset{1}{\xi'^i}\ \underset{r}{\xi'^j} = 0 \quad (r = 1, \ldots, m), \tag{42}$$

and is clearly necessary. To show that it is sufficient, we write (42) in the form

$$\overset{s}{\omega}_{ij}\ \underset{1}{\xi'^i}\ \underset{r}{\xi'^j} = 0 \quad (r = 1, \ldots, m;\ s = 1, \ldots, q),$$

which expresses that, for any given $\underset{1}{\xi'^i}$, each of the q covariant vectors $\overset{s}{\omega}_{ij}\,\xi'^i$ is either a zero vector or is perpendicular to the m-space defined by the vectors $\underset{r}{\xi'^j}$ $(r = 1, \ldots, m)$, and therefore perpendicular to V_m. But $\overset{s}{\omega}_{ij}\ \underset{1}{\xi'^i}$ (for fixed s) is a V_m-vector; hence we require

$$\overset{s}{\omega}_{ij}\ \underset{1}{\xi'^i} = 0 \quad (s = 1, \ldots, q),$$

and therefore, since $\underset{1}{\xi'^i}$ is arbitrary,

$$\overset{s}{\omega}_{ij} = 0 \quad (s = 1, \ldots, q),$$

i.e.

$$\Omega_{ij}^{\ a} = 0.$$

In particular, putting $m = 2$ in II, we have

THEOREM III. *A necessary and sufficient condition for a geodesic V_2 to be a hyperplane is that $\underset{2}{\kappa}$ vanishes for every curve in V_2.*

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* Hayden, *loc. cit.*, 27.

† *Ibid.*, 28 (11).

5. V_l *in* V_m *in* V_n.

5.1. *Introductory.* While the geometry of curves in a sub-space (V_1 in V_m in V_n) has been fairly extensively developed, at any rate for Riemannian spaces, the more general theory of sub-spaces of a sub-space (V_l in V_m in V_n) has as yet received little attention, and it does not appear to have been noticed that, just as a hyperplane (or, in the Riemannian case, a geodesic sub-space) is the spatial analogue of a geodesic line, so there is a spatial analogue of asymptotic lines.

Our first object is to form the equation [viz. equation (51) *post*] for the general theory, corresponding to equation (39) for a curve. We find that it contains two terms on the right-hand side, the vanishing of one of which gives a hyperplane of V_m, while the vanishing of the other gives the spatial analogue that we require (§ 6).

5.2. *Induced tensors and connection.* Let V_m be a sub-space immersed in V_n, and let X_l be an l-dimensional manifold contained in V_m and defined by equations

$$x'^i = x'^i(x''^1, \ldots, x''^l) \quad (i = 1, \ldots, m);$$

x''^a ($a = 1, \ldots, l$) being coordinates* in X_l. If we write

$$\psi_a^{\cdot i} \equiv \frac{\partial x'^i}{\partial x''^a}, \quad \phi_a^{\cdot z} \equiv \frac{\partial x^\alpha}{\partial x''^a}, \tag{43}$$

then

$$\phi_a^{\cdot \alpha} = \theta_i^{\cdot \alpha} \psi_a^{\cdot i}. \tag{44}$$

If $T_{\alpha\beta\gamma\ldots}$ is any covariant V_n-tensor at a point of X_l and $T'_{ijk\ldots}$ the induced V_m-tensor, the X_l-tensor derived from $T'_{ijk\ldots}$ by induction is

$$T'_{ijk\ldots} \psi_a^{\cdot i} \psi_b^{\cdot j} \psi_c^{\cdot k},$$

and the X_l-tensor derived directly from $T_{\alpha\beta\gamma\ldots}$ by induction is

$$T_{\alpha\beta\gamma\ldots} \phi_a^{\cdot \alpha} \phi_b^{\cdot \beta} \phi_c^{\cdot \gamma} \ldots.$$

On account of (44) these are identical and there is therefore no ambiguity in speaking of the induced X_l-tensor derived from $T\ldots$.

We obtain then the induced metric tensor

$$g''_{ab} \equiv g_{\alpha\beta} \phi_a^{\cdot \alpha} \phi_b^{\cdot \beta} \equiv g'_{ij} \psi_a^{\cdot i} \psi_b^{\cdot j}, \tag{45}$$

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* In this section the Latin indices a, b, c, ... (at the beginning of the alphabet) take values 1, ..., l; the Latin indices i, j, k, ... take values 1, ..., m; Greek indices, as hitherto, take values 1, ..., n.

and, if g''^{ab} is the reciprocal element to g'_{ab} in the determinant $|g'_{ab}|$, we define

$$\psi^a_{\cdot i} \equiv g''^{ab} g'_{ij} \psi_b^{\cdot j}, \quad \phi^a_{\cdot \alpha} \equiv g''^{ab} g_{a\beta} \phi_b^{\cdot \beta}. \tag{46}$$

It is easily verified that

$$\phi^a_{\cdot \alpha} = \theta^i_{\cdot \alpha} \psi^a_{\cdot i}. \tag{47}$$

The induced torsion tensor is

$$S''^{\cdot\cdot a}_{bc} \equiv S'^{\cdot\cdot i}_{jk} \psi^a_{\cdot i} \psi_b^{\cdot j} \psi_c^{\cdot k} \equiv S^{\cdot\cdot\alpha}_{\beta\gamma} \phi^a_{\cdot \alpha} \phi_b^{\cdot \beta} \phi_c^{\cdot \gamma}, \tag{48}$$

and g'_{ab}, $S''^{\cdot\cdot a}_{bc}$ determine a connection Γ''^{a}_{bc}. If, therefore, V_l is immersed in V_m it is also immersed in V_n, and its connection is the induced connection Γ''^{a}_{bc}.

5.3. *Definition of a new absolute differential operator for tensors over V_l.* In the ordinary theory of curves in a sub-space we have to deal with tensors which have a two-fold tensor character, viz. with respect to the sub-space and to the enveloping space. In considering a V_l in a V_m in a V_n we are concerned with tensors which have a three-fold tensor character, and we shall therefore extend the definition of absolute differentiation as follows.

Let $T^{a_1 \ldots i_1 \ldots a_1 \ldots}_{\beta_1 \ldots j_1 \ldots b_1 \ldots}$ be a set of quantities over V_l having the three-fold tensor character indicated by the three types of indices, and let $\Gamma^{\alpha}_{\beta\gamma}$, Γ'^{i}_{jk}, Γ''^{a}_{bc} be the coefficients of connection of V_n, V_m, V_l respectively. Then we define the *absolute differential* $DT^{\ldots}_{\ldots}$, of the tensor $T^{\ldots}_{\ldots}$, by the expression

$$\begin{aligned} DT^{\ldots}_{\ldots} = dT^{\ldots}_{\ldots} &+ \sum_\tau \Gamma^{\alpha_\tau}_{\lambda\gamma} \begin{pmatrix} \lambda \\ \alpha_\tau \end{pmatrix} T^{\ldots}_{\ldots} dx^\gamma - \sum_\tau \Gamma^{\lambda}_{\beta_\tau \gamma} \begin{pmatrix} \beta_\tau \\ \lambda \end{pmatrix} T^{\ldots}_{\ldots} dx^\gamma \\ &+ \sum_t \Gamma'^{i_t}_{lk} \begin{pmatrix} l \\ i_t \end{pmatrix} T^{\ldots}_{\ldots} dx'^k - \sum_t \Gamma'^{l}_{j_t k} \begin{pmatrix} j_t \\ l \end{pmatrix} T^{\ldots}_{\ldots} dx'^k \\ &+ \sum_e \Gamma''^{a_e}_{fc} \begin{pmatrix} f \\ a_e \end{pmatrix} T^{\ldots}_{\ldots} dx''^c - \sum_e \Gamma''^{f}_{b_e c} \begin{pmatrix} b_e \\ f \end{pmatrix} T^{\ldots}_{\ldots} dx''^c, \end{aligned} \tag{49}$$

where $\begin{pmatrix} \lambda \\ \alpha_\tau \end{pmatrix} T^{\ldots}_{\ldots}$ denotes the result of replacing α_τ by λ in $T^{\ldots}_{\ldots}$, with similar meanings for the other brackets.

If $T^{\ldots}_{\ldots}$ is defined along a curve in V_l, the absolute derivative along the curve is defined as $DT^{\ldots}_{\ldots}/Ds$, where, since s is a scalar,

$$Ds = ds.$$

If $T^{\cdots}_{\cdots}$ is defined over a continuous region of V_l,

$$DT^{\cdots}_{\cdots} = \nabla_c T^{\cdots}_{\cdots} dx''^c,$$

where the operator ∇_c is given by

$$\begin{aligned}\nabla_c = \frac{\partial}{\partial x''^c} &+ \sum_\tau \phi_c^{\cdot\gamma} \Gamma^{a_\tau}_{\lambda\gamma}\binom{\lambda}{a_\tau} - \sum_\tau \phi_c^{\cdot\gamma} \Gamma^{\lambda}_{\beta_\tau\gamma}\binom{\beta_\tau}{\lambda} \\ &+ \sum_t \psi_c^{\cdot k} \Gamma''^{i_t}_{lk}\binom{l}{i_t} - \sum_t \psi_c^{\cdot k} \Gamma'^{l}_{j_t k}\binom{j_t}{l} \\ &+ \sum_e \Gamma''^{a_e}_{fc}\binom{f}{a_e} - \sum_e \Gamma''^{f}_{b_e c}\binom{b_e}{f}. \end{aligned} \tag{50}$$

For tensors which are invariant to coordinate transformations in V_l, the operator ∇_k is defined as hitherto; for such a tensor

$$\nabla_c T^{\cdots}_{\cdots} = \psi_c^{\cdot k} \nabla_k T^{\cdots}_{\cdots}.$$

For tensors which are invariant to coordinate transformations both in V_l and in V_m, ∇_a is the ordinary Ricci differentiator; for such tensors

$$\nabla_c T^{\cdots}_{\cdots} = \phi_c^{\cdot\gamma} \nabla_\gamma T^{\cdots}_{\cdots}.$$

5.4. *Fundamental relation between the curvature tensors.* Let

$\Omega^{\cdots\alpha}_{ij}$ denote the curvature tensor of V_m in V_n,

$\Omega'^{\cdots k}_{ab}$,, ,, ,, V_l in V_m,

$\Omega''^{\cdots a}_{ab}$,, ,, ,, V_l in V_n;

i.e. $\Omega^{\cdots\alpha}_{ij} = \nabla_i \theta_j^{\cdot\alpha}$, $\Omega'^{\cdots k}_{ab} = \nabla_a \psi_l^{\cdot k}$, $\Omega''^{\cdots a}_{ab} = \nabla_a \phi_b^{\cdot a}$.

Then, operating on (44) with ∇_a, we have the fundamental relation

$$\boxed{\Omega''^{\cdots a}_{ab} = \Omega'^{\cdots i}_{ab}\theta_i^{\cdot a} + \Omega^{\cdots a}_{ij}\psi_a^{\cdot i}\psi_b^{\cdot j}.} \tag{51}$$

When $l = 1$ this reduces to (39) if we choose s as the single coordinate in V_1.

5.41. If ξ''^a is a unit vector in V_l, and ξ'^i, ξ^α are its components when regarded as a V_m-vector or as a V_n-vector,

$$\xi'^i = \xi''^a \psi_a^{\cdot i}, \quad \xi^\alpha = \xi'^i \theta_i^{\cdot\alpha} = \xi''^a \phi_a^{\cdot\alpha}.$$

Multiplying (51) by $\xi''^a \eta''^b$, and summing for a, b, we see that the

following theorem is true for *any* V_l:

THEOREM I. *If two directions are conjugate* in V_l in V_n they are also conjugate in V_l in V_m and in V_m in V_n.*

But two directions conjugate in V_l in V_m are not, in general, conjugate in V_l in V_n.

From Theorem I we have

THEOREM II. *Every asymptotic line of V_l in V_n is also an asymptotic line of V_l in V_m and of V_m in V_n.*

If L''_{abcd} is the relative curvature tensor of V_l in V_n, and L'_{abcd} the relative curvature tensor of V_l in V_m, we obtain from (51), using (13), the relation

$$\boxed{L''_{abcd} = L'_{abcd} + L_{ijkl}\psi_a^{\cdot i}\psi_b^{\cdot j}\psi_c^{\cdot k}\psi_d^{\cdot l},} \tag{52}$$

i.e.

THEOREM III. *The relative curvature tensor of V_l in V_n is the sum of the relative curvature tensor of V_l in V_m and the induced relative curvature tensor of V_m in V_n.*

5.5. *Normal spaces.* The m^2 vectors $\Omega_{ij}^{\cdot\cdot a}$ $(i, j = 1, \ldots, m)$, which are all orthogonal to V_m, are not necessarily independent. If at a point P only m_1 of them are linearly independent they determine a space of m_1 dimensions $(m_1 \leqslant n-m)$, which we shall call the *first normal space of V_m relative to V_n* at P.†

Similarly the l^2 vectors $\Omega''_{ab}{}^{\cdot\cdot a}$ $(a, b = 1, \ldots, l)$ at a point of V_l determine a space, say of dimension l_1 $(\leqslant n-l)$, orthogonal to V_l, viz. the first normal space of V_l relative to V_n; while the l^2 V_m-vectors $\Omega'_{ab}{}^{\cdot\cdot i}$ $(a, b = 1, \ldots, l)$ determine a space, say of dimension L_1 $(\leqslant m-l)$, orthogonal to V_l, viz. the first normal space of V_l relative to V_m.

Suppose that m_1, l_1, L_1 refer to the same point of V_l. It is not evident *a priori* that $L_1 \leqslant l_1$, but it is immediately deducible from (51). For the vector $\Omega'_{ab}{}^{\cdot\cdot i}$ (for fixed a, b) when regarded as a vector in V_n has components $\Omega'_{ab}{}^{\cdot\cdot i}\theta_i^{\cdot a}$, and hence the number of independent vectors $\Omega'_{ab}{}^{\cdot\cdot i}\theta_i^{\cdot a}$ (for

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* We have hitherto defined conjugate directions only for a V_{n-1} in V_n. For a V_m in V_n we define conjugate directions by the relations

$$\Omega_{(ij)}^{\cdot\cdot a}\xi'^i\eta'^j = 0 \quad (a = 1, \ldots, n),$$

which are also equivalent to the single relation

$$\Omega_{(ij)}^{\cdot\cdot a}\,\Omega_{(kl)\,a}\,\xi'^i\eta'^j\xi'^k\eta'^l = 0$$

† *Cf.* Duschek und Mayer, *loc. cit.*, 2, 202.

variable a, b) is equal to L_1. The number of independent vectors $\Omega_{ij}{}^{a}\psi_{a}{}^{i}\psi_{b}{}^{j}$ is less than or equal to m_1; let us call the number p. Then, since each of the vectors of the first set is tangential to V_m, while each of those of the second set is normal to V_m, it follows from (51) that the number of independent vectors $\Omega''_{ab}{}^{a}$ is greater than or equal to L_1 and p, but less than or equal to L_1+p. Hence

$$\boxed{L_1 \leqslant l_1 \leqslant L_1+m_1.} \tag{53}$$

5.51. For fixed a, b the vectors on the right of (51) are the components of $\Omega''_{ab}{}^{x}$ tangential and normal, respectively, to V_m. The tangential component vanishes for every a, b if, and only if,

$$\Omega'_{ab}{}^{i} = 0.$$

Hence *a necessary and sufficient condition for V_l to be a hyperplane of V_m is that its first normal space relative to V_n is perpendicular to V_m.*

(Compare the corresponding property for geodesics.)

6. *Asymptotic sub-spaces.*

6.1. *Definition.* The vanishing of the second term on the right of (51), for every a, b, expresses that the first normal space of V_l relative to V_n is tangential to V_m. For $l = 1$ this defines an asymptotic line (of the first order) of V_m in V_n. We shall therefore define V_l to be an *asymptotic sub-space (of the first order) of V_m in V_n* if its first normal space relative to V_n is tangential to V_m at every point.

From (51), a necessary and sufficient condition for an asymptotic sub-space is

$$\boxed{\Omega_{ij}{}^{a}\psi_{a}{}^{i}\psi_{b}{}^{j} = 0.} \tag{54}$$

An alternative necessary and sufficient condition is

$$\boxed{\Omega''_{ab}{}^{a} = \Omega'_{ab}{}^{i}\theta_{i}{}^{a},} \tag{55}$$

i.e. its first normal space relative to V_n is identical with its first normal space relative to V_m.

6.2. *Some properties.* Multiplying (54) and (55) by $\xi''^{a}\eta''^{b}$, and summing for a, b, we have the following properties of an asymptotic sub-

space (V_l) of V_m in V_n:

I. *Every pair of directions in V_l is conjugate in V_m in V_n.*

II. *If two directions are conjugate in V_l in V_m they are also conjugate in V_l in V_n.*

Hence also

III. *Every curve in V_l is an asymptotic line of V_m in V_n.*

IV. *Every asymptotic line of V_l in V_m is also an asymptotic line of V_l in V_n.*

[These properties are not sufficient to define an asymptotic sub-space. From I or III we can only conclude that

$$\Omega_{ij}^{\ \ a}\psi_{(a}^{\ \ i}\psi_{b)}^{\ \ j}=0,$$

i.e. that

$$\Omega_{(ij)}^{\ \ a}\psi_{a}^{\ \ i}\psi_{b}^{\ \ j}=0.$$

From II or IV we can conclude even less.]

Further, it follows from (54) that

$$L_{ijkl}\psi_{a}^{\ \ i}\psi_{b}^{\ \ j}\psi_{c}^{\ \ k}\psi_{d}^{\ \ l}=0,$$

whence

V. *The induced relative curvature tensor of V_m in V_n vanishes at every point of V_l.*

If V_m is a hyperplane of V_n equation (54) is satisfied identically for every V_l. Thus

VI. *Every sub-space of a hyperplane (V_m) of V_n is an asymptotic sub-space of V_m in V_n.*

This is to be compared with the fact that every curve in a geodesic sub-space is an asymptotic line; in that case the word "hyperplane" can be replaced by the less exacting "geodesic sub-space" on account of the symmetry introduced into (51) when $l=1$.

6.3. *Normal spaces of higher orders.* We shall denote

$$\Omega_{ijk}^{\ \ \ a}\equiv\nabla_i\Omega_{jk}^{\ \ a},\quad \Omega_{ijkl}^{\ \ \ \ a}\equiv\nabla_i\Omega_{jkl}^{\ \ \ a},\quad \text{etc.};$$

$$\Omega'_{abc}^{\ \ \ \ i}\equiv\nabla_a\Omega_{bc}^{\ \ i},\quad \text{etc.};$$

$$\Omega''_{abc}^{\ \ \ \ a}\equiv\nabla_a\Omega''_{bc}^{\ \ a},\quad \text{etc.};$$

and the vector space at a point of V_m determined by the aggregate of the vectors

$$\theta_{i_1}^{\ \alpha}, \quad \Omega_{i_1 i_2}^{\ \ \alpha}, \quad \Omega_{i_1 i_2 i_3}^{\ \ \ \alpha}, \quad \ldots, \quad \Omega_{i_1 i_2 \ldots i_p}^{\ \ \ \ \ \alpha}$$

$(i_1, i_2, \ldots, i_p = 1, \ldots, m)$ we shall call the p-th *osculating space of V_m relative to V_n* at that point*. The first osculating space is the tangent m-space; the second osculating space is the space (of dimension $m+m_1$) determined by the tangent space and the first normal space at the point.

In general, each of the m^3 vectors $\Omega_{ijk}^{\ \ \alpha}$ $(i, j, k = 1, \ldots, m)$ at a point of V_m has a component perpendicular to the second osculating space at the point. These components are not necessarily independent, and if $m_2 (\leqslant n-m-m_1)$ of them are linearly independent they determine a space of m_2 dimensions, which we shall call the *second normal space of V_m relative to V_n*. Proceeding in this way we define normal spaces of higher orders, which are mutually perpendicular. Thus the p-th *normal space of V_m relative to V_n* is defined as the vector space of greatest dimensions contained in the $(p+1)$-th osculating space and perpendicular to the p-th osculating space.

The osculating spaces and higher normal spaces of V_l relative to V_n and of V_l relative to V_m are defined in a similar manner.

6.4. *Asymptotic sub-spaces of higher orders.* From analogy with the case when $l = 1$, we now define V_l as an *asymptotic sub-space of order p of V_m in V_n* if its first p normal spaces relative to V_n are all tangential to V_m; *i.e.* if its $(p+1)$-th osculating space relative to V_n is tangential to V_m; *i.e.* if†

$$[D; p] \qquad \Omega''_{a_1 a_2 \ldots a_{r+1}}{}^{\alpha}_{s} \chi_{\alpha} = 0 \quad (r = 1, \ldots, p; \ s = 1, \ldots, q).$$

6.41. We can prove that each of the sets of conditions $[C; p]$, $[A; p]$, given below, is necessary and sufficient for an asymptotic sub-space of order p:

$$[C; p] \qquad \boxed{\Omega_{ij}^{\ \ \alpha} \psi_{a_1}^{\ \ i} \Omega'_{a_2 \ldots a_{r+1}}{}^{j} = 0} \qquad (r = 1, \ldots, p),$$

$$[A; p] \qquad \boxed{\Omega''_{a_1 \ldots a_{r+1}}{}^{\alpha} = \Omega'_{a_1 \ldots a_{r+1}}{}^{i} \theta_i^{\ \alpha}} \qquad (r = 1, \ldots, p).$$

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* Duschek und Mayer, *loc cit.* (2), 202 (*p-ter Schmiegraum*); Struik, *loc. cit.*, 110 [*Krümmungsgebiet $(p-1)$-ter Ordnung*].

† The labels $[D; p]$, $[C; p]$, etc., indicate references to the corresponding formulae for asymptotic lines in our previous paper (already cited); thus, for example, $[C; p]$ is the generalization for $l > 1$ of the formula $[C; p]$ of our previous paper.

The latter expresses that the first p normal spaces of V_l relative to V_n are identical respectively with the first p normal spaces relative to V_m.

We shall outline the proof, which follows along the same lines as that for asymptotic lines in our previous paper.

Suppose that $[C;\ p]$ and $[A;\ p]$ are both satisfied for an asymptotic sub-space of order p. Since an asymptotic sub-space of order $p+1$ is, *ipso facto*, asymptotic of order p, we have, on differentiating the last of the relations $[A;\ p]$,

$$\Omega''^{\cdots\cdots\alpha}_{a_1\ldots a_{p+2}} = \Omega'^{\cdots\cdots i}_{a_1\ldots a_{p+2}}\theta_i^\alpha + \Omega_{ij}^{\ \ \alpha}\psi_{a_1}^{\ i}\,\Omega'^{\cdots\cdots j}_{a_2\ldots a_{p+2}}. \tag{56}$$

Multiplying by $\overset{s}{\chi}_\alpha$ and using $[D;\ p+1]$, it follows that

$$\overset{s}{\omega}_{ij}\psi_{a_1}^{\ i}\,\Omega'^{\cdots\cdots j}_{a_2\ldots a_{p+2}} = 0 \quad (s = 1, \ldots, q),$$

whence

$$\Omega_{ij}^{\ \ \alpha}\psi_{a_1}^{\ i}\,\Omega'^{\cdots\cdots j}_{a_2\ldots a_{p+1}} = 0, \tag{57}$$

and (56) reduces to

$$\Omega''^{\cdots\cdots\alpha}_{a_1\ldots a_{p+2}} = \Omega'^{\cdots\cdots i}_{a_1\ldots a_{p+2}}\theta_i^\alpha. \tag{58}$$

Equations (57) and (58), together with $[C;\ p]$ and $[A;\ p]$, constitute $[C;\ p+1]$ and $[A;\ p+1]$. Thus, if $[C;\ p]$ and $[A;\ p]$ are necessary for an asymptotic sub-space of order p, then $[C;\ p+1]$ and $[A;\ p+1]$ are necessary for an asymptotic sub-space of order $p+1$. But we have shown that $[C;\ 1]$ and $[A;\ 1]$ are necessary for an asymptotic sub-space of the first order, whence the result follows for $p = 2, 3, \ldots$.

That $[A;\ p]$ is sufficient is obvious, since, on multiplying throughout by $\overset{s}{\chi}_\alpha$ $(s = 1, \ldots, q)$, we obtain $[D;\ p]$.

To show that $[C;\ p]$ is sufficient we again proceed by induction. Suppose that $[C;\ p]$ is sufficient for an asymptotic sub-space of order p. Then $[C;\ p+1]$ defines an asymptotic sub-space of order p with the additional property

$$\Omega_{ij}^{\ \ \alpha}\psi_{a_1}^{\ i}\,\Omega'^{\cdots\cdots j}_{a_2\ldots a_{p+2}} = 0.$$

Substituting this in (56) we have (58), which, together with $[A;\ p]$, constitutes $[A;\ p+1]$, and shows that $[C;\ p+1]$ defines an asymptotic sub-space of order $p+1$. But $[C,\ 1]$ is sufficient for an asymptotic sub-space of the first order, whence the theorem is true for $p = 2, 3, \ldots$.

6.42. Finally we can prove that

Every curve in an asymptotic sub-space of order p (of V_m in V_n) is an asymptotic line of order p of V_m in V_n.

(Compare § 6.2 for $p = 1$.)

For along any curve in *any* V_l,

$$\frac{D}{Ds}\underset{1}{\xi^a} = \phi_{\cdot a}^{\cdot a}\frac{D}{Ds}\underset{1}{\xi''^a} + \Omega''_{\cdot\cdot ab}{}^{\cdot a}\underset{1}{\xi''^a}\underset{1}{\xi''^b},$$

$$\frac{D^2}{Ds^2}\underset{1}{\xi^a} = \phi_{\cdot a}^{\cdot a}\frac{D^2}{Ds^2}\underset{1}{\xi''^a} + (2\Omega''_{\cdot\cdot ab}{}^{\cdot a} + \Omega''_{\cdot\cdot ba}{}^{\cdot a})\underset{1}{\xi''^a}\frac{D}{Ds}\underset{1}{\xi''^b} + \Omega''_{\cdot\cdot\cdot abc}{}^{\cdot a}\underset{1}{\xi''^a}\underset{1}{\xi''^b}\underset{1}{\xi''^c},$$

and so on.

Generally it is clear that $D^r\underset{1}{\xi^a}/Ds^r$ is expressed linearly in terms of the vectors $\phi_{\cdot a}^{\cdot a}$, $\Omega''_{\cdot\cdot ab}{}^{\cdot a}$, $\Omega''_{\cdot\cdot\cdot abc}{}^{\cdot a}$, ..., $\Omega''_{\cdot\cdot\cdots\cdots a_1 a_2 \ldots a_{r+1}}{}^{\cdot a}$, and therefore lies in the $(r+1)$-th osculating space of V_l relative to V_n. But the first p normals relative to V_n of a curve in V_l lie in the $(p+1)$-th osculating space of the curve relative to V_n, which is determined by

$$\underset{1}{\xi^a}, \quad \frac{D}{Ds}\underset{1}{\xi^a}, \quad \frac{D^2}{Ds^2}\underset{1}{\xi^a}, \quad \ldots, \quad \frac{D^p}{Ds^p}\underset{1}{\xi^a},$$

and therefore lie in the $(p+1)$-th osculating space of V_l relative to V_n.

The result follows immediately.

SOME TWO-DIMENSIONAL DIFFUSION PROBLEMS WITH CIRCULAR SYMMETRY

By S. GOLDSTEIN.

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Examples of the application of Heaviside's operational method* to two-dimensional diffusion problems with circular symmetry are given below†. The problems considered are mostly hydrodynamical, but occasionally problems in the conduction of heat are included, and, in some cases, the same analysis applies to more than one problem.

The most usual method for the solution of problems similar to those considered here involves the use of an infinite series of normal functions or an infinite integral to represent an arbitrary function. Compared with this method, the operational method has the advantages: (1) that previous knowledge of the form of the series or integral representation of an arbitrary function is not necessary; (2) that the working is easier; (3) that the solution as a series of normal functions or an integral is not suitable for computation or for exhibiting the character of the solution after the lapse of only a short interval of time, and a form of the solution specially suitable for these circumstances is easily obtainable from the operational solution. One or other of these advantages is illustrated in almost every paragraph below; several paragraphs illustrate them all. (1) is illustrated in particular in §§ 3, 6, 9, 14, 15 and 16 (§ 3 is a good illustration, since I was completely ignorant of formula (52) when I wrote it). Where I have stated an infinite series or integral representation of an arbitrary function, I have done so to stress the fact that previous

* For a general account of the method, and a bibliography, see H. Jeffreys, *Operational methods in mathematical physics* (Cambridge Tracts in Mathematics and Mathematical Physics, No. 23).

† For other applications to such problems, see Jeffreys, *loc. cit.*, 74, 75; Crossley, *Proc. Camb. Phil. Soc.*, 24 (1928), 234, 235, and 480–488; Bromwich, *Proc. London Math. Soc.* (2) 31 (1930), 209–216.

knowledge thereof was not necessary for the solution of the problem under consideration.

The use of Green's functions (Carslaw, *The conduction of heat*, ch. 10) also does away with any necessity of previous knowledge of the form of the series or integral representation of an arbitrary function; the advantages (2) and (3) remain with the operational method; and, in additional, the values of the Green's functions for several problems have been found by the use of contour integrals by a method which is really equivalent to the operational method. Further, the use of Green's functions to solve problems in the conduction of heat in which the boundary, for example, is maintained at a given non-zero temperature, does not give the solution directly in its simplest form (for an example see below), and it is much simpler to use contour integrals or the operational method to solve such problems directly.

Since the operational solution can be interpreted by Bromwich's integral, contour integrals can clearly be used instead of Heaviside's operators; but the gain in conciseness and ease of manipulation in using the latter is very striking.

The paragraph headings provide a summary of the contents of the paper; where the heading is not sufficiently explanatory, the problem to be solved is stated in the opening words of the paragraph.

Not all the results given below are new, nor even as many of them as I believed to be new when I obtained them. But so far as I know, none of them have previously been obtained by the same method and from the same point of view.

The paragraph headings are collected together below for convenience.

1. The motion of viscous fluid inside an infinite rotating circular cylinder.
2. The motion of viscous fluid between two infinite rotating circular cylinders.
3. The motion of viscous fluid outside an infinite rotating circular cylinder.
4. Diffusion of a concentrated line vortex or instantaneous line source of heat.
5. Comparison with instantaneous point source and instantaneous plane source of heat.
6. Any given initial distribution of temperature. Neumann's integral.
7. Diffusion of a vortex with finite circular section.
8. Diffusion of a cylindrical sheet of vorticity.
9. Any given initial symmetrical distribution of vorticity.
10. Diffusion of a concentrated line vortex with a boundary at $r = a$.
11. Instantaneous line source of heat, with a boundary at $r = a$, maintained at zero temperature.
12. Diffusion of a cylindrical sheet of vorticity with an external coaxial boundary.
13. Any initial symmetrical distribution of vorticity with an external coaxial boundary.
14. Heat conduction in a circular cylinder with the surface impervious to heat.
15. Diffusion of vorticity in the space outside a circular cylinder.
16. The conduction of heat in the space outside a circular cylinder.
17. The solution of the equation of heat conduction for the space outside a circular cylinder when the surface temperature is $\cos at$ or $\sin at$ for $t > 0$.

§ 1 solves a well-known problem. The operational solution and the interpretation by the partial fraction rule as a series of normal functions are required for §§ 10 and 12; the asymptotic expressions for the velocity and vorticity for small t are new, so far as I know. [In future, "new" will be used for the last six words of the previous sentence.] § 2 contains the only discussion in the paper of a problem for an annular space; while "new," the series of normal functions has probably been given previously, without the asymptotic expressions for small t; § 3 is new. § 4 contains the fundamental result from which the others are derived; the main operational representation obtained there should, in my opinion, be included in any list of operators. § 5 is an interpolation. §§ 6 and 9 stress the inversion of the usual procedure when the operational method is used. §§ 7 and 8 solve problems previously considered by Terazawa (§ 8, by Rayleigh also); the consideration of the circulation, which gives the total amount of vorticity within any circle, and the limiting forms for large and small t, are new. §§ 10, 12, and 13 are new. § 11 is included to show a second method of considering problems in which there is a concentrated source and a boundary, and for the sake of the treatment for small t; § 14 for comparison with § 13, and to show that previous knowledge of the correct form of the Dini expansion of an arbitrary function is not necessary. § 15 and the second half of § 16 are new; Carslaw gives the Green's function for the outside of a cylinder (*The conduction of heat*, 196) from which (216) is easily deduced, and (217) is apparently well known. § 17 is new.

Except for §§ 5 and 7, the various paragraphs are connected, but no attempt has been made to give a complete account. One omission may be mentioned. No attempt is made to follow up the work of § 2 in connection with annular spaces; this could easily be done on the lines of §§ 15 and 16, and, in the conduction of heat, the conditions at the two boundaries may be supposed either the same or different from one another. The results might be of some mathematical interest.

A few words may be written in connection with § 17 and the use of Green's functions to solve problems in the conduction of heat in which the boundary is maintained at a given non-zero temperature. The solution of the equation of heat conduction for the outside of a cylinder, when the surface is maintained at a given temperature, can be found on the lines of § 3. Thus if the surface is maintained at unit temperature from $t = 0$ onwards, the temperature is given by

$$\zeta = \frac{K_0(qr)}{K_0(qa)} = 1 + \frac{2}{\pi}\int_0^\infty e^{-\nu u^2 t}\,\frac{J_0(ur)\,Y_0(ua) - J_0(ua)\,Y_0(ur)}{J_0^2(ua) + Y_0^2(ua)}\,\frac{du}{u},$$

with the notation used in the following paragraphs. This problem has previously been considered by Rayleigh (*Scientific papers*, 6, 64), and Nicholson [*Proc. Roy. Soc.* (A), 100 (1921), 226-240]. Rayleigh used a Fourier integral; Nicholson used Weber's integral [(217) below]. The method suggested here should be compared with theirs. Notice also that if we use the Green's function for the outside of a cylinder (Carslaw, *loc. cit.*) in the general formula (4) on p. 172 of Carslaw's book, we find

$$\zeta = \frac{2}{\pi}\int_0^\infty (e^{-\nu u^2 t}-1)\,\frac{J_0(ur)\,Y_0(ua)-J_0(ua)\,Y_0(ur)}{J_0^2(ua)+Y_0^2(ua)}\,\frac{du}{u},$$

which is seen to be the same as the result above only after the evaluation of an integral. A similar point arises when using the Green's function for the inside of a cylinder whose surface is maintained at a given temperature, or for any such problem.

Interesting results can also be found by taking the surface temperature to be some other given function of the time; the results when it is $\cos at$ or $\sin at$ for $t > 0$ are given in § 17 without proof. The point in connection with the use of Green's function arises in the examples of § 17 as well, but the example above is simpler, and the problem worth mention on its own account.

The problems considered here are all two-dimensional. I have elsewhere used the same method in connection with a problem relating to a finite cylinder.

1. *The motion of viscous fluid inside an infinite rotating circular cylinder.*

At time $t = 0$ the circular cylinder $r = a$, filled with viscous fluid, begins suddenly to rotate with unit velocity. It is assumed that the motion remains two-dimensional and symmetrical about the axis. If v is the velocity at the time t at distance r from the axis, then

$$\frac{\partial v}{\partial t} = \nu\left[\frac{\partial^2 v}{\partial r^2}+\frac{1}{r}\frac{\partial v}{\partial r}-\frac{v}{r^2}\right], \tag{1}$$

where ν is the kinematic viscosity of the fluid. The initial value of v is zero, and if we replace $\partial/\partial t$ by p, and put

$$p = \nu q^2, \tag{2}$$

the auxiliary equation is

$$\frac{\partial^2 v}{\partial r^2}+\frac{1}{r}\frac{\partial v}{\partial r}-\frac{v}{r^2}-q^2 v = 0. \tag{3}$$

The solution that remains finite when $r = 0$ and is equal to 1 when $r = a$ is

$$v = \frac{I_1(qr)}{I_1(qa)}. \tag{4}$$

This is an even function of q, and therefore a single-valued function of p. The numerator has no poles. The denominator has zeros when $q = \pm ik$, where

$$J_1(ka) = 0. \tag{5}$$

Apply the partial fraction rule*

$$\frac{f(p)}{F(p)} = \frac{f(0)}{F(0)} + \sum_{\alpha} \frac{f(\alpha)}{\alpha F'(\alpha)} e^{\alpha t}, \tag{6}$$

where the α are the zeros of $F(p)$; it is assumed that they are all simple and none of them zero, and the summation is over all the zeros. Since the two values $\pm ik$ of q both give the same value $-\nu k^2$ of p, only the values $+ik$ of q need be considered.

Since
$$p \frac{d}{dp} = \tfrac{1}{2} q \frac{d}{dq} \tag{7}$$

and
$$J_1'(ka) = J_0(ka), \tag{8}$$

when $J_1(ka) = 0$, the result is

$$v = \frac{r}{a} + 2 \sum_{k} \frac{J_1(kr)}{ka\, J_0(ka)} e^{-\nu k^2 t}, \tag{9}$$

a known result.

For large or moderate values of $\nu t/a^2$ the series converges rapidly, but for small values of $\nu t/a^2$ the convergence is too slow for the series to be of any use. To find a suitable approximation to v in these circumstances, return to (4). If $\nu t/r^2$ is small, we may treat qr as large, and use the asymptotic expansions of $I_1(qr)$ and $I_1(qa)$. This gives

$$\begin{aligned} v &= \frac{e^{qr}(2\pi qa)^{\frac{1}{2}}[1 - \{3/(8qr)\} + \ldots]}{e^{qa}(2\pi qr)^{\frac{1}{2}}[1 - \{3/(8qa)\} + \ldots]} \\ &= \left(\frac{a}{r}\right)^{\frac{1}{2}} e^{-q(a-r)} \left[1 - \frac{1}{q}\left(\frac{3}{8r} - \frac{3}{8a}\right) + \ldots\right]. \end{aligned} \tag{10}$$

Now it is known that†

$$q^m e^{-qx} = \left(\frac{2}{\pi}\right)^{\frac{1}{2}} (2\nu t)^{-\frac{1}{2}m} e^{-x^2/(8\nu t)} D_{m-1}\left(\frac{x}{\sqrt{(2\nu t)}}\right), \tag{11}$$

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* Jeffreys, *loc. cit.* (2nd ed., 1931), 11 and 33.

† See a forthcoming paper by Goldstein.

where $D_m(z)$ satisfies the differential equation*

$$\frac{d^2}{dz^2} D_m(z) + (m + \tfrac{1}{2} - \tfrac{1}{4}z^2) D_m(z) = 0. \tag{12}$$

For large positive values of z, $D_m(z)$ has the asymptotic expansion

$$D_m(z) \sim e^{-\frac{1}{4}z^2} z^m \left\{1 - \frac{m(m-1)}{2z^2} + \frac{m(m-1)(m-2)(m-3)}{2 \cdot 4z^4} - \ldots\right\}. \tag{13}$$

It satisfies the recurrence relation

$$D_{m+1}(z) - zD_m(z) + mD_{m-1}(z) = 0; \tag{14}$$

and

$$D_0(z) = e^{-\frac{1}{4}z^2}, \quad D_{-1}(z) = (\tfrac{1}{2}\pi)^{\frac{1}{2}} e^{\frac{1}{4}z^2} \{1 - \operatorname{erf}(z/\sqrt{2})\}, \tag{15}$$

where

$$\operatorname{erf} x = \frac{2}{\sqrt{\pi}} \int_0^x e^{-u^2} du, \tag{16}$$

and is tabulated by Jahnke and Emde, and by Dale; so that D_{-2}, D_{-3}, etc., are easily found, and could in fact be easily tabulated, from the recurrence relation (14).

Term by term interpretation of (10) therefore gives

$$v = \left(\frac{2a}{\pi r}\right)^{\frac{1}{2}} e^{-(a-r)^2/(8\nu t)} \left\{D_{-1}\left(\frac{a-r}{\sqrt{(2\nu t)}}\right) - \left(\frac{3}{8r} - \frac{3}{8a}\right)(2\nu t)^{\frac{1}{2}} D_{-2}\left(\frac{a-r}{\sqrt{(2\nu t)}}\right) + \ldots\right\}. \tag{17}$$

If we retain only these first two terms, we may write this as

$$v = \left(\frac{a}{r}\right)^{\frac{1}{2}} \left\{1 - \operatorname{erf}\left(\frac{a-r}{2\sqrt{(\nu t)}}\right)\right\} - \tfrac{3}{8}\left(\frac{a}{r}\right)^{\frac{1}{2}} \left(\frac{1}{r} - \frac{1}{a}\right)$$
$$\times \left[2\sqrt{\left(\frac{\nu t}{\pi}\right)} \exp\left(-\frac{(a-r)^2}{4\nu t}\right) - (a-r)\left\{1 - \operatorname{erf}\left(\frac{a-r}{2\sqrt{(\nu t)}}\right)\right\}\right]. \tag{18}$$

[We may note that the interpretations

$$e^{-qx} = 1 - \operatorname{erf}\left(\frac{x}{2\sqrt{(\nu t)}}\right), \tag{19}$$

$$qe^{-qx} = \frac{1}{\sqrt{(\pi\nu t)}} e^{-x^2/(4\nu t)}, \tag{20}$$

$$\frac{1}{q} e^{-qx} = 2\sqrt{\left(\frac{\nu t}{\pi}\right)} e^{-x^2/(4\nu t)} - x\left\{1 - \operatorname{erf}\left(\frac{x}{2\sqrt{(\nu t)}}\right)\right\}, \tag{21}$$

which are special cases of (11), were given by Jeffreys (*loc. cit.*, p. 112).]

* See Whittaker and Watson, *Modern analysis* (1920), 347 *et seq.*

If $a-r$ is not small, we may approximate by using (13); but (17) or (18) apply whether $a-r$ is small or not, so long as r is not small.

If $\nu t/a^2$ is small, but $\nu t/ar$ is large, we may use the asymptotic expansion of $I_1(qa)$ and the ascending series for $I_1(qr)$. This gives

$$v = e^{-qa}(2\pi qa)^{\frac{1}{2}}\left(1+\frac{3}{8qa}+\ldots\right)\left(\frac{qr}{2}+\frac{q^3r^3}{2^2.4}+\ldots\right). \tag{22}$$

Each term may then be interpreted by (11), and we may approximate to the functions $D_m(z)$ by means of (13). Thus the first approximation for points near the axis is

$$v = (\tfrac{1}{2}\pi a)^{\frac{1}{2}} rq^{\frac{3}{2}} e^{-qa} = a^{\frac{1}{2}} r(2\nu t)^{-\frac{3}{4}} e^{-a^2/(8\nu t)} D_{\frac{1}{2}}\left(\frac{a}{\sqrt{(2\nu t)}}\right) \sim \frac{ar}{2\nu t} e^{-a^2/(4\nu t)}. \tag{23}$$

For intermediate values of $(\nu t)^{\frac{1}{2}}/r$, neither the ascending nor the descending series may be substituted for $I_0(qr)$. The approximations appear to be too complicated to be of much use, but also appear to be unnecessary for ordinary computation of the value of v.

If ζ is the vorticity and Γ the circulation,

$$\Gamma = 2\pi rv = 2\pi \int_0^r r\zeta\, dr, \tag{24}$$

and therefore

$$\zeta = \frac{1}{r}\frac{d}{dr}(rv) = \frac{qI_0(qr)}{I_1(qa)}. \tag{25}$$

The application of the partial fraction rule gives

$$\zeta = \frac{2}{a} + \frac{2}{a}\sum_k \frac{J_0(kr)}{J_0(ka)} e^{-\nu k^2 t},$$

where the summation is over all the roots of (5). For small values of $\nu t/r^2$,

$$\begin{aligned}\zeta &= q\,\frac{e^{qr}(2\pi qa)^{\frac{1}{2}}\{1+1/(8qr)\}}{e^{qa}(2\pi qr)^{\frac{1}{2}}\{1-3/(8qa)\}} \\ &= \left(\frac{a}{r}\right)^{\frac{1}{2}} qe^{-q(a-r)}\left\{1+\frac{1}{8q}\left(\frac{1}{r}+\frac{3}{a}\right)\right\} \\ &= \left(\frac{a}{r}\right)^{\frac{1}{2}}\left[\frac{1}{\sqrt{(\pi\nu t)}}\, e^{-(a-r)^2/(4\nu t)} + \tfrac{1}{8}\left(\frac{1}{r}+\frac{3}{a}\right)\left\{1-\operatorname{erf}\left(\frac{a-r}{2\sqrt{(\nu t)}}\right)\right\}\right].\end{aligned} \tag{27}$$

For points on the axis, when $\nu t/a^2$ is small,

$$\zeta = (2\pi a)^{\frac{1}{2}} q^{\frac{3}{2}} e^{-qa} = 2a^{\frac{1}{2}}(2\nu t)^{-\frac{3}{4}} e^{-a^2/(8\nu t)} D_{\frac{1}{2}}\left(\frac{a}{\sqrt{(2\nu t)}}\right) \sim \frac{a}{\nu t} e^{-a^2/(4\nu t)}. \tag{28}$$

These results will be required later.

2. *The motion of viscous fluid between two infinite rotating circular cylinders.*

At time $t=0$, the circular cylinders $r=a$ and $r=b$ $(b>a)$, the space between which is filled with viscous fluid, begin suddenly to rotate with angular velocities ω and Ω respectively. With the same assumptions as before, the auxiliary equation for the velocity v is the same as before [equation (3)]. The boundary conditions are $v=a\omega$ when $r=a$ and $v=b\Omega$ when $r=b$; and the solution is

$$v=a\omega\frac{K_1(qb)\,I_1(qr)-I_1(qb)\,K_1(qr)}{K_1(qb)\,I_1(qa)-I_1(qb)\,K_1(qa)}+b\Omega\frac{I_1(qa)\,K_1(qr)-K_1(qa)\,I_1(qr)}{I_1(qa)\,K_1(qb)-K_1(qa)\,I_1(qb)}. \quad (29)$$

It is easy to verify that this solution is a single-valued even function of q, and therefore a single-valued function of p. We may therefore apply the partial fraction rule. The numerator has no poles; the denominator has zeros when $q=\pm ik$, where

$$J_1(ka)\,Y_1(kb)-Y_1(ka)\,J_1(kb)=0, \quad (30)$$

since

$$I_1(ix)=iJ_1(x), \quad K_1(ix)=-\tfrac{1}{2}\pi\,[J_1(x)-iY_1(x)], \quad (31)$$

the notation being that of Watson's *Bessel functions*. McMahon's formula for the large roots of (30) is reproduced by Gray, Mathews, and MacRobert (*Bessel functions*, p. 261); and Watson (p. 507) gives a reference to some seemingly scanty tables. The roots are all real.

If now we write

$$\left.\begin{aligned} f(p)&=K_1(qb)\,I_1(qr)-I_1(qb)\,K_1(qr)\\ F(p)&=K_1(qb)\,I_1(qa)-I_1(qb)\,K_1(qa)\end{aligned}\right\}, \quad (32)$$

we have

$$\frac{f(0)}{F(0)}=\frac{r/b-b/r}{a/b-b/a}=\frac{a}{r}\,\frac{b^2-r^2}{b^2-a^2}, \quad (33)$$

and, when $q=ik$,

$$f(p)=\tfrac{1}{2}\pi\{J_1(kb)\,Y_1(kr)-J_1(kr)\,Y_1(kb)\}, \quad (34)$$

$$p\frac{d}{dp}F(p)=\tfrac{1}{2}q\frac{d}{dq}F(q)=-\tfrac{1}{4}k\pi\,[a\{J_1'(ka)\,Y_1(kb)-J_1(kb)\,Y_1'(ka)\}$$
$$+b\{J_1(ka)\,Y_1'(kb)-J_1'(kb)\,Y_1(ka)\}]. \quad (35)$$

Now from (30) it follows that

$$\frac{J_1(ka)}{J_1(kb)}=\frac{Y_1(ka)}{Y_1(kb)}=\lambda \text{ (say)}, \quad (36)$$

and therefore (35) is equal to

$$-\tfrac{1}{4}k\pi\left[\frac{a}{\lambda}\{J_1'(ka)\,Y_1(ka)-J_1(ka)\,Y_1'(ka)\}\right.$$
$$\left.+b\lambda\{J_1(kb)\,Y_1'(kb)-J_1'(kb)\,Y_1(kb)\}\right]. \quad (37)$$

Since
$$J_1(x)\,Y_1'(x)-J_1'(x)\,Y_1(x)=2/(\pi x), \quad (38)$$

this is equal to

$$-\tfrac{1}{4}k\pi\left[-\frac{2a}{\pi\lambda ka}+\frac{2b\lambda}{\pi kb}\right]=\tfrac{1}{2}\left(\frac{1}{\lambda}-\lambda\right)=\frac{J_1^2(kb)-J_1^2(ka)}{2J_1(ka)\,J_1(kb)}. \quad (39)$$

The application of the partial fraction rule to the first term of (29) therefore gives

$$\frac{\omega a^2}{r}\frac{b^2-r^2}{b^2-a^2}+\pi a\omega\sum_k\frac{J_1(kb)\,Y_1(kr)-J_1(kr)\,Y_1(kb)}{J_1^2(kb)-J_1^2(ka)}\,J_1(ka)\,J_1(kb)\,e^{-\nu k^2t}. \quad (40)$$

The second term is obtained from this by interchanging a and b and ω and Ω. The interpretation of (29) is, therefore,

$$v=\frac{\omega a^2}{r}\frac{b^2-r^2}{b^2-a^2}+\pi a\omega\sum_k\frac{J_1(kb)\,Y_1(kr)-J_1(kr)\,Y_1(kb)}{J_1^2(kb)-J_1^2(ka)}\,J_1(ka)\,J_1(kb)\,e^{-\nu k^2t}$$
$$+\frac{\Omega b^2}{r}\frac{r^2-a^2}{b^2-a^2}-\pi b\Omega\sum_k\frac{J_1(ka)\,Y_1(kr)-J_1(kr)\,Y_1(ka)}{J_1^2(kb)-J_1^2(ka)}\,J_1(ka)\,J_1(kb)\,e^{-\nu k^2t}. \quad (41)$$

When $\nu t/a^2$ is small, we have

$$K_1(qb)\,I_1(qa)-I_1(qb)\,K_1(qa)$$
$$=\frac{e^{-(b-a)q}}{2q(ab)^{\frac{1}{2}}}\left[1-\frac{3}{8qa}+\frac{3}{8qb}+\ldots\right]-\frac{e^{(b-a)q}}{2q(ab)^{\frac{1}{2}}}\left[1-\frac{3}{8qb}+\frac{3}{8qa}+\ldots\right]. \quad (42)$$

The first term of (29), therefore, gives, approximately,

$$a\omega\left(\frac{a}{r}\right)^{\frac{1}{2}}\frac{e^{(b-r)q}\{1-3/(8qb)+3/(8qr)\}-e^{-(b-r)q}\{1-3/(8qr)+3/(8qb)\}}{e^{(b-a)q}\{1-3/(8qb)+3/(8qa)\}-e^{-(b-a)q}\{1-3/(8qa)+3/(8qb)\}}. \quad (43)$$

If we neglect $e^{-2(b-a)q}$, we find that this is equal to

$$a\omega\left(\frac{a}{r}\right)^{\frac{1}{2}}e^{-(r-a)q}\left\{1-\frac{3}{8qb}+\frac{3}{8qr}-e^{-2(b-r)q}\left(1-\frac{3}{8qr}+\frac{3}{8qb}\right)\right\}\left\{1+\frac{3}{8qb}-\frac{3}{8qa}\right\}. \quad (44)$$

If, for the sake of simplicity, we further neglect $q^{-1}e^{-2(b-r)q}$, this reduces to

$$a\omega\left(\frac{a}{r}\right)^{\frac{1}{2}}e^{-(r-a)q}\left\{1-\frac{3}{8qa}+\frac{3}{8qr}\right\}-a\omega\left(\frac{a}{r}\right)^{\frac{1}{2}}e^{-(2b-a-r)q}. \tag{45}$$

The second term can be similarly treated, and the result is

$$\begin{aligned}v=a\omega\left(\frac{a}{r}\right)^{\frac{1}{2}}&\left\{1-\operatorname{erf}\left(\frac{r-a}{2\sqrt{(\nu t)}}\right)+\left[\frac{3}{8r}-\frac{3}{8a}\right]\left[2\sqrt{\left(\frac{\nu t}{\pi}\right)}e^{-(r-a)^2/(4\nu t)}\right.\right.\\&\left.\left.-(r-a)\left\{1-\operatorname{erf}\left(\frac{r-a}{2\sqrt{(\nu t)}}\right)\right\}\right]-\left\{1-\operatorname{erf}\left(\frac{2b-a-r}{2\sqrt{(\nu t)}}\right)\right\}\right\}\\+b\Omega\left(\frac{b}{r}\right)^{\frac{1}{2}}&\left\{1-\operatorname{erf}\left(\frac{b-r}{2\sqrt{(\nu t)}}\right)+\left[\frac{3}{8b}-\frac{3}{8r}\right]\left[2\sqrt{\left(\frac{\nu t}{\pi}\right)}e^{-(b-r)^2/(4\nu t)}\right.\right.\\&\left.\left.-(b-r)\left\{1-\operatorname{erf}\left(\frac{b-r}{2\sqrt{(\nu t)}}\right)\right\}\right]-\left\{1-\operatorname{erf}\left(\frac{b+r-2a}{2\sqrt{(\nu t)}}\right)\right\}\right\}.\end{aligned} \tag{46}$$

3. *The motion of viscous fluid outside an infinite rotating circular cylinder.*

At time $t=0$ the circular cylinder $r=a$, the space outside which is filled with viscous fluid, begins to rotate with unit velocity. The appropriate solution of equation (3) is now

$$v=\frac{K_1(qr)}{K_1(qa)}. \tag{47}$$

The interpretation by Bromwich's integral is

$$v=\frac{1}{2\pi i}\int_L\frac{e^{zt}}{z}\frac{K_1(Zr)}{K_1(Za)}\,dz, \tag{48}$$

where $|\arg z|\leqslant\pi$, $z=\nu Z^2$, and L is a curve from $c-i\infty$ to $c+i\infty$, where c is finite and positive, with all the singularities of the integrand on the same side of the curve as $-\infty$. The integrals along two infinitely large quadrants, from $\arg z=\frac{1}{2}\pi$ to $\arg z=\pi$, and from $\arg z=-\frac{1}{2}\pi$ to $\arg z=-\pi$, vanish; and therefore the path L may be replaced by a path $CDEF$, where CD is the negative real axis ($\arg z=-\pi$) from $-\infty$ to $-\epsilon$, DE is the circle $|z|=\epsilon$ from $\arg z=-\pi$ to $\arg z=\pi$, EF is the negative real axis ($\arg z=+\pi$) from $-\epsilon$ to $-\infty$, and ϵ tends to zero in the limit. The contribution from the circle is a/r in the limit. On CD, put $z=\nu u^2e^{-i\pi}$, so that $Z=ue^{-\frac{1}{2}i\pi}=u/i$. Then, since

$$K_1(x/i)=-\tfrac{1}{2}\pi\,[J_1(x)+iY_1(x)], \tag{49}$$

the contribution from CD is

$$\frac{1}{\pi i}\int_{\infty}^{0} e^{-\nu t u^2}\,\frac{J_1(ur)+i\,Y_1(ur)}{J_1(ua)+i\,Y_1(ua)}\,\frac{du}{u}. \tag{50}$$

The contribution from EF is the integral between the same limits (divided by πi) of minus the expression conjugate to the integrand in (50). The result is, therefore,

$$v=\frac{a}{r}+\frac{2}{\pi}\int_{0}^{\infty} e^{-\nu t u^2}\,\frac{J_1(ur)\,Y_1(ua)-Y_1(ur)\,J_1(ua)}{J_1^{\,2}(ua)+Y_1^{\,2}(ua)}\,\frac{du}{u}. \tag{51}$$

If we put $t=0$, this leads to the result

$$\frac{2}{\pi}\int_{0}^{\infty}\frac{Y_1(ur)\,J_1(ua)-J_1(ur)\,Y_1(ua)}{J_1^{\,2}(ua)+Y_1^{\,2}(ua)}\,\frac{du}{u}=\frac{a}{r}\quad(r>a), \tag{52}$$

since the integral converges*.

For small values of $\nu t/a^2$, we use the asymptotic expansions of $K_1(qr)$ and $K_1(qa)$. These give

$$\begin{aligned} v=\frac{K_1(qr)}{K_1(qa)}&=\left(\frac{a}{r}\right)^{\frac{1}{2}} e^{-q(r-a)}\left[1+\frac{3}{8qr}-\frac{3}{8qa}+\ldots\right]\\ &=\left(\frac{a}{r}\right)^{\frac{1}{2}}\left\{1-\operatorname{erf}\left(\frac{r-a}{2\sqrt{(\nu t)}}\right)+\left[\frac{3}{8r}-\frac{3}{8a}\right]\right.\\ &\qquad\left.\times\left[2\sqrt{\left(\frac{\nu t}{\pi}\right)}\exp\left(-\frac{(r-a)^2}{4\nu t}\right)-(r-a)\left\{1-\operatorname{erf}\left(\frac{r-a}{2\sqrt{(\nu t)}}\right)\right\}\right]\right\} \end{aligned} \tag{53}$$

approximately.

When $t\to\infty$, $v\to a/r$. The next approximation for large values of t is found by using the ascending series for K_1, namely

$$K_1(x)=\frac{1}{x}+(\log\tfrac{1}{2}x+\gamma)\,I_1(x)-\tfrac{1}{2}\left\{\frac{x}{2}+\frac{x^3}{2^2\,.\,4}(1+1+\tfrac{1}{2})+\ldots\right\}, \tag{54}$$

where γ is Euler's constant. A term q^{2n} in v, where n is integral, will give no contribution to the result, and the second approximation is

$$v=\frac{a}{r}\,\frac{1+\frac{1}{2}q^2r^2\log\frac{1}{2}qr}{1+\frac{1}{2}q^2a^2\log\frac{1}{2}qa},$$

$$v=\frac{a}{r}\,[1+\tfrac{1}{2}q^2r^2\log\tfrac{1}{2}qr-\tfrac{1}{2}q^2a^2\log\tfrac{1}{2}qa]. \tag{55}$$

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* See Titchmarsh, *Proc. London Math. Soc.* (2), 22 (1923), 17. I am indebted to Dr. W. N. Bailey for drawing my attention to this paper.

By Bromwich's integral,

$$\tfrac{1}{2}q^2 r^2 \log \tfrac{1}{2}qr = \frac{1}{2\pi i}\int_{CDEF} \frac{e^{zt}}{z}\, \tfrac{1}{2}Z^2 r^2 \log \tfrac{1}{2}Zr\, dz. \tag{56}$$

The integral along the circle vanishes in the limit, and the integrals along CD and EF together give

$$-\tfrac{1}{2}r^2 \int_0^\infty e^{-\nu t u^2}\, u\, du = -\frac{r^2}{4\nu t}. \tag{57}$$

The second approximation is, therefore,

$$v = \frac{a}{r}\left[1 - \frac{r^2 - a^2}{4\nu t}\right]. \tag{58}$$

It can be shown that

$$(\tfrac{1}{2}qr)^{2n} \log \tfrac{1}{2}qr = \tfrac{1}{2}\left(-\frac{r^2}{4\nu t}\right)^n \Gamma(n), \tag{59}$$

and further approximations are, therefore, easy to obtain.

4. *Diffusion of a concentrated line vortex, or instantaneous line source of heat.*

Suppose that at time $t = 0$ we have an isolated line vortex of strength K concentrated in the axis. The motion being in circles about the axis, the vorticity ζ satisfies the equation

$$\frac{\partial \zeta}{\partial t} = \nu\left(\frac{\partial^2 \zeta}{\partial r^2} + \frac{1}{r}\frac{\partial \zeta}{\partial r}\right). \tag{60}$$

The same equation would apply in the theory of the conduction of heat if ζ were the temperature and ν the thermometric conductivity.

Since ζ is initially zero everywhere except at the origin, the auxiliary equation is

$$\frac{\partial^2 \zeta}{\partial r^2} + \frac{1}{r}\frac{\partial \zeta}{\partial r} - q^2 \zeta = 0, \tag{61}$$

where $p = \nu q^2$ as before. The solution for an unbounded fluid is

$$\zeta = AK_0(qr). \tag{62}$$

If Γ is the circulation,

$$\Gamma = 2\pi \int_0^r r\zeta dr = 2\pi A \int_0^r rK_0(qr)\, dr$$

$$= -\frac{2\pi A}{q}\int_0^r \frac{d}{dr}\left[rK_1(qr)\right] dr = \frac{2\pi A}{q^2}\left[1 - qr\, K_1(qr)\right]. \tag{63}$$

When $t \to 0$, *i.e.* when $q \to \infty$, the circulation must tend to K for all values of r except $r = 0$. Hence we must have

$$2\pi A = Kq^2, \tag{64}$$

$$\zeta = \frac{K}{2\pi} q^2 K_0(qr), \qquad \Gamma = K\,[1 - qr\,K_1(qr)]. \tag{65}$$

By Bromwich's integral,

$$\zeta = \frac{K}{4\pi^2 i} \int_{CDEF} \frac{e^{zt}}{z} Z^2 K_0(Zr)\, dz, \tag{66}$$

where $|\arg z| \leqslant \pi$, $z = \nu Z^2$, as on p. 60, and $CDEF$ is the path defined there. The integral along the circle is zero in the limit. Along CD, the integral, without the factor $K/(4\pi^2 i)$, is found to be

$$-\pi \int_{\infty}^{0} e^{-\nu t u^2} u\,[i\,J_0(ur) - Y_0(ur)]\, du, \tag{67}$$

on putting $z = \nu u^2 e^{-i\pi}$ on CD, and using the relation

$$K_0(x/i) = \tfrac{1}{2}\pi i\,[J_0(x) + i\,Y_0(x)]. \tag{68}$$

The integral along EF is minus the conjugate of this. Hence

$$\zeta = \frac{K}{2\pi} \int_0^{\infty} e^{-\nu t u^2} u\, J_0(ur)\, du$$

$$= \frac{K}{4\pi\nu t} e^{-r^2/(4\nu t)} \tag{69}$$

by Weber's first exponential integral (Watson's *Bessel functions*, p. 393). This is a well-known result*.

Note that
$$q^2 K_0(qr) = \frac{1}{2\nu t} e^{-r^2/(4\nu t)}. \tag{70}$$

Also
$$\Gamma = 2\pi \int_0^r r\zeta\, dr = K(1 - e^{-r^2/(4\nu t)}), \tag{71}$$

and, therefore,
$$qrK_1(qr) = e^{-r^2/(4\nu t)}. \tag{72}$$

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* Rayleigh, *Papers* 6, 53; Carslaw, *The conduction of heat*, 152; Lamb, *Hydrodynamics*, 560.

5. *Comparison with instantaneous point source and instantaneous plane source of heat.*

Using ζ for temperature, ν for thermometric conductivity, R for distance from the origin, we have, for a unit source of heat at the origin,

$$\frac{\partial}{\partial t}(R\zeta) = \nu q^2 R\zeta = \nu \frac{\partial^2}{\partial R^2}(R\zeta), \tag{73}$$

and, therefore,

$$R\zeta = Ae^{-qR}, \tag{74}$$

where

$$\int_0^R 4\pi R^2 \zeta \, dR = 4\pi A \int_0^R Re^{-qR} dR = 4\pi A \left(-R\frac{e^{-qR}}{q} - \frac{e^{-qR}}{q^2} + \frac{1}{q^2}\right) \tag{75}$$

tends to 1 as q tends to ∞, so that

$$4\pi A = q^2, \tag{76}$$

$$\zeta = \frac{1}{4\pi R} q^2 e^{-qR}. \tag{77}$$

Now

$$qe^{-qR} = \frac{1}{\sqrt{(\pi\nu t)}} e^{-R^2/(4\nu t)}. \tag{78}$$

(Jeffreys, *loc. cit.*, 112.)

Differentiate with respect to R. Then

$$q^2 e^{-qR} = \frac{R}{2\sqrt{(\pi\nu^3 t^3)}} e^{-R^2/(4\nu t)}, \tag{79}$$

$$\zeta = \frac{1}{\{2\sqrt{(\pi\nu t)}\}^3} e^{-R^2/(4\nu t)}. \tag{80}$$

Similarly, for a plane unit source,

$$\frac{\partial \zeta}{\partial t} = \nu q^2 \zeta = \nu \frac{\partial^2 \zeta}{\partial x^2}, \tag{81}$$

so that

$$\zeta = Ae^{-qx} \tag{82}$$

for $x > 0$, where

$$2\int_0^x \zeta \, dx = \frac{2A}{q}(1 - e^{-qx}) \tag{83}$$

tends to 1 as q tends to ∞. Hence

$$2A = q, \tag{84}$$

$$\zeta = \tfrac{1}{2}qe^{-qx} = \frac{1}{2\sqrt{(\pi\nu t)}}\, e^{-x^2/(4\nu t)}. \tag{85}$$

Compare (85) and (79) with (70). The results (80) and (85) are, of course, well known. They are due to Fourier and Kelvin.

6. *Any given initial distribution of temperature. Neumann's integral.*

Use r, θ for plane polar coordinates, and at $t = 0$ let ζ (the temperature) be equal to $F(r, \theta)$, the problem being supposed two-dimensional. The element $\varpi\, d\phi\, d\varpi$ at (ϖ, ϕ) can be treated as a line source of strength $F(\varpi, \phi)\, \varpi\, d\phi\, d\varpi$. The distance of this element from the point (r, θ) is

$$\rho = \{\varpi^2 + r^2 - 2\varpi r \cos(\phi - \theta)\}^{\frac{1}{2}}, \tag{86}$$

and its contribution to the temperature at (r, θ) is

$$\frac{F(\varpi, \phi)}{2\pi}\, \varpi\, d\phi\, d\varpi\, q^2 K_0(q\rho). \tag{87}$$

The temperature at (r, θ), being the sum of all such contributions, is

$$\begin{aligned}\zeta &= \frac{1}{2\pi}\int_0^\infty \int_{-\pi}^{\pi} F(\varpi, \phi)\, \varpi q^2 K_0(q\rho)\, d\phi\, d\varpi \\ &= \frac{1}{2\pi}\int_0^\infty e^{-\nu t u^2} u\, du \int_0^\infty \int_{-\pi}^{\pi} F(\varpi, \phi)\, J_0(u\rho)\, \varpi\, d\phi\, d\varpi,\end{aligned} \tag{88}$$

with suitable restrictions on F, where ν is the thermometric conductivity.

If we take $t = 0$, we find that

$$F(r, \theta) = \frac{1}{2\pi}\int_0^\infty u\, du \int_0^\infty \int_{-\pi}^{\pi} F(\varpi, \phi)\, J_0(u\rho)\, \varpi\, d\phi\, d\varpi, \tag{89}$$

and this is Neumann's integral. (Watson's *Bessel functions*, 453, 470-475.)

The hydrodynamical equations are not linear, and this paragraph does not apply to the hydrodynamical problem. The next paragraph applies either to the conduction of heat or the diffusion of vorticity; for brevity's sake, we shall refer only to the diffusion of vorticity.

7. Diffusion of a vortex with finite circular section.

Suppose that, when $t=0$, the vorticity ζ is equal to 1 for $r<a$, and to 0 for $r>a$. The auxiliary equation is

$$\left.\begin{aligned}\frac{\partial^2\zeta}{\partial r^2}+\frac{1}{r}\frac{\partial\zeta}{\partial r}-q^2\zeta &= -q^2 \quad (r<a)\\ &= 0 \quad (r>a)\end{aligned}\right\}, \tag{90}$$

so that

$$\left.\begin{aligned}\zeta &= 1+AI_0(qr) \quad (r<a)\\ &= BK_0(qr) \quad (r>a)\end{aligned}\right\}, \tag{91}$$

ζ being finite on the axis and at infinity. ζ and its derivatives must be continuous at $r=a$; if we make ζ and $\partial\zeta/\partial r$ continuous, the continuity of the higher derivatives will follow from the differential equation. Hence we must have

$$\left.\begin{aligned}AI_0(qa)-BK_0(qa)+1&=0\\ AI_0'(qa)-BK_0'(qa)&=0\end{aligned}\right\}. \tag{92}$$

Since
$$K_0(qa)\,I_0'(qa)-I_0(qa)\,K_0'(qa)=1/(qa), \tag{93}$$

and
$$K_0'(qa)=-K_1(qa), \quad I_0'(qa)=I_1(qa), \tag{94}$$

the solution of (92) is

$$A=-qa\,K_1(qa), \quad B=qa\,I_1(qa), \tag{95}$$

so that

$$\left.\begin{aligned}\zeta &= 1-qa\,K_1(qa)\,I_0(qr) \quad (r<a)\\ &= qa\,I_1(qa)\,K_0(qr) \quad (r>a)\end{aligned}\right\}. \tag{96}$$

Interpreting by Bromwich's integral, we find, for $r<a$,

$$\zeta=\frac{1}{2\pi i}\int_{CDEF}\frac{e^{zt}}{z}\,[1-Za\,K_1(Za)\,I_0(Zr)]\,dz, \tag{97}$$

where $|\arg z|\leqslant\pi$, $z=\nu Z^2$, as on p. 60, and $CDEF$ is the path defined there. The integral along the circle is zero in the limit, since the factor in brackets in the integrand is zero when Z is zero. The integral along CD, without the factor $1/(2\pi i)$, is found to be

$$2\int_\infty^0\frac{e^{-\nu tu^2}}{u}\,[1+iua\,K_1(ua/i)\,I_0(ur/i)]\,du$$
$$=2\int_\infty^0\frac{e^{-\nu tu^2}}{u}\,[1-\tfrac{1}{2}\pi iua\,J_0(ur)\{J_1(ua)+iY_1(ua)\}]\,du, \tag{98}$$

on putting $z = -\nu u^2$, $Z = ue^{-\frac{1}{2}\pi i}$, along CD. The integral along EF is minus the conjugate of this, and the result is

$$\zeta = a\int_0^\infty e^{-\nu t u^2} J_0(ur)\, J_1(ua)\, du. \tag{99}$$

If we put $t = 0$, and remember the initial value of ζ, we find one of Weber's integrals (Watson's *Bessel functions*, 406, eq. 9). The same result (99) is obtained for $r > a$. The result (99) was obtained by a different method by Terazawa*, who obtained series for the integral by means of a substitution based on the addition theorem for $J_1(u\rho)$, where

$$\rho^2 = a^2 + r^2 - 2ar\cos\theta, \tag{100}$$

and θ is any real angle. We proceed to find these series by direct transformation of the operational solution (96).

If $r < a$, and we define ψ by the relations

$$\rho\cos\psi = a - r\cos\theta, \quad \rho\sin\psi = r\sin\theta, \tag{101}$$

so that if, in a triangle ABC, $AB = a$, $BC = r$, angle $ABC = \theta$, then $AC = \rho$ and angle $BAC = \psi$, we have the formula

$$K_1(q\rho)\cos\psi = \sum_{m=-\infty}^{\infty} K_{m+1}(qa)\, I_m(qr)\cos m\theta. \tag{102}$$

(Watson, *Bessel functions*, 361). Therefore

$$qaK_1(qa)\, I_0(qr) = \frac{1}{\pi}\int_0^\pi qaK_1(q\rho)\cos\psi\, d\theta$$

$$= \frac{a}{\pi}\int_0^\pi q\rho K_1(q\rho)\,\frac{a - r\cos\theta}{\rho^2}\, d\theta. \tag{103}$$

Now
$$q\rho K_1(q\rho) = e^{-\rho^2/(4\nu t)} \tag{104}$$

by (72); and, for $r < a$,

$$\frac{a - r\cos\theta}{\rho^2} = \frac{1}{a}\sum_{m=0}^{\infty}\left(\frac{r}{a}\right)^m \cos m\theta. \tag{105}$$

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* *Report of the Aeronautical Research Institute, Tokyo Imperial University*, 1 (1922), No. 4, 99.

Hence

$$qaK_1(qa)\,I_0(qr) = \frac{1}{\pi}\int_0^{\pi} e^{-\rho^2/(4\nu t)} \sum_{m=0}^{\infty} \left(\frac{r}{a}\right)^m \cos m\theta\, d\theta$$

$$= e^{-(r^2+a^2)/(4\nu t)}\,\frac{1}{\pi} \sum_{m=0}^{\infty} \left(\frac{r}{a}\right)^m \int_0^{\pi} e^{-ar\cos\theta/(2\nu t)} \cos m\theta\, d\theta$$

$$= e^{-(r^2+a^2)/(4\nu t)} \sum_{m=0}^{\infty} \left(\frac{r}{a}\right)^m I_m\left(\frac{ar}{2\nu t}\right), \tag{106}$$

and

$$\zeta = 1 - e^{-(r^2+a^2)/(4\nu t)} \sum_{m=0}^{\infty} \left(\frac{r}{a}\right)^m I_m\left(\frac{ar}{2\nu t}\right). \tag{107}$$

For $r > a$,

$$K_1(q\rho) \begin{matrix}\cos\psi \\ \sin\psi\end{matrix} = \sum_{m=-\infty}^{\infty} K_{m+1}(qr)\, I_m(qa) \begin{matrix}\cos m\theta \\ \sin m\theta\end{matrix}, \tag{108}$$

so that

$$K_1(q\rho)\cos(\psi+\theta) = \sum_{m=-\infty}^{\infty} K_{m+1}(qr)\, I_m(qa)\cos(m+1)\theta. \tag{109}$$

Hence

$$K_0(qr)\, I_{-1}(qa) = K_0(qr)\, I_1(qa) = \frac{1}{\pi}\int_0^{\pi} K_1(q\rho)\cos(\psi+\theta)\, d\theta, \tag{110}$$

and

$$qaI_1(qa)\, K_0(qr) = \frac{a}{\pi}\int_0^{\pi} q\rho K_1(q\rho)\,\frac{\cos(\psi+\theta)}{\rho}\, d\theta. \tag{111}$$

In the triangle ABC, defined above, $\psi+\theta$ is the supplement of the angle ACB, and

$$\cos(\psi+\theta) = \frac{r\cos\theta - a}{\rho}. \tag{112}$$

Hence

$$\frac{\cos(\psi+\theta)}{\rho} = \frac{r\cos\theta - a}{\rho^2} = \frac{1}{r}\sum_{m=1}^{\infty}\left(\frac{a}{r}\right)^{m-1}\cos m\theta \tag{113}$$

for $r > a$, and

$$\zeta = qaI_1(qa)\, K_0(qr) = \sum_{m=1}^{\infty}\left(\frac{a}{r}\right)^m \frac{1}{\pi}\int_0^{\pi} e^{-\rho^2/(4\nu t)}\cos m\theta\, d\theta$$

$$= e^{-(r^2+a^2)/(4\nu t)} \sum_{m=1}^{\infty}\left(\frac{a}{r}\right)^m \frac{1}{\pi}\int_0^{\pi} e^{-ar\cos\theta/(2\nu t)}\cos m\theta\, d\theta$$

$$= e^{-(r^2+a^2)/(4\nu t)} \sum_{m=1}^{\infty}\left(\frac{a}{r}\right)^m I_m\left(\frac{ar}{2\nu t}\right). \tag{114}$$

When $r = a$, (107) and (114) must be equal, from which we easily deduce that

$$\zeta = \tfrac{1}{2}\left[1 - e^{-a^2/(2\nu t)} I_0\left(\frac{a^2}{2\nu t}\right)\right] \tag{115}$$

when $r = a$, so that as $t \to 0$, $\zeta \to \tfrac{1}{2}$, a result which also follows immediately by putting $r = a$ and letting $q \to \infty$ in (96). When $r = 0$,

$$\zeta = 1 - qaK_1(qa) = 1 - e^{-a^2/(4\nu t)}. \tag{116}$$

When $\nu t/r^2$ is small, and $r < a$,

$$\zeta = 1 - qa\,K_1(qa)\,I_0(qr) = 1 - \tfrac{1}{2}\left(\frac{a}{r}\right)^{\frac{1}{2}} e^{-q(a-r)}\left[1 + \frac{1}{q}\left(\frac{3}{8a} + \frac{1}{8r}\right) + \ldots\right]$$

$$= 1 - \tfrac{1}{2}\left(\frac{a}{r}\right)^{\frac{1}{2}}\left\{1 - \operatorname{erf}\left(\frac{a-r}{2\sqrt{(\nu t)}}\right) + \left[\frac{3}{8a} + \frac{1}{8r}\right]\right.$$

$$\left. \times\left[2\sqrt{\left(\frac{\nu t}{\pi}\right)} e^{-(a-r)^2/(4\nu t)} - (a-r)\left\{1 - \operatorname{erf}\left(\frac{a-r}{2\sqrt{(\nu t)}}\right)\right\}\right]\right\} \tag{117}$$

approximately, if r is not too small.

When $\nu t/r^2$ is small, and $r > a$,

$$\zeta = qaI_1(qa)\,K_0(qr) = \tfrac{1}{2}\left(\frac{a}{r}\right)^{\frac{1}{2}} e^{-q(r-a)}\left[1 - \frac{1}{q}\left(\frac{3}{8a} + \frac{1}{8r}\right) + \ldots\right]$$

$$= \tfrac{1}{2}\left(\frac{a}{r}\right)^{\frac{1}{2}}\left\{1 - \operatorname{erf}\left(\frac{r-a}{2\sqrt{(\nu t)}}\right) - \left[\frac{3}{8a} + \frac{1}{8r}\right]\right.$$

$$\left. \times\left[2\sqrt{\left(\frac{\nu t}{\pi}\right)} e^{-(r-a)^2/(4\nu t)} - (r-a)\left\{1 - \operatorname{erf}\left(\frac{r-a}{2\sqrt{(\nu t)}}\right)\right\}\right]\right\} \tag{118}$$

approximately.

Again, if t is large, we may write, for $r < a$,

$$\zeta = 1 - qaK_1(qa)\,I_0(qr)$$

$$= 1 - I_0(qr)\left\{1 + qa(\log \tfrac{1}{2}qa + \gamma)\,I_1(qa) - \tfrac{1}{2}\left[\frac{q^2a^2}{2} + \frac{q^4a^4}{2^2.4}(1+1+\tfrac{1}{2}) + \ldots\right]\right\}$$

$$= -qa \log \tfrac{1}{2}qa I_1(qa)\,I_0(qr), \tag{119}$$

since even powers of q contribute nothing to the result.

Now

$$qaI_1(qa)\,I_0(qr) = 2\sum_{m=0}^{\infty} \frac{(\frac{1}{2}qa)^{2m+2}}{m!\,(m+1)!}\,{}_2F_1\left(-m, -m-1;\ 1;\ \frac{r^2}{a^2}\right) \tag{120}$$

(*cf.* Watson's *Bessel functions*, 148) and

$$(\tfrac{1}{2}qa)^{2n}\log\tfrac{1}{2}qa=\tfrac{1}{2}\left(-\frac{a^2}{4\nu t}\right)^n\Gamma(n) \tag{121}$$

if n is integral [*cf.* (59)]. Hence

$$\begin{aligned}\zeta&=\sum_{m=0}^{\infty}\frac{(-)^m}{(m+1)!}\left(\frac{a^2}{4\nu t}\right)^{m+1}{}_2F_1\left(-m,\ -m-1;\ 1;\ \frac{r^2}{a^2}\right)\\&=\frac{a^2}{4\nu t}-\frac{1}{2!}\left(\frac{a^2}{4\nu t}\right)^2\left(1+2\,\frac{r^2}{a^2}\right)+\frac{1}{3!}\left(\frac{a^2}{4\nu t}\right)^3\left(1+6\,\frac{r^2}{a^2}+3\,\frac{r^4}{a^4}\right)-\ldots.\end{aligned} \tag{122}$$

Similarly, for $r>a$, we may write

$$\zeta=qaI_1(qa)\,K_0(qr)=-qa\log\tfrac{1}{2}qr\,I_1(qa)\,I_0(qr), \tag{123}$$

and this is the same as (119), since $\log(r/a)\,qaI_1(qa)\,I_0(qr)$ contributes nothing to the result. The same formula (122) therefore holds as before.

The circulation Γ is worthy of some attention. For $r<a$, we have

$$\begin{aligned}\Gamma&=2\pi\int_0^r\zeta r\,dr=2\pi\int_0^r r[1-qaK_1(qa)\,I_0(qr)]dr\\&=\pi r^2-2\pi arK_1(qa)\,I_1(qr).\end{aligned} \tag{124}$$

At $r=a$,
$$\Gamma=\pi a^2-2\pi a^2K_1(qa)\,I_1(qa), \tag{125}$$

and for $r>a$,

$$\begin{aligned}\Gamma&=\pi a^2-2\pi a^2\,K_1(qa)\,I_1(qa)+2\pi\int_a^r qaI_1(qa)\,K_0(qr)\,r\,dr\\&=\pi a^2-2\pi a^2\,K_1(qa)\,I_1(qa)+2\pi aI_1(qa)\,[aK_1(qa)-rK_1(qr)]\\&=\pi a^2-2\pi ar\,I_1(qa)\,K_1(qr).\end{aligned} \tag{126}$$

By Bromwich's integral, for $r<a$,

$$\Gamma=\frac{1}{2\pi i}\int_{CDEF}\frac{e^{zt}}{z}\,[\pi r^2-2\pi arK_1(Za)\,I_1(Zr)]\,dz, \tag{127}$$

and, proceeding as before, we find that this reduces to

$$\Gamma=2\pi ar\int_0^\infty e^{-\nu tu^2}\,J_1(ur)\,J_1(ua)\,\frac{du}{u}. \tag{128}$$

The same expression (128) is found for $r>a$. Now

$$\int_0^\pi\frac{J_1(u\rho)}{u\rho}\sin^2\theta\,d\theta=\pi\,\frac{J_1(ur)\,J_1(ua)}{u^2\,ar} \tag{129}$$

(Watson's *Bessel functions*, 367), where ρ is given by (100), and therefore

$$\Gamma = 2a^2r^2\int_0^\infty\int_0^\pi e^{-\nu t u^2}J_1(u\rho)\,\frac{\sin^2\theta}{\rho}\,du\,d\theta. \tag{130}$$

The order of integration may be changed, and, since

$$\int_0^\infty e^{-\nu t u^2}J_1(u\rho)\,du = \frac{1}{\rho}\,(1-e^{-\rho^2/(4\nu t)}) \tag{131}$$

(Watson's *Bessel functions*, 394), we have

$$\Gamma = 2a^2r^2\int_0^\pi (1-e^{-\rho^2/(4\nu t)})\,\frac{\sin^2\theta}{\rho^2}\,d\theta. \tag{132}$$

It is easy to prove that, when $r > a$,

$$\frac{\sin^2\theta}{\rho^2} = \frac{1}{2ar}\sum_{m=0}^{\infty}\left\{\left(\frac{a}{r}\right)^m[\cos(m-1)\theta-\cos(m+1)\theta]\right\}, \tag{133}$$

so that

$$\begin{aligned}\Gamma &= ar\sum_{m=0}^{\infty}\left(\frac{a}{r}\right)^m\int_0^\pi[1-e^{-\rho^2/(4\nu t)}]\,[\cos(m-1)\theta-\cos(m+1)\theta]\,d\theta\\
&= \pi a^2-ar\,e^{-(r^2+a^2)/(4\nu t)}\sum_{m=0}^{\infty}\left(\frac{a}{r}\right)^m\\
&\qquad\qquad\times\int_0^\pi e^{-(ar\cos\theta)/(2\nu t)}\,[\cos(m-1)\theta-\cos(m+1)\theta]\,d\theta\\
&= \pi a^2-\pi ar\,e^{-(r^2+a^2)/(4\nu t)}\sum_{m=0}^{\infty}\left(\frac{a}{r}\right)^m\left[I_{m-1}\left(\frac{ar}{2\nu t}\right)-I_{m+1}\left(\frac{ar}{2\nu t}\right)\right]\\
&= \pi a^2-\pi ar\,e^{-(r^2+a^2)/(4\nu t)}\sum_{m=0}^{\infty}\left(\frac{a}{r}\right)^m\frac{4m\nu t}{ar}\,I_m\left(\frac{ar}{2\nu t}\right)\\
&= \pi a^2-4\pi\nu t\,e^{-(r^2+a^2)/(4\nu t)}\sum_{m=1}^{\infty}m\left(\frac{a}{r}\right)^m I_m\left(\frac{ar}{2\nu t}\right).\end{aligned} \tag{134}$$

Similarly, when $r < a$,

$$\Gamma = \pi r^2-4\pi\nu t\,e^{-(r^2+a^2)/(4\nu t)}\sum_{m=1}^{\infty}m\left(\frac{r}{a}\right)^m I_m\left(\frac{ar}{2\nu t}\right). \tag{135}$$

When $r = a$, it can be shown that (134) and (135) both give

$$\Gamma = \pi a^2-\pi a^2\,e^{-(a^2)/(2\nu t)}\left[I_0\left(\frac{a^2}{2\nu t}\right)+I_1\left(\frac{a^2}{2\nu t}\right)\right]. \tag{136}$$

For small values of $\nu t/r^2$, formulae are easily deduced by the usual method. Thus, for $r < a$,

$$\Gamma = \pi r^2 - \pi(ar)^{\frac{1}{2}}\left[2\sqrt{\left(\frac{\nu t}{\pi}\right)}e^{-(a-r)^2/(4\nu t)} - (a-r)\left\{1-\operatorname{erf}\left(\frac{a-r}{2\sqrt{(\nu t)}}\right)\right\}\right]+\ldots, \tag{137}$$

if r is not too small; and, for $r > a$,

$$\Gamma = \pi a^2 - \pi(ar)^{\frac{1}{2}}\left[2\sqrt{\left(\frac{\nu t}{\pi}\right)}e^{-(r-a)^2/(4\nu t)} - (r-a)\left\{1-\operatorname{erf}\left(\frac{r-a}{2\sqrt{(\nu t)}}\right)\right\}\right]+\ldots. \tag{138}$$

Formulae analogous to (122) are also easily obtainable.

8. *Diffusion of a cylindrical sheet of vorticity.*

Suppose that, at time $t = 0$, there is, at $r = c$, a cylindrical sheet of vorticity of strength σ per unit length. The values of the vorticity and circulation can then be obtained from the results of the last paragraph by differentiation with respect to a. It is, however, more instructive to obtain them by integration from the results for an initial concentrated line vortex. Although the hydrodynamical equations are not linear, the motion is here symmetrical about the axis, and results obtained by integration in this way will be correct.

The element of the sheet at (c, θ) will give, at the point $(r, 0)$, a value of the vorticity

$$\zeta = \frac{\sigma c}{2\pi} q^2 K_0(q\rho)\, d\theta = \frac{\sigma c}{4\pi\nu t} e^{-\rho^2/(4\nu t)}\, d\theta, \tag{139}$$

where

$$\rho^2 = c^2 + r^2 - 2cr\cos\theta. \tag{140}$$

By integration, the value of ζ at the point $(r, 0)$ at time t, due to the diffusion of the complete sheet, will be

$$\zeta = \frac{\sigma c}{2\pi}\int_{-\pi}^{\pi} q^2 K_0(q\rho)\, d\theta \tag{141}$$

$$= \frac{\sigma c}{4\pi\nu t} e^{-(r^2+c^2)/(4\nu t)}\int_{-\pi}^{\pi} e^{-(rc\cos\theta)/(2\nu t)}\, d\theta$$

$$= \frac{\sigma c}{2\nu t} e^{-(r^2+c^2)/(4\nu t)} I_0\left(\frac{rc}{2\nu t}\right). \tag{142}$$

The motion being symmetrical, the result holds for all points at a distance r from the axis*.

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* The formula (142) was given by Rayleigh, *loc. cit.*, 53, and Terazawa, *loc. cit.*, 106.

Now, when $r > c$,

$$K_0(q\rho) = \sum_{n=-\infty}^{\infty} K_m(qr)\, I_m(qc) \cos m\theta, \tag{143}$$

and, when $r < c$,

$$K_0(q\rho) = \sum_{m=-\infty}^{\infty} K_m(qc)\, I_m(qr) \cos m\theta \tag{144}$$

(Watson's *Bessel functions*, 361), and hence

$$\left.\begin{aligned} \zeta = \frac{\sigma c}{2\pi}\int_{-\pi}^{\pi} q^2 K_0(q\rho)\, d\theta &= \sigma c q^2 K_0(qr)\, I_0(qc) \quad (r > c) \\ &= \sigma c q^2 K_0(qc)\, I_0(qr) \quad (r < c) \end{aligned}\right\}. \tag{145}$$

The interpretation by Bromwich's integral gives

$$\zeta = \sigma c \int_0^{\infty} e^{-\nu t u^2} J_0(ur)\, J_0(uc)\, u\, du, \tag{146}$$

whether $r \lessgtr c$, and comparison with (142) gives an integral due to Weber (Watson's *Bessel functions*, 395), which is easily generalized by taking a periodic distribution for ζ (though the result would then not apply to the hydrodynamical problem).

For the circulation, we find

$$\Gamma = 2\pi \int_0^r r\zeta\, dr = 2\pi\sigma c q r\, K_0(qc)\, I_1(qr) \tag{147}$$

for $r < c$, and

$$\begin{aligned} \Gamma &= 2\pi\sigma c^2 q K_0(qc)\, I_1(qc) + 2\pi \int_c^r \sigma c q^2 K_0(qr)\, I_0(qc)\, r\, dr \\ &= 2\pi\sigma c^2 q K_0(qc)\, I_1(qc) + 2\pi\sigma c q\, I_0(qc)\, [cK_1(qc) - rK_1(qr)] \\ &= 2\pi\sigma c\, [1 - qr I_0(qc)\, K_1(qr)] \end{aligned} \tag{148}$$

for $r > c$. The value of $\Gamma/(2\pi\sigma c)$ is, therefore, the same as the value of ζ in (96), with r in place of a, and c in place of r. The interpretation by Bromwich's integral is then immediately obtained from (99); also we find, from (107) and (114), that

$$\left.\begin{aligned} \frac{\Gamma}{2\pi\sigma c} &= 1 - e^{-(r^2+c^2)/(4\nu t)} \sum_{m=0}^{\infty} \left(\frac{c}{r}\right)^m I_m\left(\frac{cr}{2\nu t}\right) \quad (c < r) \\ &= e^{-(r^2+c^2)/(4\nu t)} \sum_{m=1}^{\infty} \left(\frac{r}{c}\right)^m I_m\left(\frac{cr}{2\nu t}\right) \quad (c > r) \end{aligned}\right\}. \tag{149}$$

The physical significance of the substitution employed by Terazawa in finding these series now becomes clear. For the velocity at the point $(r, 0)$ due to the element of the sheet at (c, θ) is

$$\frac{\sigma a d\theta}{2\pi\rho}[1-e^{-\rho^2/(4\nu t)}]; \tag{150}$$

the component at right angles to the radius vector is found by multiplying this by

$$\cos\psi = \frac{r-c\cos\theta}{\rho}, \tag{151}$$

where ψ is the angle between the radius vector to $(r, 0)$ and the line joining (c, θ) to $(r, 0)$; the total velocity due to the ring is found by integrating this component with respect to θ from $-\pi$ to π; and the circulation by multiplying by $2\pi r$. Hence

$$\frac{\Gamma}{2\pi\sigma c} = \frac{r}{2\pi}\int_{-\pi}^{\pi}(1-e^{-\rho^2/(4\nu t)})\left(\frac{r-c\cos\theta}{\rho^2}\right)d\theta, \tag{152}$$

and on expanding $(r-c\cos\theta)/\rho^2$ in a Fourier series, and integrating term by term, we get (149).

9. *Any given initial symmetrical distribution of vorticity.*

If initially $\zeta = f(r)$, then in (145) we must put $\sigma = f(c)\,dc$, and integrate with respect to c from 0 to ∞. In this way we get

$$\zeta = q^2 K_0(qr)\int_0^r f(c)\,I_0(qc)\,c\,dc + q^2 I_0(qr)\int_r^\infty f(c)\,K_0(qc)\,c\,dc, \tag{153}$$

which is the same result as we get by integrating the auxiliary equation

$$\frac{\partial^2\zeta}{\partial r^2}+\frac{1}{r}\frac{\partial\zeta}{\partial r}-q^2\zeta = -q^2 f(r), \tag{154}$$

by the method of variation of parameters. The interpretation of (153) by Bromwich's integral is

$$\zeta = \int_0^\infty e^{-\nu t u^2}\,u\,du\int_0^\infty f(c)\,J_0(ur)\,J_0(uc)\,c\,dc, \tag{155}$$

with suitable restrictions on f. With $t = 0$ we find that

$$f(r) = \int_0^\infty u\,du\int_0^\infty f(c)\,J_0(ur)\,J_0(uc)\,c\,dc, \tag{156}$$

and this is a special case of Neumann's integral (Watson's *Bessel functions*, 453–464).

The value of ζ can also be obtained by replacing σ by $f(c)\,dc$ in (142), and integrating with respect to c from 0 to ∞, if the integral converges.

10. *Diffusion of a concentrated line vortex, with a boundary at $r = a$.*

Suppose that at time $t = 0$ we have an isolated line vortex of strength K concentrated in the axis, and that at $r = a$ there is a solid boundary, so that the velocity v and circulation Γ must vanish there. If there were no boundary, the solution would be

$$\zeta = \frac{K}{2\pi} q^2 K_0(qr), \quad \Gamma = K[1 - qrK_1(qr)], \tag{157}$$

and at $r = a$ we should have

$$v = \frac{K}{2\pi a}[1 - qaK_1(qa)]. \tag{158}$$

If then we superpose on (157) the solution obtained by giving the boundary a velocity equal and opposite to this, we shall not disturb the initial conditions, and shall satisfy the boundary conditions. If now the velocity of the boundary is Heaviside's unit function $H(t)$, which is unity when t is positive, and zero when t is negative, we know that

$$\Gamma = 2\pi rv = 2\pi r I_1(qr)/I_1(qa). \tag{159}$$

If we multiply this by the left side of (158), and subtract from the expression on the left in the second equation of (157), we find the correct operational solution of our problem, namely

$$\begin{aligned}\Gamma &= K[1 - qrK_1(qr)] - \frac{Kr}{a}\frac{I_1(qr)}{I_1(qa)}[1 - qaK_1(qa)] \\ &= K - K\frac{r}{a}\frac{I_1(qr)}{I_1(qa)} - \frac{Kqr}{I_1(qa)}[K_1(qr)I_1(qa) - I_1(qr)K_1(qa)].\end{aligned} \tag{160}$$

The corresponding value of the vorticity is

$$\begin{aligned}\zeta &= \frac{K}{2\pi} q^2 K_0(qr) - \frac{Kq}{2\pi a}\frac{I_0(qr)}{I_1(qa)}[1 - qaK_1(qa)] \\ &= \frac{Kq^2}{2\pi I_1(qa)}[K_0(qr)I_1(qa) + I_0(qr)K_1(qa)] - \frac{KqI_0(qr)}{2\pi a I_1(qa)}.\end{aligned} \tag{161}$$

These are both single-valued, even functions of q, and therefore single-valued functions of p, and we may interpret by the partial fraction rule. $I_1(qr)/I_1(qa)$ and $qI_0(qr)/I_1(qa)$ have already been interpreted. [See (4) and (9), and (25) and (26).] Proceeding similarly, we find that, if

$$J_1(ka) = 0, \tag{162}$$

then

$$\frac{q[K_1(qr)\,I_1(qa) - I_1(qr)\,K_1(qa)]}{I_1(qa)}$$

$$= \frac{1}{r} - \frac{r}{a^2} + \sum_k \frac{ik[K_1(ikr)\,I_1(ika) - I_1(ikr)\,K_1(ika)]}{\frac{1}{2}ikaI_1'(ika)}\, e^{-\nu k^2 t}$$

$$= \frac{1}{r} - \frac{r}{a^2} + \frac{\pi}{a} \sum_k \frac{J_1(kr)\,Y_1(ka)}{J_1'(ka)}\, e^{-\nu k^2 t}$$

$$= \frac{1}{r} - \frac{r}{a^2} - \frac{2}{a^2} \sum_k \frac{J_1(kr)}{kJ_0^2(ka)}\, e^{-\nu k^2 t}, \tag{163}$$

since
$$J_1(ka)\,Y_1'(ka) - J_1'(ka)\,Y_1(ka) = 2/(\pi ka), \tag{164}$$

and therefore, when $J_1(ka) = 0$, so that $J_1'(ka) = J_0(ka)$, we have

$$Y_1(ka) = -2/[\pi ka\, J_0(ka)]. \tag{165}$$

The summation in (163) is over all the roots of (162). The interpretation of (160) by the partial fraction rule therefore gives

$$\Gamma = K - K\frac{r}{a}\left\{\frac{r}{a} + 2\sum_k \frac{J_1(kr)}{kaJ_0(ka)}\, e^{-\nu k^2 t}\right\}$$

$$- Kr\left\{\frac{1}{r} - \frac{r}{a} - \frac{2}{a^2}\sum_k \frac{J_1(kr)}{kJ_0^2(ka)}\, e^{-\nu k^2 t}\right\}$$

$$= \frac{2Kr}{a^2} \sum_k \frac{J_1(kr)}{J_0^2(ka)}\,[1 - J_0(ka)]\, e^{-\nu k^2 t}. \tag{166}$$

Similarly, the interpretation of (161) gives

$$\zeta = \frac{K}{\pi a^2} \sum_k \frac{J_0(kr)}{J_0^2(ka)}\,[1 - J_0(ka)]\, e^{-\nu k^2 t}. \tag{167}$$

When $\nu t/a^2$ is small, the term

$$\frac{Kqr I_1(qr)\,K_1(qa)}{I_1(qa)} \tag{168}$$

in Γ may be ignored, and, approximately,

$$\Gamma = K\,[1-qr\,K_1(qr)]-\frac{Kr}{a}\,\frac{I_1(qr)}{I_1(qa)}$$

$$= K(1-e^{-r^2/(4\nu t)})-K\left(\frac{r}{a}\right)^{\frac{1}{2}}\left\{1-\mathrm{erf}\left(\frac{a-r}{2\sqrt{(\nu t)}}\right)\right\}$$

$$+\frac{3K}{8}\left(\frac{r}{a}\right)^{\frac{1}{2}}\left(\frac{1}{r}-\frac{1}{a}\right)\left[2\sqrt{\left(\frac{\nu t}{\pi}\right)}\,e^{-(a-r)^2/(4\nu t)}-(a-r)\left\{1-\mathrm{erf}\left(\frac{a-r}{2\sqrt{(\nu t)}}\right)\right\}\right] \tag{169}$$

by (72) and (18), if r is not too small.

Similarly, when $\nu t/a^2$ is small, the term

$$\frac{Kq^2\,I_0(qr)\,K_1(qa)}{2\pi I_1(qa)} \tag{170}$$

in ζ may be ignored, and approximately

$$\zeta = \frac{K}{2\pi}\,q^2\,K_0(qr)-\frac{Kq}{2\pi a}\,\frac{I_0(qr)}{I_1(qa)}$$

$$= \frac{K}{4\pi\nu t}e^{-r^2/(4\nu t)}-\frac{K}{2\pi(ar)^{\frac{1}{2}}}\left[\frac{1}{\sqrt{(\pi\nu t)}}e^{-(r-a)^2/(4\nu t)}+\frac{1}{8}\left(\frac{1}{r}+\frac{3}{a}\right)\left\{1-\mathrm{erf}\left(\frac{a-r}{2\sqrt{(\nu t)}}\right)\right\}\right] \tag{171}$$

by (70) and (27), if r is not too small.

These formulae give all that is required for numerical computation of Γ and ζ. Physically, it is clear that, while positive vorticity is diffusing outwards from the axis, negative vorticity will diffuse inwards from the boundary; and this will cause a decrease in the vorticity at any point of the fluid. It may, however, be of interest to examine the effect of the boundary numerically; below, tables are given of Γ and of ζ, both with and without a boundary. It is simplest to take the radius of the boundary as the unit of length, and adjust the value of ν accordingly. Thus if t is measured in seconds, then, for air at standard temperature and pressure, $\nu = 0{\cdot}132$ if $a = 1$ cm.; if $a = n$ cm. we must take $\nu = 0{\cdot}132/n^2$; and similarly for other fluids.

With $a = 1$, the values of Γ/K for various values of r and νt are given in Table I; the values of $2\pi\zeta/K$ are given in Table II. The figures in italics give the values with a boundary; the others the values in an infinite fluid.

In an infinite fluid, the value of Γ depends only on $r^2/4\nu t$, so that values of Γ/K for values of r and νt not included in Table I are easily found.

TABLE I. Values of Γ/K.

νt \ r	0·2	0·4	0·6	0·8	1
0·01	0·63 *0·63*	0·98 *0·98*	1·00 *1·00*	1·00 *0·86*	1·00 *0*
0·02	0·39 *0·39*	0·86 *0·86*	0·99 *0·96*	1·00 *0·74*	1·00 *0*
0·04	0·22 *0·22*	0·63 *0·61*	0·89 *0·76*	0·98 *0·56*	1·00 *0*
0·08	0·12 *0·10*	0·39 *0·32*	0·68 *0·45*	0·86 *0·34*	0·96 *0*
0·16	0·06 *0·03*	0·22 *0·10*	0·43 *0·15*	0·63 *0·11*	0·79 *0*
0·32	0·03 *0·00*	0·12 *0·01*	0·25 *0·01*	0·39 *0·01*	0·54 *0*
0·64	0·02 *0·00*	0·06 *0·00*	0·13 *0·00*	0·22 *0·00*	0·32 *0*

TABLE II. Values of $2\pi\zeta/K$.

νt \ r	0	0·1	0·2	0·3	0·4	0·6	0·8	1·0
0·01	50 *50*	38·94 *38·94*	18·39 *18·39*	5·26 *5·26*	0·92 *0·92*	0·01 *−0·13*	0·00 *−2·41*	0·00 *−6·14*
0·02	25 *25*	22.06 *22·06*	15·16 *15·16*	8·12 *8·11*	3·38 *3·31*	0·28 *−0·45*	0·01 *−2·88*	0·00 *−4·49*
0·04	12·5 *12·48*	11·74 *11·66*	9·74 *9·55*	*	4·60 *3·97*	1·32 *−0·15*	0·23 *−2·50*	0·02 *−3·20*
0·08	6·25 *5·67*	6·06 *	5·52 *4·77*	*	3·79 *2·58*	2·03 *0·18*	0·85 *−1·50*	0·27 *−2·07*
0·16	3·125 *1·73*	*	2·94 *1.48*	*	2·43 *0·85*	1·78 *0·10*	1·15 *−0.49*	0·66 *−0·70*
0·32	1·56 *0·17*	*	1·51 *0·15*	*	1·38 *0·09*	1·18 *0·01*	0·95 *−0·05*	0·71 *−0·07*
0·64	0·78 *	*	0·77 *	*	0·73 *	0·68 *	0·61 *	0·53 *

* Not calculated.

Thus the value for $r = 0{\cdot}1$, $\nu t = 0{\cdot}01$, is the same as for $r = 0{\cdot}2$, $\nu t = 0{\cdot}04$, namely $0{\cdot}22$, and so on. Also, Γ has its maximum when ζ is zero. Graphs of Γ/K and $2\pi\zeta/K$ are now easily drawn; I have drawn such graphs, but there are rather a large number of them, and it does not seem worth while to reproduce them, since they are easily drawn from the tables above by anyone interested.

I may add that (166) and (169) give practically the same values of Γ/K when $\nu t = 0{\cdot}01$, for all the values of r considered; but to find $2\pi\zeta/K$, (171) had to be used for $\nu t = 0{\cdot}01$ and $0{\cdot}02$, and (167) and (171) in conjunction for $\nu t = 0{\cdot}04$.

11. *Instantaneous line source of heat, with a boundary at $r = a$ maintained at zero temperature.*

A line source of heat of unit strength is instantaneously generated along the axis at time $t = 0$. The temperature ζ was everywhere previously zero, and the temperature of the boundary $r = a$ is maintained zero (Carslaw, *The conduction of heat*, 247).

The auxiliary equation for ζ is

$$\frac{\partial^2 \zeta}{\partial r^2} + \frac{1}{r}\frac{\partial \zeta}{\partial r} - q^2 \zeta = 0, \tag{172}$$

and the solution which vanishes when $r = a$ is

$$\zeta = C[I_0(qa)\, K_0(qr) - I_0(qr)\, K_0(qa)]. \tag{173}$$

Then

$$2\pi \int_0^r r\zeta\, dr = \frac{2\pi C I_0(qa)\,[1 - qrK_1(qr)]}{q^2} - \frac{2\pi C r K_0(qa)\, I_1(qr)}{q} \tag{174}$$

and when q tends to ∞, this must tend to 1. Hence we take

$$C = \frac{q^2}{2\pi I_0(qa)}, \tag{175}$$

$$\zeta = \frac{q^2}{2\pi}\,\frac{I_0(qa)\, K_0(qr) - I_0(qr)\, K_0(qa)}{I_0(qa)}. \tag{176}$$

This is a single-valued function of p; and if

$$J_0(ka) = 0, \tag{177}$$

interpretation by the partial fraction rule gives

$$\zeta = \frac{1}{2\pi} \sum_k \frac{-k\{I_0(ika)\,K_0(ikr) - I_0(ikr)\,K_0(ika)\}}{\frac{1}{2}ikaI_0'(ika)}\, e^{-\nu k^2 t}$$

$$= -\frac{1}{2a^2} \sum_k \frac{ka}{J_0'(ka)}\, J_0(kr)\, Y_0(ka)\, e^{-\nu k^2 t}$$

$$= \frac{1}{\pi a^2} \sum_k \frac{J_0(kr)}{J_1^2(ka)}\, e^{-\nu k^2 t}, \tag{178}$$

since
$$J_0(ka)\, Y_0'(ka) - J_0'(ka)\, Y_0(ka) = 2/(\pi ka), \tag{179}$$

and therefore, when $J_0(ka) = 0$,

$$Y_0(ka) = -2/[\pi ka\, J_0'(ka)] = +2/[\pi ka\, J_1(ka)]. \tag{180}$$

The summation in (178) is over all the roots of (177). The series (178) is Carslaw's result.

For small values νt we have approximately

$$\zeta = \frac{1}{2\pi} q^2 K_0(qr) - \frac{1}{2\pi} \frac{q^2 I_0(qr)\, K_0(qa)}{I_0(qa)}$$

$$= \frac{1}{2\pi} q^2 K_0(qr) - \frac{1}{\sqrt{(8\pi r)}}\, q^{\frac{3}{2}}\, e^{-q(2a-r)}$$

$$= \frac{1}{4\pi\nu t}\, e^{-r^2/(4\nu t)} - \frac{1}{2\pi r^{\frac{1}{2}}}\, e^{-(2a-r)^2/(8\nu t)} (2\nu t)^{-\frac{3}{4}} D_{\frac{1}{2}}\left(\frac{2a-r}{\sqrt{(2\nu t)}}\right)$$

$$\sim \frac{1}{4\pi\nu t}\, e^{-r^2/(4\nu t)} - \frac{1}{2\pi r^{\frac{1}{2}}}\, e^{-(2a-r)^2/(4\nu t)} (2\nu t)^{-\frac{3}{4}} \frac{(2a-r)^{\frac{1}{2}}}{(2\nu t)^{\frac{1}{4}}}$$

$$= \frac{1}{4\pi\nu t}\, e^{-r^2/(4\nu t)} - \frac{1}{4\pi\nu t} \left(\frac{2a-r}{r}\right)^{\frac{1}{2}} e^{-(2a-r)^2/(4\nu t)}, \tag{181}$$

if r is not too small.

The second term in (181) is of the same order as the term neglected in (170). It is of the same order as the first term in (181) if r is nearly equal to a, but then, for small values of νt, the first term is itself very small. And for other values of r, the correction due to the boundary is negligible for small values of νt.

12. *Diffusion of a cylindrical sheet of vorticity, with an external coaxial boundary.*

Suppose that, at time $t = 0$, there is, at $r = c$, a cylindrical sheet of vorticity, of strength σ per unit length, and at $r = a$, where $a > c$, there

is a solid boundary at which the velocity must vanish. In the absence of a boundary, we should have

$$\Gamma = 2\pi\sigma c\,[1-qrI_0(qc)\,K_1(qr)] \quad (r>c) \tag{182}$$

[see (148)], and the velocity at $r=a$ would be

$$v = \frac{\sigma c}{a}\,[1-qaI_0(qc)\,K_1(qa)]. \tag{183}$$

Proceeding as in § 10, we see that we must subtract

$$\frac{2\pi r\sigma c}{a}\,\frac{I_1(qr)}{I_1(qa)}\,[1-qa\,I_0(qc)\,K_1(qa)] \tag{184}$$

from the circulation as given by (147) and (148), and

$$\frac{\sigma c}{a}\,\frac{qI_0(qr)}{I_1(a)}\,[1-qaI_0(qc)\,K_1(qa)]$$

from the vorticity as given by (145). The results are, therefore,

$$\left.\begin{aligned}\zeta &= \sigma cq^2\,I_0(qc)\,\frac{K_0(qr)\,I_1(qa)+I_0(qr)\,K_1(qa)}{I_1(qa)} - \frac{\sigma c}{a}\,\frac{qI_0(qr)}{I_1(qa)} \quad (r>c)\\ &= \sigma cq^2\,I_0(qr)\,\frac{K_0(qc)\,I_1(qa)+I_0(qc)\,K_1(qa)}{I_1(qa)} - \frac{\sigma c}{a}\,\frac{qI_0(qr)}{I_1(qa)} \quad (r<c)\end{aligned}\right\}. \tag{185}$$

$$\left.\begin{aligned}\Gamma &= 2\pi\sigma c\,\left\{1-qr\,\frac{I_0(qc)}{I_1(qa)}\,[I_1(qa)\,K_1(qr)-I_1(qr)\,K_1(qa)]-\frac{r}{a}\,\frac{I_1(qr)}{I_1(qa)}\right\} \quad (r>c)\\ &= 2\pi\sigma c\,\left\{qr\,\frac{I_1(qr)}{I_1(qa)}\,[K_0(qc)\,I_1(qa)+I_0(qc)\,K_1(qa)]-\frac{r}{a}\,\frac{I_1(qr)}{I_1(qa)}\right\} \quad (r<c)\end{aligned}\right\}. \tag{186}$$

Interpretation by the partial fraction rule gives

$$\Gamma = \frac{4\pi\sigma cr}{a^2}\,\Sigma_k\,\frac{J_1(kr)}{kJ_0{}^2(ka)}\,[J_0(kc)-J_0(ka)]\,e^{-\nu k^2 t}, \tag{187}$$

$$\zeta = \frac{2\sigma c}{a^2}\,\Sigma_k\,\frac{J_0(kr)}{J_0{}^2(ka)}\,[J_0(kc)-J_0(ka)]\,e^{-\nu k^2 t}, \tag{188}$$

whether $r \lessgtr c$, where the summation is over the roots of $J_1(ka)=0$. [*Cf.* (9), (26), (163), (166), and (167).] Approximations for small values of νt can be found as before.

13. *Any initial distribution of vorticity with an external boundary.*

If initially $\zeta = f(r)$, and there is a solid boundary at $r = a$, we find the operational form of the solution required by replacing σ by $f(c)\,dc$, and integrating with respect to c from 0 to a, in (185) and (186). The interpretation by the partial fraction rule then gives

$$\Gamma = \frac{4\pi r}{a^2} \sum_k \frac{J_1(kr)}{kJ_0^2(ka)}\, e^{-\nu k^2 t} \int_0^a f(c)\,[J_0(kc) - J_0(ka)]\,c\,dc, \tag{189}$$

and

$$\zeta = \frac{2}{a^2} \sum_k \frac{J_0(kr)}{J_0^2(ka)}\, e^{-\nu k^2 t} \int_0^a f(c)\,[J_0(kc) - J_0(ka)]\,c\,dc. \tag{190}$$

Putting $t = 0$ in the last result suggests that

$$f(r) = \frac{2}{a^2} \sum_k \frac{J_0(kr)}{J_0^2(ka)} \int_0^a f(c)\,[J_0(kc) - J_0(ka)]\,c\,dc. \tag{191}$$

Now putting $t = 0$ in (26) gives

$$\sum_k \frac{J_0(kr)}{J_0(ka)} = -1, \tag{192}$$

and therefore (191) is equivalent to

$$f(r) = \frac{2}{a^2} \int_0^a f(c)\,c\,dc + \frac{2}{a^2} \sum_k \frac{J_0(kr)}{J_0^2(ka)} \int_0^a f(c)\,J_0(kc)\,c\,dc, \tag{193}$$

which is a special case of the Dini expansion of an arbitrary function. (Watson's *Bessel functions*, chap. 18, especially § 18.3 *et seq.*)

14. *Heat conduction in a circular cylinder, with the surface impervious to heat.*

A problem closely related to that considered in the preceding paragraph is the flow of heat in a circular cylinder, with the surface $r = a$ impervious to heat. We begin with the consideration of an instantaneous cylindrical surface source of strength σ per unit length, situated at $r = c (c < a)$. If there were no boundary, the temperature ζ would be given by

$$\zeta = \sigma c q^2 K_0(qr)\, I_0(qc) \quad (r > c) \tag{194}$$

as in (145); this makes

$$\frac{\partial \zeta}{\partial r} = -\sigma c q^3 K_1(qa)\, I_0(qc) \tag{195}$$

at $r = a$. If $\zeta = 0$ initially, and $\partial\zeta/\partial r = H(t)$ at $r = a$, it is easily seen that the solution of the equation of heat conduction is

$$\zeta = I_0(qr)/qI_1(qa). \tag{196}$$

To obtain the solution of our problem, we must therefore subtract

$$-\sigma c q^2 K_1(qa)\, I_0(qc)\, I_0(qr)/I_1(qa) \tag{197}$$

from the solution for no boundary; and this gives

$$\left.\begin{aligned}\zeta &= \sigma c q^2 I_0(qc)[K_0(qr)\, I_1(qa)+I_0(qr)\, K_1(qa)]/I_1(qa) \quad (r>c)\\ &= \sigma c q^2 I_0(qr)[K_0(qc)\, I_1(qa)+I_0(qc)\, K_1(qa)]/I_1(qa) \quad (r<c)\end{aligned}\right\}. \tag{198}$$

The interpretation by the partial fraction rule is

$$\zeta = \frac{2\sigma c}{a^2}\left\{1+\sum_k \frac{J_0(kr)\, J_0(kc)}{J_0{}^2(ka)}\, e^{-\nu k^2 t}\right\}, \tag{199}$$

where the summation is over the roots of $J_1(ka)=0$, and ν is the thermometric conductivity. If $\zeta = f(r)$ at $t=0$, we must write $f(c)\,dc$ for σ in (198), and integrate with respect to c from 0 to a. The interpretation by the partial fraction rule then gives

$$\zeta = \frac{2}{a^2}\int_0^a f(c)\, c\, dc + \frac{2}{a^2}\sum_k \frac{J_0(kr)}{J_0{}^2(ka)}\, e^{-\nu k^2 t}\int_0^a f(c)\, J_0(kc)\, c\, dc. \tag{200}$$

A solution is given by Carslaw to this problem (*The conduction of heat*, 118), which is the same as (200) without the first term. The error arises from assuming a wrong form for the Dini expansion of an arbitrary function.

Putting $t=0$ in (200) gives (193) more simply than before.

We pass over the consideration of the diffusion of vorticity in the enclosed space between two coaxial circular cylinders—a problem of some mathematical interest, but capable of discussion by methods similar to those already given—and proceed to consider finally the diffusion of vorticity in the space outside a given cylinder.

15. *Diffusion of vorticity in the space outside a circular cylinder.*

Suppose, first, that at time $t=0$ there is, at $r=c$, a cylindrical sheet of vorticity of strength σ per unit length, and that at $r=a$, where $a<c$, there is a solid boundary at which the velocity must vanish. In the absence of a boundary, we should have

$$\Gamma = 2\pi\sigma\, c\, qr\, K_0(qc)\, I_1(qr) \quad (r<c) \tag{201}$$

from (147), and the velocity at $r=a$ would be

$$v = \sigma c q\, K_0(qc)\, I_1(qa). \tag{202}$$

For $v = H(t)$ at $r = a$, we know that

$$\Gamma = 2\pi r\, K_1(qr)/K_1(qa), \quad \zeta = -q\, K_0(qr)/K_1(qa). \tag{203}$$

Hence, to find the solution of our problem, we must multiply these last two expressions by the right side of (202), and subtract the products from (147) and (148), and from (145) respectively. In this way we get

$$\left.\begin{aligned}\Gamma &= 2\pi\sigma c\, qrK_0(qc)\,[I_1(qr)\,K_1(qa) - K_1(qr)\,I_1(qa)]/K_1(qa) \qquad (r<c)\\ &= 2\pi\sigma c\{1 - qrK_1(qr)\,[I_0(qc)\,K_1(qa) + K_0(qc)\,I_1(qa)]/K_1(qa)\} \quad (r>c)\end{aligned}\right\} \tag{204}$$

and

$$\left.\begin{aligned}\zeta &= \sigma cq^2 K_0(qc)\,[I_0(qr)\,K_1(qa) + K_0(qr)\,I_1(qa)]/K_1(qa) \quad (r<c)\\ &= \sigma cq^2 K_0(qr)\,[I_0(qc)\,K_1(qa) + K_0(qc)\,I_1(qa)]/K_1(qa) \quad (r>c)\end{aligned}\right\}. \tag{205}$$

Interpretation by Bromwich's integral gives

$$\Gamma = 2\pi\sigma cr \int_0^\infty e^{-\nu t u^2} \times \frac{[J_1(ua)\,Y_0(uc) - Y_1(ua)\,J_0(uc)][J_1(ua)\,Y_1(ur) - Y_1(ua)\,J_1(ur)]}{J_1^2(ua) + Y_1^2(ua)}\, du \tag{206}$$

and

$$\zeta = \sigma c \int_0^\infty e^{-\nu t u^2} \times \frac{[J_1(ua)\,Y_0(uc) - Y_1(ua)\,J_0(uc)][J_1(ua)\,Y_0(ur) - J_0(ur)\,Y_1(ua)]}{J_1^2(ua) + Y_1^2(ua)}\, u\, du. \tag{207}$$

If, when $t = 0$, we have $\zeta = f(r)$, then in (204) and (205) we must replace σ by $f(c)\,dc$, and integrate with respect to c from a to ∞. With suitable restrictions on f, interpretation by Bromwich's integral then gives

$$\Gamma = 2\pi r \int_0^\infty e^{-\nu t u^2} du \int_a^\infty f(c) \times \frac{[J_1(ua)\,Y_0(uc) - Y_1(ua)\,J_0(uc)][J_1(ua)\,Y_1(ur) - Y_1(ua)\,J_1(ur)]}{J_1^2(ua) + Y_1^2(ua)}\, c\, dc, \tag{208}$$

$$\zeta = \int_0^\infty e^{-\nu t u^2} u\, du \int_a^\infty f(c) \times \frac{[J_1(ua)\,Y_0(uc) - Y_1(ua)\,J_0(uc)][J_1(ua)\,Y_0(ur) - J_0(ur)\,Y_1(ua)]}{J_1^2(ua) + Y_1^2(ua)}\, c\, dc. \tag{209}$$

Putting $t = 0$ in (209) gives us the result

$$f(r) = \int_0^\infty u\,du \int_a^\infty f(c) \times \frac{[J_1(ua)\,Y_0(uc) - Y_1(ua)\,J_0(uc)][J_1(ua)\,Y_0(ur) - J_0(ur)\,Y_1(ua)]}{J_1^2(ua) + Y_1^2(ua)}\,c\,dc. \tag{210}$$

Again, putting $t = 0$ in (206) gives

$$\left.\begin{aligned} \int_0^\infty \frac{[J_1(ua)\,Y_0(uc) - Y_1(ua)\,J_0(uc)][J_1(ua)\,Y_1(ur) - Y_1(ua)\,J_1(ur)]}{J_1^2(ua) + Y_1^2(ua)}\,du & \\ = 0 \quad (r < c) & \\ = \frac{1}{r} \quad (r < c) & \end{aligned}\right\}, \tag{211}$$

the integral being convergent. The connection of (211) with (210) is reminiscent of Sonine's proof of Hankel's integral formula. (See Watson's *Bessel functions*, 454.)

16. *The conduction of heat in the space outside a circular cylinder.*

Suppose first that at $r = c$ we have an instantaneous cylindrical surface source of strength σ per unit length, and that at $r = a$ $(a < c)$ the temperature ζ is maintained zero. Without the boundary we should have ζ given by (145), and the value of ζ at $r = a$ would be

$$\sigma c q^2 K_0(qc)\,I_0(qa). \tag{212}$$

If $\zeta = H(t)$ at $r = a$, then $\zeta = K_0(qr)/K_0(qa)$. Hence the solution to our problem is given by subtracting

$$\sigma c q^2 K_0(qc)\,I_0(qa)\,K_0(qr)/K_0(qa) \tag{213}$$

from (145), and is

$$\left.\begin{aligned} \zeta &= \sigma c q^2 K_0(qc)\,[I_0(qr)\,K_0(qa) - I_0(qa)\,K_0(qr)]/K_0(qa) \quad (r < c) \\ &= \sigma c q^2 K_0(qr)\,[I_0(qc)\,K_0(qa) - I_0(qa)\,K_0(qc)]/K_0(qa) \quad (r > c) \end{aligned}\right\}. \tag{214}$$

The interpretation by Bromwich's integral is

$$\zeta = \sigma c \int_0^\infty e^{-\nu t u^2} \times \frac{[J_0(ua)\,Y_0(uc) - J_0(uc)\,Y_0(ua)][J_0(ua)\,Y_0(ur) - J_0(ur)\,Y_0(ua)]}{J_0^2(ua) + Y_0^2(ua)}\,u\,du. \tag{215}$$

If $\zeta = f(r)$ initially, then*

$$\zeta = \int_0^\infty e^{-\nu t u^2} u\,du \int_a^\infty f(c) \times \frac{[J_0(ua)\,Y_0(uc) - J_0(uc)\,Y_0(ua)]\,[J_0(ua)\,Y_0(ur) - J_0(ur)\,Y_0(ua)]}{J_0^2(ua) + Y_0^2(ua)}\,c\,dc \tag{216}$$

and

$$f(r) = \int_0^\infty u\,du \int_a^\infty f(c) \times \frac{[J_0(ua)\,Y_0(uc) - J_0(uc)\,Y_0(ua)]\,[J_0(ua)\,Y_0(ur) - J_0(ur)\,Y_0(ua)]}{J_0^2(ua) + Y_0^2(ua)}\,c\,dc \tag{217}$$

with suitable restrictions†.

If the boundary were impervious to heat, the result would be the same as in the last paragraph, but we can generalize further by assuming radiation at the boundary into a medium at zero temperature. Then the boundary condition is

$$-\frac{\partial \zeta}{\partial r} + h\zeta = 0 \tag{218}$$

at $r = a$, where h is the relative emissivity and is positive. Now if ζ were given by (145), then at $r = a$ we should have

$$-\frac{\partial \zeta}{\partial r} + h\zeta = +\sigma c q^2 K_0(qc)\,[h\,I_0(qa) - q\,I_1(qa)]. \tag{219}$$

If $-\partial\zeta/\partial r + h\zeta = H(t)$ at $r = a$, then, for the space outside,

$$\zeta = K_0(qr)/\{q\,K_1(qa) + h\,K_0(qa)\}. \tag{220}$$

Hence for an instantaneous cylindrical surface source of strength σ per unit length, at $t = 0$ and at $r = c$, the solution to our problem is found by subtracting the products of the right sides of (219) and (220) from (145), and is

$$\left.\begin{aligned} \zeta &= \frac{\sigma c q^2 K_0(qc)}{qK_1(qa) + hK_0(qa)}\,[q\{I_0(qr)\,K_1(qa) + K_0(qr)\,I_1(qa)\} \\ &\qquad + h\{I_0(qr)\,K_0(qa) - K_0(qr)\,I_0(qa)\}] \quad (r < c) \\ &= \frac{\sigma c q^2 K_0(qr)}{qK_1(qa) + hK_0(qa)}\,[q\{I_0(qc)\,K_1(qa) + K_0(qc)\,I_1(qa)\} \\ &\qquad + h\{I_0(qc)\,K_0(qa) - K_0(qc)\,I_0(qa)\}] \quad (r > c) \end{aligned}\right\}. \tag{221}$$

* This result can be deduced from the formula given by Carslaw, *The conduction of heat*, 196, for the Green's function for the space outside an infinite cylinder.

† This is a special case of an integral due to Weber, *Math. Annalen*, 6 (1873), 146–154, and rediscovered by Orr, *Proc. Royal Irish Acad.*, 27 A (1909), 222–225. See also Titchmarsh, *Proc. London Math. Soc.* (2), 22 (1923), 15–28.

If we write

$$C_0(z) = Y_0(z)\,[u\,J_1(ua)+hJ_0(ua)]-J_0(z)\,[u\,Y_1(ua)+h\,Y_0(ua)], \quad (222)$$

the interpretation by Bromwich's integral is

$$\zeta = \sigma c \int_0^\infty e^{-\nu t u^2} \frac{C_0(uc)\,C_0(ur)}{[uJ_1(ua)+hJ_0(ua)]^2+[u\,Y_1(ua)+h\,Y_0(ua)]^2}\,u\,du. \quad (223)$$

Then, if $\zeta = f(r)$ initially,

$$\zeta = \int_0^\infty e^{-\nu t u^2}\,u\,du \int_a^\infty f(c)\,\frac{C_0(uc)\,C_0(ur)}{[uJ_1(ua)+hJ_0(ua)]^2+[u\,Y_1(ua)+h\,Y_0(ua)]^2}\,c\,dc, \quad (224)$$

with suitable restrictions, and

$$f(r) = \int_0^\infty u\,du \int_a^\infty f(c)\,\frac{C_0(uc)\,C_0(ur)}{[uJ_1(ua)+hJ_0(ua)]^2+[u\,Y_1(ua)+h\,Y_0(ua)]^2}\,c\,dc. \quad (225)$$

I know of no adequate independent discussion of this formula and its generalization to cylinder functions of any order.

17. *The solution of the equation of heat conduction for the space outside a circular cylinder, when the surface temperature is* $\cos at$ *or* $\sin at$ *for* $t > 0$.

a is an arbitrary parameter; the surface temperature (at $r = a$) is zero for $t < 0$; the initial temperature is zero for $r > a$. If, for $t > 0$, the surface temperature is

$$\cos at = \frac{p^2}{p^2+a^2} \quad \text{or} \quad \sin at = \frac{ap}{p^2+a^2} \quad (226)$$

the operational solution is

$$\zeta = \frac{p^2}{p^2+a^2}\,\frac{K_0(qr)}{K_0(qa)} \quad \text{or} \quad \frac{ap}{p^2+a^2}\,\frac{K_0(qr)}{K_0(qa)}, \quad (227)$$

respectively. The interpretations are

$$\zeta = \cos at\,\frac{\ker r\,\sqrt{(a/\nu)}\,\ker a\,\sqrt{(a/\nu)}+\mathrm{kei}\,r\,\sqrt{(a/\nu)}\,\mathrm{kei}\,a\,\sqrt{(a/\nu)}}{\ker^2 a\,\sqrt{(a/\nu)}+\mathrm{kei}^2 a\,\sqrt{(a/\nu)}}$$

$$+\sin at\,\frac{\ker r\,\sqrt{(a/\nu)}\,\mathrm{kei}\,a\,\sqrt{(a/\nu)}-\mathrm{kei}\,r\,\sqrt{(a/\nu)}\,\ker a\,\sqrt{(a/\nu)}}{\ker^2 a\,\sqrt{(a/\nu)}+\mathrm{kei}^2 a\,\sqrt{(a/\nu)}}$$

$$+\frac{2\nu^2}{\pi}\int_0^\infty e^{-\nu u^2 t}\,\frac{u^3}{a^2+\nu^2 u^4}\,\frac{J_0(ur)\,Y_0(ua)-J_0(ua)\,Y_0(ur)}{J_0^2(ua)+Y_0^2(ua)}\,du, \quad (228)$$

or

$$\zeta = \sin at \frac{\ker r\sqrt{(a/\nu)}\,\ker a\sqrt{(a/\nu)}+\mathrm{kei}\, r\sqrt{(a/\nu)}\,\mathrm{kei}\, a\sqrt{(a/\nu)}}{\ker^2 a\sqrt{(a/\nu)}+\mathrm{kei}^2 a\sqrt{(a/\nu)}}$$

$$-\cos at \frac{\ker r\sqrt{(a/\nu)}\,\mathrm{kei}\, a\sqrt{(a/\nu)}-\mathrm{kei}\, r\sqrt{(a/\nu)}\,\ker a\sqrt{(a/\nu)}}{\ker^2 a\sqrt{(a/\nu)}+\mathrm{kei}^2 a\sqrt{(a/\nu)}}$$

$$-\frac{2a\nu}{\pi}\int_0^\infty e^{-\nu u^2 t}\frac{u}{a^2+\nu^2 u^4}\frac{J_0(ur)\,Y_0(ua)-J_0(ua)\,Y_0(ur)}{J_0^2(ua)+Y_0^2(ua)}\,du \quad (229)$$

respectively, where

$$K_0(e^{\frac{1}{4}\pi i}z) = \ker z + i\,\mathrm{kei}\, z. \quad (230)$$

(Watson's *Bessel functions*, 81.)

With $\nu = 1$, it may be deduced that

$$-\frac{2}{\pi}\int_0^\infty \frac{u^3}{a^2+u^4}\frac{J_0(ur)\,Y_0(ua)-J_0(ua)\,Y_0(ur)}{J_0^2(ua)+Y_0^2(ua)}$$

$$= \frac{\ker r\sqrt{a}\,\ker a\sqrt{a}+\mathrm{kei}\, r\sqrt{a}\,\mathrm{kei}\, a\sqrt{a}}{\ker^2 a\sqrt{a}+\mathrm{kei}^2 a\sqrt{a}} \quad (231)$$

and

$$-\frac{2}{\pi}\int_0^\infty \frac{u}{a^2+u^4}\frac{J_0(ur)\,Y_0(ua)-J_0(ua)\,Y_0(ur)}{J_0^2(ua)+Y_0^2(ua)}$$

$$= \frac{\ker r\sqrt{a}\,\mathrm{kei}\, a\sqrt{a}-\mathrm{kei}\, r\sqrt{a}\,\ker a\sqrt{a}}{\ker^2 a\sqrt{a}+\mathrm{kei}^2 a\sqrt{a}} \quad (232)$$

or, in one equation,

$$-\frac{2}{\pi}\int_0^\infty \frac{u}{u^2-ia}\frac{J_0(ur)\,Y_0(ua)-J_0(ua)\,Y_0(ur)}{J_0^2(ua)+Y_0^2(ua)}\,du = \frac{K_0(e^{\frac{1}{4}\pi i}r\sqrt{a})}{K_0(e^{\frac{1}{4}\pi i}a\sqrt{a})} \quad (r>a), \quad (233)$$

a result easily generalized by considering

$$\int \frac{1}{\lambda-ia}\frac{K_n(\lambda^{\frac{1}{2}}r)}{K_n(\lambda^{\frac{1}{2}}a)}\,d\lambda \quad (234)$$

round an appropriate contour.

A SPECIAL COLLINEATION IN n-SPACE, WITH PARTICULAR REFERENCE TO $n = 6$.

By W. Saddler.

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The properties of the general collineation have been very fully developed from various aspects: its algebraical details from the invariant point of view are well summarised by E. Study*. It is intended to consider here a collineation built up from two dual linear complexes and to exhibit its connection, in respect to its covariants and invariants, with the general case; the algebra involved is necessarily more complex since "compound symbols" are used. The characteristic equation of the matrix of the transformation has all its factors but one, squares, and thus the collineation will have double lines and "double spaces" instead of the normal fixed points. The equations of such double lines will be found in the case of $n = 6$. A corresponding case has been worked out by R. Weitzenböck† for $n = 4$ by the use of "complex symbols", but the parallelism with the general theory is thereby lost. The algebra involves applications of Sylvester's *Identity in determinant theory*.

1. A brief summary of the general form in n-space will be given to simplify the discussion.

Represent the collineation by T, where

$$T = a_x u_\alpha = b_x u_\beta = \ldots = 0,$$

and where

$$a_x = a_1 x_1 + a_2 x_2 + \ldots + a_{n+1} x_{n+1},$$

$$u_\alpha = u_1 \alpha_1 + u_2 \alpha_2 + \ldots + u_{n+1} \alpha_{n+1}.$$

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* *Einleitung in die Theorie der invarianten linearer Transformationen* (1923), 170–207.

† R. Weitzenböck, *Rend. di Palermo*, 31 (1911), 301.

Then

$$T^0 = E = u_x,$$
$$T^2 = a_x b_a u_\beta,$$
$$T^3 = a_x b_a c_\beta u_\gamma,$$
$$\ldots \quad \ldots \quad \ldots$$
$$T^{n+1} - J_1 T^n + J_2 T^{n-1} \ldots + (-1)^{n+1} J_{n+1} T^0 = 0.$$

The J's are the invariants of the form and are such that

$$J_{n+1} = \frac{1}{(n+1)!} (abc \ldots k_{n+1})(\alpha\beta \ldots),$$

$$J_n = \frac{1}{n!} (abcd \ldots l_n | \alpha\beta\gamma \ldots),$$

where each bracket of J_{n+1} contains $n+1$ symbols and each part of J_n contains n symbols; also

$$(abcd \ldots | \alpha\beta\gamma\delta \ldots)$$

is the determinant

$$\begin{vmatrix} a_\alpha & a_\beta & a_\gamma & a_\delta & \ldots \\ b_\alpha & b_\beta & b_\gamma & b_\delta & \ldots \\ \ldots & \ldots & & \ldots & \ldots \\ \ldots & \ldots & & \ldots & \ldots \end{vmatrix}.$$

The characteristic determinant of the matrix of T is given by

$$|\lambda E - T| = \lambda^{n+1} - J_1 \lambda^n + \ldots (-1)^{n+1} J_{n+1}.$$

Defining E_1, E_2, ..., by

$$E_1 = \frac{(T-\lambda_2 E)(T-\lambda_3 E) \ldots (T-\lambda_{n+1} E)}{(\lambda_1-\lambda_2)(\lambda_1-\lambda_3) \ldots (\lambda_1-\lambda_{n+1})},$$

the canonical forms for the various powers of T are

$$T^0 = E = E_1 + E_2 + \ldots + E_{n+1},$$
$$T = \lambda_1 E_1 + \lambda_2 E_2 + \ldots + \lambda_{n+1} E_{n+1},$$
$$T^2 = \lambda_1^2 E_1 + \lambda_2^2 E_2 + \ldots + \lambda_{n+1}^2 E_{n+1},$$
$$\ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots$$

the λ's being the roots of the characteristic equation. The covariants of T, which may be expressed in terms of the powers of T, are given by forms of the type

$$(abcu | \alpha\beta\gamma x).$$

The canonical expression for T is

$$T = \lambda_1 X_1 U_1 + \lambda_2 X_2 U_2 + \ldots + \lambda_{n+1} X_{n+1} U_{n+1}.$$

2. Consider a linear complex in n-space as typified* by

$$(aa' \,|\, xy) = (bb' \,|\, xy) = \ldots = a_x a_y' - a_x' a_y = 0, \tag{i}$$

where $(aa')_{ij}$ and $(bb')_{ij}$ are equivalent symbols, and x, y are cogredient point variables.

The complex in prime coordinates may be written as $(aa' uvw \ldots) = 0$. The number of symbols in the bracket being $n+1$, of which $u, v, w, \ldots$ are primes, dual to points x.

Similarly, a space complex is

$$(\alpha\alpha' \,|\, uv) = (\beta\beta' \,|\, uv) = \ldots = (u_\alpha v_{\alpha'} - u_{\alpha'} v_\alpha) = 0. \tag{ii}$$

Now the null prime of the point y in (i) is

$$a_x a_y' - a_x' a_y = 0,$$

with coordinates

$$a_i a_y' - a_i' a_y \quad (i = 1, 2, \ldots, n+1), \tag{iii}$$

and the null point of the prime v in (ii) is

$$u_\alpha v_{\alpha'} - u_{\alpha'} v_\alpha = 0,$$

with coordinates

$$\alpha_i v_{\alpha'} - \alpha_{i'} v_\alpha. \tag{iv}$$

The focal point of the complex (i) is the point with coordinates S_i, where

$$(aa' bb' \ldots u) = u_S = 0, \tag{v}$$

and the focal prime of (ii) is S_i', or

$$(\alpha\alpha' \beta\beta' \ldots x) = S_x' = 0. \tag{vi}$$

There exists the identity

$$\dot{a}_x \dot{a}_S' \equiv 0 = a_x a_S' - a_x' a_S, \tag{vii}$$

where the dot notation represents a determinantal permutation†. This result follows immediately by interchange of equivalent symbols and an identity of the type

$$(abcd \ldots) p_x = \Sigma (pbcd \ldots) a_x.$$

* Turnbull, *The theory of determinants, matrices, and invariants* (1928), 86.

† Turnbull, *loc. cit.*

The null prime of any point y thus goes through S, while S itself has an indeterminate null prime.

Dually, for the second complex,

$$(aa'|uS') \equiv 0, \quad \text{for all } u,$$

and thus the null points of any prime v lie in S', while the null point of S' is indeterminate.

3. Consider now the null prime of a point y in the first complex to be incident with the null point of a prime v in the second complex.

These are given by

$$a_x a_y' - a_x' a_y = 0,$$

$$u_\alpha v_{\alpha'} - u_{\alpha'} v_\alpha = 0,$$

so that the incidence condition is

$$(aa'|\alpha' y) v_\alpha - (aa'|\alpha y) v_{\alpha'} = 0. \tag{i}$$

There is thus a collineation of the form $p_y v_\pi = 0$ connecting y and v; this turns out to be by no means general, as its properties now discussed will show.

Represent it by T and take x and u as the variables; thus

$$T = (aa'|\dot{\alpha}' x) u_{\dot{\alpha}'}. \tag{ii}$$

Identities useful in reducing or simplifying the algebra are typified by

$$(abc|xyz) = \dot{a}_x \dot{b}_y \dot{c}_z = (ab|yz) c_x + (ab|zx) c_y + (ab|xy) c_z, \tag{iii}$$

with a corresponding dual expression, and

$$\sum_{q=1} (-1)^q (a_1 \dots a_i \dot{b}_1 \dots \dot{b}_q | x_1 \dots x_i \bar{y}_1 \dots \bar{y}_q) \times (\dot{b}_{q+1} \dots \dot{b}_i | \bar{y}_{q+1} \dots \bar{y}_i)$$
$$= (a_1 \dots a_i | y_1 \dots y_i)(b_1 \dots b_i | x_1 \dots x_i) - (a_1 \dots a_i | x_1 \dots x_i)(b_1 \dots b_i | y_1 \dots y_i).$$

This is Sylvester's Identity in its most general form, the dot and bar denoting determinantal permutations independently. An example useful in the sequence may with advantage be quoted.

$$\begin{aligned}
&(aa'\,\beta\beta'\,y \;\; aa'\,bb'\,u) \times (\gamma\gamma'\,\delta\delta'\,z|cc'\,dd'\,v) \\
&-(aa'\,\beta\beta'\,y|cc'\,dd'\,v) \times (\gamma\gamma'\,\delta\delta'\,z|aa'\,bb'\,u) \\
&= -(aa'\,\beta\beta'\,y\bar{\gamma}|cc'\,dd'\,v\dot{a}) \times (\bar{\gamma}'\,\bar{\delta}\bar{\delta}'\,\bar{z}|\dot{a}'\,\dot{b}\dot{b}'\,\dot{u}) \\
&+(aa'\,\beta\beta'\,y\bar{\gamma}\bar{\gamma}'|cc'\,dd'\,v\dot{a}\dot{a}') \times (\bar{\delta}\bar{\delta}'\,\bar{z} \;\; \dot{b}\dot{b}'\,\dot{u}),
\end{aligned} \tag{iv}$$

other terms vanishing in virtue of § 2 (vii).

Now consider the collineation T; in virtue of § 2 (i),

$$\begin{aligned} T(x) &= (aa'u\,|\,aa'x) - (aa'\,|\,aa')u_x \\ &= T_1 \qquad\qquad -A_1 T_0, \end{aligned} \tag{v}$$

where T_1 is a typical covariant and A_1 an invariant.

Thus the point x and its transform by T and by T_1 are collinear.

$$\begin{aligned} T^2(x) &= (aa'\,|\dot{a}'x)(bb'\,|\beta'\dot{a})\,u_\beta - (aa'\,|\dot{a}'x)(bb'\,|\beta\dot{a})\,u_{\beta'} \\ &= -(aa'\,|\,\dot{a}'x)\,u_a(bb'\,|\beta\beta') + (aa'\,|\,\dot{a}'x)(bb'u\,|\beta\beta'\dot{a}) \\ &= -T(x)\,A_1 + a_{a'}a_{x'}(bb'u\,|\beta\beta'\dot{a}) - a_x a'_{\dot{a}'}(bb'u\,|\beta\beta'\dot{a}). \end{aligned}$$

By a second use of the identity (iii) and the interchange of equivalent symbols, this leads to

$$T^2(x) = -T(x)\,A_1 - \frac{1}{2^2}(aa'\,bb'\;\; aa'\,\beta\beta')\,u_x + \frac{1}{2^2}(aa'\,bb'u\,|\,aa'\,\beta\beta'x)$$

$$= -TA_1 - \frac{1}{2^2}T_0 A_2 + \frac{1}{2^2}T_2, \tag{vi}$$

where $$A_2 = (aa'\,bb'\,|\,aa'\,\beta\beta');$$

$$T_2 = (aa'\,bb'u\,|\,aa'\,\beta\beta'x) = -T_1 A_1 + \left(A_1^2 - \frac{A_2}{2^2}\right)T_0 + \frac{1}{2^2}T_2.$$

By a similar process we obtain a series of invariants of the type

$$A_3 = (aa'\,bb'\,cc'\,|\,aa'\,\beta\beta'\,\gamma\gamma'),$$

$$T_3 = (aa'\,bb'\,cc'u\,|\,aa'\,\beta\beta'\,\gamma\gamma'x).$$

These forms are all in agreement with the type of concomitants of the general collineation in § 1.

It is not difficult to show that

$$T\,T_2 = -\frac{1}{3^2}A_3\,T_0 + \frac{1}{3^2}T_3,$$

$$T\,T_n = -\frac{1}{n^2}A_n\,T_0 + \frac{1}{n^2}T_n, \tag{vii}$$

which gives the rule for obtaining successive values of the operation T:

$$T^3 = -T^2 A_1 - \frac{1}{2^2}TA_2 - \frac{1}{2^2\,3^2}T_0 A_3 + \frac{1}{2^2\,3^2}T_3, \tag{viii}$$

$$T^{r+1} = -T^r A_1 - \frac{1}{(2!)^2}T^{r-1}A_2 - \frac{1}{(3!)^2}T^{r-2}A_3 \ldots + \frac{1}{(r!)^2}T_{r+1}.$$

Consider the simpler case of six dimensions—then the possible number of symbols in a bracket factor is seven.

Thus
$$T_3 = (a\alpha'\,\beta\beta'\,\gamma\gamma'\,x)(a\alpha'\,bb'\,cc'\,u)$$
$$= S_x'\,u_S,$$
$$T\,T_3(x) = (aa'\,|\,\dot{a}'\,x)\,(\beta\beta'\,\gamma\gamma'\,\delta\delta'\,\dot{a})\,.\,(aa'\,bb'\,cc'\,u)$$
$$= a_{\dot{a}'}(\beta\beta'\,\gamma\gamma'\,\delta\delta'\,\dot{a})\,a_x'\,.\,(aa'\,bb'\,cc'\,u)$$
$$-a_{\dot{a}}'(\beta\beta'\,\gamma\gamma'\,\delta\delta'\,\dot{a})\,a_x\,.\,(aa'\,bb'\,cc'\,u)$$
$$\equiv 0 \qquad \text{(ix)}$$
$$= (aa'\,|\,\dot{a}S')\,a_x'\,.\,u_S,$$

or
$$(aa'\,|\,\dot{a}S')\,a_x' \equiv 0\,;$$

similarly,
$$(aa'\,|\,\dot{a}'\,S)\,u_{\dot{a}} \equiv 0.$$

Thus
$$T(S) \equiv 0,$$

or S is a singular point of the transformation and each point $T(y)$ lies in S'; S' is invariant with respect to the collineation.

When the invariant A_3 vanishes, the projection of the point x by S on to the region S' is given by the covariant $T_3 = 0$.

Again, since
$$TT_3 = 0, \quad \text{from (ix)},$$
$$T^4 = -T^3 A_1 - \frac{1}{(2!)^2}\,T^2 A_2 - \frac{1}{(3!)^2}\,TA_3,$$

or
$$T\left(T^3 + T^2 A_1 + \frac{1}{2^2}\,TA_2 + \frac{1}{2^2 3^2}\,T^0 A_3\right) = 0. \qquad \text{(x)}$$

But, since T, T^2, T^3 can all be expressed in terms of T_0, T_1, T_2, T_3, we can express all powers of the form T in terms of the four fundamental types; for instance, if

$$T^n = p_n T_0 + q_n T_1 + r_n T_2 + s_n T_3,$$
$$T^{n+1} = p_{n+1} T_0 + q_{n+1} T_1 + r_{n+1} T_2 + s_{n+1} T_3$$
$$= TT^n$$
$$= p_n T + q_n TT_1 + r_n TT_2 + s_n TT_3$$
$$= p_n(T_1 - AT_0) + q_n\left(-\frac{1}{2^2}\,A_2 T_0 + \frac{1}{2^2}\,T_2\right) + r_n\left(-\frac{1}{3^2}\,A_3 T_0 + \frac{1}{3^2}\,T_3\right).$$

Hence
$$p_{n+1} = -A_1 p_n - \frac{1}{2^2} A_2 q_n - \frac{1}{3^2} A_3 r_n,$$
$$q_{n+1} = p_n,$$
$$r_{n+1} = \frac{1}{2^2} r_n,$$
$$s_{n+1} = \frac{1}{3^2} r_n,$$
or
$$p_{n+1} = -A_1 p_n - \frac{1}{2^2} A_2 p_{n-1} - \frac{1}{2^2 3^2} A_3 p_{n-2}.$$

This gives the law of recurrence for the difference equation and enables one to calculate the coefficients in the powers of T in terms of the invariants.

A similar method holds for $2n$ dimensions.

Thus, since
$$T T_n = 0,$$
$$T^r = p_r T_0 + q_r T_1 + \ldots + t_r T_n \quad (r > n),$$
we find that
$$p_{r+1} = -A_1 p_r - \frac{1}{2^2} A_2 p_{r-1} \ldots - \frac{1}{(n!)^2} A_n p_{r+1-n},$$
with corresponding equations for the other coefficients.

Consider now the characteristic equation of the form T in space of six dimensions; we have observed that, for
$$T = a_x u_\alpha = b_x u_\beta = \ldots,$$
the *characteristic equation* has the form
$$|T - \lambda E| = 0,$$
or
$$T^7 - \frac{J_1}{1!} T^6 + \frac{J_2}{2!} T^5 \ldots + \frac{J_7}{7!} T^0 = 0,$$
where
$$J_1 = a_\alpha, \quad J_2 = (ab|\alpha\beta), \quad J_3 = (abc|\alpha\beta\gamma), \quad \ldots,$$
and the law of recurrence connecting these with the invariants of T, T^2, T^3, ..., namely
$$P_1 = a_\alpha, \quad P_2 = a_\beta b_\alpha, \quad P_3 = a_\beta b_\gamma c_\alpha, \quad \ldots$$
is
$$J_7 = J_6 P_1 - 6 J_5 P_2 + 6 \cdot 5 J_4 P_3 \ldots + 6! P_7.$$

Now, in our case, the invariants P_1, P_2, P_3, ..., are found to be

$$-2A_1, \quad 2A_1^2-A_2, \quad -2A_1^3+\frac{3}{2}A_1A_2-\frac{1}{6}A_3,$$

$$2A_1^4-2A_1^2A_2+\frac{2}{9}A_1A_3-\frac{1}{4}A_2^2,$$

$$-2A_1^5+\frac{5}{2}A_1^3A_2-\frac{5}{18}A_1^2A_3-\frac{5}{8}A_1A_2^2+\frac{5}{72}A_2A_3,$$

$$2A_1^6-3A_1^4A_2+\frac{9}{8}A_1^2A_2^2-\frac{1}{6}A_1A_2A_3-\frac{1}{16}A_2^3+\frac{1}{3}A_1^3A_3+\frac{1}{216}A_3^2,$$

$$-2A_1^7+\frac{7}{2}A_1^5A_2-\frac{7}{4}A_1^3A_2^2+\frac{7}{24}A_1^2A_2A_3+\frac{7}{32}A_1A_2^2$$

$$-\frac{7}{18}A_1^4A_3-\frac{7}{18.36}A_1A_3^2-\frac{7}{8.36}A_2^2A_3.$$

Hence our invariants J_1, J_2, ... are found to be

$$-2A_1, \quad 2A_1^2+A_2, \quad -3A_1A_2-\tfrac{1}{3}A_3, \quad \tfrac{3}{2}A_2^2+\tfrac{4}{3}A_1A_3, \quad -\tfrac{5}{3}A_2A_3, \quad \tfrac{5}{9}A_3^2, \quad 0.$$

Note that in this case the J_{n+1} (*i.e.* J_7) vanishes.

Substituting these values, the characteristic equation satisfied by T is found to be

$$T\left(T^3+T^2A_1+\frac{1}{2^2}TA_2+\frac{1}{2^2\,3^2}T^0A_3\right)^2=0. \qquad \text{(xi)}$$

But we have already established (x), namely

$$T\left(T^3+T^2A_1+\frac{1}{2^2}TA_2+\frac{1}{2^2\,3^2}T^0A_3\right)=0.$$

Thus the characteristic equation of our form T satisfies the two equations

$$T(T-\lambda_1)^2(T-\lambda_2)^2(T-\lambda_3)^2=0 \quad \text{and} \quad T(T-\lambda_1)(T-\lambda_2)(T-\lambda_3)=0.$$

Hence, geometrically, the transformation has a singular point corresponding to the root $\lambda=0$, which is the focal point S, and dually has the focal prime S', in addition to three double lines, instead of the usual fixed points of the general collineation; all these double lines lie in the prime S' and dually the "double spaces" contain the point S. The extension to the case of $2n$ dimensions is immediate.

Before proceeding to find the actual equation of these double lines, certain remarks on the concomitants T_1, T_2, ... will be made. First, in connection with the invariants, consider T_r, where

$$T_r = (\alpha\alpha'\beta\beta'\ldots x|\alpha\alpha' bb'\ldots u),$$

there being r symbols $\alpha\alpha'$ convolved.

It is known that the invariant a_α of $a_x u_\alpha$ is equal to

$$\Sigma \frac{\partial^2}{\partial x_i \partial u_i}(a_x u_\alpha) \quad (i = 1, \ldots, n);$$

hence, expanding T_r as

$$u_x A_r + \Sigma \dot{a}_x(\alpha\alpha'\beta\beta'\ldots|\dot{a}'\dot{b}\dot{b}'\ldots u),$$

and using the differential operator, we find that the invariant is

$$(n+1-r)A_r.$$

When $r = n$, this simply gives A_n.

Thus the invariant of T_1, in space of 6, where

$$T_1 = T + A_1 u_x,$$

is

$$-2A_1 + 7A_1 = 5A_1.$$

Secondly, the characteristic equation for T_1 can be written down from that of T, which is

$$T\left(T^3 + A_1 T^2 + \frac{A_2}{2^2} T + \frac{A_3}{2^2 3^2} T^0\right)^2 = 0,$$

by merely transforming this by means of the substitution

$$T = T_1 - A T^0,$$

viz.

$$(T_1 - A_1 T^0)\left\{T_1^3 - 2T_1^2 A_1 + T_1\left(A_1^2 + \frac{A_2}{2^2}\right) - \left(\frac{A_1 A_2}{2^2} - \frac{A_3}{36}\right) T_0\right\}^2 = 0.$$

Hence T_1 will still have fixed double lines but no longer a singular point. It will also leave the point S and the prime S' invariant; similarly for T_2 and T_3, since each of these is compounded of parts which leave S and S' invariant.

4. In the general collineation (*cf.* § 1), the canonical form for T in $\{n\}$ has the form

$$T = \lambda_1 E_1 + \lambda_2 E_2 + \ldots + \lambda_{n+1} E_{n+1}$$
$$= \lambda_1 X_1 U_1 + \ldots + \lambda_{n+1} X_{n+1} U_{n+1},$$

where E_1, etc., are irrational covariants of the form and may be written as

$$E_1 = P_x U_\pi,$$

where $P_x = 0$ is now a real prime and $U_\pi = 0$ a point. But, when the characteristic equation satisfies § 3 (x), (xi), the canonical form for T in space $\{6\}$ is

$$T = \lambda_1 E_1' + \lambda_3 E_3' + \lambda_5 E_5', \tag{1}$$

$$T^0 = E_1' + E_3' + E_5',$$

where

$$E_1' = \frac{T^3 - T^2(\lambda_3+\lambda_5) + T\lambda_3\lambda_5}{\lambda_1(\lambda_1-\lambda_3)(\lambda_1-\lambda_5)},$$

and, since [§ 3 (viii)]

$$T^3 + T^2 A_1 + T\tfrac{1}{2}A_2 + T^0 \tfrac{1}{36} A_3 - \tfrac{1}{36} T_3 = 0,$$

E_1', save for a numerical factor, reduces to

$$T\lambda_1^2 - T^0(A_2\tfrac{1}{4}\lambda_1 + \tfrac{1}{36}A_3) + T_2\tfrac{1}{4}\lambda_1 + \tfrac{1}{36}T_3. \tag{2}$$

The problem thus reduces to getting this into a factorisable form*

$$E_1' = P_x u_A + Q_x u_B, \tag{3}$$

where the double line is the join of the points AB and the "double space" the intersection of $P_x = 0$ and $Q_x = 0$.

Let

$$E_1' = R_x u_C.$$

Now the line of intersection of the two primes

$$R_y C_i, \quad R_z' C_i' \tag{4}$$

will be

$$\tfrac{1}{2}(RR'|yz)(CC'|uv) = 0, \tag{5}$$

which, when applied to

$$P_x u_A + Q_n u_B,$$

gives

$$(AB|xy)(PQ|uv) = 0.$$

Thus we must take E_1' in the form [§ 4 (2)] and try by means of (4) and (5) to split it up into the two factors required. In an analogous manner to (4), we thus, from

$$T\lambda_1^2 - T^0(A_2\tfrac{1}{4}\lambda_1 + \tfrac{1}{36}A_3) + T_2\tfrac{1}{4}\lambda_1 + \tfrac{1}{36}T_3 = 0,$$

* Study, *loc. cit.*

form the equation

$$\begin{aligned}
&\overset{\beta\beta'}{\underset{\alpha\alpha'}{\Omega}}\lambda_1^4(aa'|\alpha' y)(bb'|\beta' z)(\alpha\beta|uv)\\
&\quad-\lambda_1^2(A_2\tfrac{1}{4}\lambda_1+\tfrac{1}{36}A_3)\overset{yz}{\underset{\alpha\alpha'}{\Omega}}(aa'|\alpha' y)(\alpha z|uv)\\
&\quad+\tfrac{1}{4}\lambda_1^3\Big[A_2\overset{yz}{\underset{\alpha\alpha'}{\Omega}}(aa'|\alpha' y)(\alpha z|uv)-\overset{\gamma'\beta\beta'\gamma}{\underset{\alpha\alpha'}{\Omega}}(aa'|\alpha' y)(\alpha\gamma'|uv)(\beta\beta'\gamma z|bb'cc')\\
&\qquad\qquad+\overset{\gamma'\beta\beta'\gamma}{\underset{\alpha\alpha'}{\Omega}}(aa'|\alpha' z)(\alpha\gamma'|uv)(\beta\beta'\gamma y|bb'cc')\Big]\\
&\quad+\tfrac{1}{36}\lambda_1^2\overset{yz}{\underset{\alpha\alpha'}{\Omega}}(aa'|\alpha' y)(\alpha\pi|uv)p_z+(A_2\tfrac{1}{4}\lambda_1+\tfrac{1}{36}A_3)^2(yz|uv)\\
&\quad-2(A_2\tfrac{1}{4}\lambda_1+\tfrac{1}{36}A_3)\lambda_1\tfrac{1}{4}A_2(yz|uv)\\
&\qquad\qquad+\{\tfrac{1}{4}(A_2\lambda_1)+\tfrac{1}{36}A_3\}\tfrac{1}{4}\lambda_1\overset{\alpha\alpha'\beta\beta'}{\underset{yz}{\Omega}}(y\beta'|uv)(\alpha\alpha'\beta z|aa'bb')\\
&\quad-\tfrac{1}{36}(A_2\tfrac{1}{4}\lambda_1+\tfrac{1}{36}A_3)\underset{yz}{\Omega}(y\pi|uv)p_z\\
&\quad+\tfrac{1}{16}\lambda_1^2\Big\{A_2^2(yz|uv)-A_2\overset{\alpha\alpha'\beta\beta'}{\underset{yz}{\Omega}}(y\beta'|uv)(\alpha\alpha'\beta z|aa'bb')\\
&\qquad\qquad+\overset{\gamma\gamma'\delta\delta'}{\underset{\alpha\alpha'\beta\beta'}{\Omega}}(\beta'\delta'|uv)(\alpha\alpha'\beta y|aa'bb')(\gamma\gamma'\delta z|cc'dd')\Big\}\\
&\quad+\tfrac{1}{144}\lambda_1\Big\{A_2\underset{yz}{\Omega}(y\pi|uv)p_z-\overset{\alpha\alpha'\beta\beta'}{\underset{yz}{\Omega}}(\beta'\pi|uv)(\alpha\alpha'\beta y|aa'bb')p_z\Big\}. \qquad (6)
\end{aligned}$$

In this $\overset{\alpha\alpha'\beta\beta'}{\underset{yz}{\Omega}}$ means a determinantal permutation of y and z and an independent one in the bracket factors for $\alpha\alpha'\beta\beta'$; eight terms in all.

$$T_3=(aa'bb'cc'u)(\alpha\alpha'\beta\beta'\gamma\gamma'x)=p_x u_\pi.$$

To make this quite clear, we shall consider briefly the part in the equation obtained from the combination of T with T_2.

Replace x by y in T and x by z in T_2, obtaining, as coefficients of u,

$$(aa'|\dot\alpha' y)\dot\alpha_i,\quad z_i(aa'bb'|\alpha\alpha'\beta\beta')-\dot\beta_i'(aa'bb'|\dot\alpha\alpha'\dot\beta z);$$

combine these and obtain

$$(aa'|\dot\alpha' y)(\dot\alpha z|uv)A_2-\overset{\gamma'\beta\beta'\gamma}{\underset{\alpha\alpha'}{\Omega}}(aa'|\alpha' y)(\alpha\gamma'|uv)(bb'cc'|\beta\beta'\gamma z).$$

In addition to this is the term arising from the interchange of y and z in T and T_2; thus we get the term in the expression whose coefficient in (6) is $\frac{1}{4}\lambda_1^3$.

Now by use of the fundamental identities of § 2 (iii) and (iv) we have the following relations, which will simplify (6):

$$(1)\quad \overset{\beta\beta'}{\underset{\alpha\alpha'}{\Omega}}(aa'|a'y)(bb'|\beta'z)(\alpha\beta|uv)$$
$$= A_1(\alpha a'|uv)(aa'|yz) - \tfrac{1}{2}(aa'uv|\alpha a'\beta\beta').(aa'|yz)$$
$$+\tfrac{1}{4}A_2(uv|yz) + \tfrac{1}{4}\overset{yz}{\underset{uv}{\Omega}}(aa'bb'v|\alpha a'\beta\beta'y)u_z$$
$$-\tfrac{1}{2}(aa'bb'|\alpha a'yz)(\beta\beta'|uv) + \tfrac{1}{4}(aa'bb'uv|\alpha a'\beta\beta'yz).$$

$$(2)\quad \overset{yz}{\underset{\alpha\alpha'}{\Omega}}(aa'|a'y)(\alpha z|uv) = A_1(uv|yz) + (aa'|yz)(\alpha a'|uv) - (aa'uv|\alpha a'yz).$$

$$(3)\quad A_2\overset{yz}{\underset{\alpha\alpha'}{\Omega}}(aa'|a'y)(\alpha z|uv) - \overset{\beta\beta'\gamma\gamma'}{\underset{\alpha\alpha',yz}{\Omega}}(aa'|a'y)(\alpha\gamma'|uv)(\beta\beta'\gamma z|bb'cc')$$
$$= -A_1\overset{yz}{\underset{uv}{\Omega}}(\alpha a'\beta\beta'z|aa'bb'v)u_y + \overset{yz}{\underset{uv}{\Omega}}(aa'u|\alpha a'y)(\beta\beta'\gamma\gamma'z|bb'cc'v).$$

$$(4)\quad \overset{yz}{\underset{\alpha\alpha'}{\Omega}}(aa'|a'y)(\alpha\pi|uv)p_z = -A_1\overset{yz}{\underset{uv}{\Omega}}p_zv_\pi u_y + \overset{yz}{\underset{uv}{\Omega}}(aa'u|\alpha a'y)p_zv_\pi.$$

$$(5)\quad -2(yz|uv)A_2 + \overset{\alpha a'\beta\beta'}{\underset{yz}{\Omega}}(y\beta'|uv)(\alpha a'\beta z|aa'bb')$$
$$= -\overset{yz}{\underset{uv}{\Omega}}(\alpha a'\beta\beta'z|aa'bb'v)u_y.$$

$$(6)\quad A_2^2(yz|uv) - A_2\overset{\alpha a'\beta\beta'}{\underset{yz}{\Omega}}(y\beta'|uv)(\alpha a'\beta z|aa'bb')$$
$$+\overset{\gamma\gamma'\delta\delta',\ yz}{\underset{\alpha a'\beta\beta'}{\Omega}}[(\beta'\delta'|uv)(\alpha a'\beta y|aa'bb')(\gamma\gamma'\delta z|cc'dd')]$$
$$= \tfrac{4}{9}A_3(\alpha a'yz|aa'uv) - \tfrac{2}{3}(\alpha a'\beta\beta'\gamma\gamma'|aa'bb'uv).(\alpha a'yz|aa'bb')$$
$$+\tfrac{4}{9}\overset{yz}{\underset{uv}{\Omega}}p_yv_\pi(\alpha a'z|aa'u) - \tfrac{2}{3}(\alpha a'\beta\beta'yz|aa'bb'cc').(\alpha a'\beta\beta'|aa'uv)$$
$$+A_2(\alpha a'\beta\beta'yz|aa'bb'uv).$$

$$(7)\quad A_2\underset{yz}{\Omega}(y\pi|uv)p_z - \overset{\alpha a'\beta\beta'}{\underset{yz}{\Omega}}(\beta'\pi|uv)(\alpha a'\beta y|aa'bb')p_z$$
$$= A_3(\alpha a'\beta\beta'yz|aa'bb'uv) - (\alpha a'\beta\beta'yz|aa'bb'cc').(\alpha a'\beta\beta'\gamma\gamma'|aa'bb'uv).$$

$$(8)\quad \overset{uv}{\underset{yz}{\Omega}}(\alpha a'\beta\beta'z|aa'bb'v)(\gamma\gamma'y|cc'u)$$
$$= -(\alpha a'\beta\beta'|aa'uv).(aa'bb'|\alpha a'yz) - \tfrac{1}{9}\overset{uv}{\underset{yz}{\Omega}}p_zv_\pi u_y$$
$$+A_2(\alpha a'yz|aa'uv) + \tfrac{1}{9}A_3(yz|uv) + A_1(\alpha a'\beta\beta'yz|aa'bb'uv)$$
$$-\tfrac{1}{3}(aa'bb'cc'|\alpha a'\beta\beta'yz).(\alpha a'|uv) - \tfrac{1}{3}(\alpha a'\beta\beta'\gamma\gamma'|aa'bb'uv)(aa'|uv).$$

As an illustration consider the result in (6); the method of procedure is as follows.

Expand the second term on the left-hand side, obtaining

$$-\underset{yz}{\overset{\beta' a a' \beta}{\Omega}}(u_y v_{\beta'} - u_{\beta'} v_y)(aa' bb' \,|\, aa' \beta z).$$

In this permutation there will only be four terms arising from $\dot{\beta}' \dot{a} \dot{a}' \dot{\beta}$. Use the fundamental identity, noticing the determinantal permutation, on

$$v_{\beta'}(aa' bb' \,|\, aa' \beta z), \quad u_{\beta'}(aa' bb' \,|aa' \beta z),$$

and obtain

$$-2A_2{}^2(yz\,|\,uv) + A_2 \underset{zy}{\overset{vu}{\Omega}}(a a' \beta\beta' z \,|\, aa' bb' v)\, u_y.$$

Do the same for the third term of the left-hand side, by expanding

$$(\beta' \delta' \,|\, uv),$$

$$\overset{\beta' a a' \beta}{\Omega}\, \dot{u}_{\beta'}(aa' \beta y \,|\, aa' bb') \,.\, \overset{\delta' \gamma\gamma' \delta}{\Omega}\, \dot{v}_{\delta'}(\gamma\gamma' \delta z \,|\, cc' dd'),$$

giving

$$\underset{uv}{\Omega}\{u_y A_2 - (aa' \beta\beta' y \,|\, aa' bb' u)\}\{v_z A_2 - (cc' dd' v \,|\, \gamma\gamma' \delta\delta' z)\}.$$

The whole expression thus reduces to

$$(aa' bb' \dot{u} \,|\, aa' \beta\beta' y)(cc' dd' \dot{v} \,|\, \gamma\gamma' \delta\delta' z), \tag{7}$$

which is the Sylvester identity type, § 2 (iv).

Consider the first part of (7), [see § 2 (iv)]; it is

$$-\underset{\gamma\gamma' \delta\delta' z}{\overset{aa' bb' u}{\Omega}}(aa' \beta\beta' y\gamma \,|\, cc' dd' va)(\gamma' \delta\delta' z \,|\, a' bb' u),$$

or $$-2\underset{\gamma\gamma'}{\overset{aa' bb' u}{\Omega}}(aa' \beta\beta' y\gamma_{|} cc' dd' va)(\gamma' \delta\delta' z \,|\, a' bb' u)$$

$$-\underset{aa' bb' u}{\Omega}(aa' \beta\beta' yz \,|\, cc' dd' va)(\gamma\gamma' \delta\delta' \,|\, a' bb' u).$$

Now

$$\underset{aa' bb' u}{\Omega}(aa' \beta\beta' yz \,|\, cc' dd' va)(\gamma\gamma' \delta\delta' \,|\, a' bb' u)$$

$$= \underset{aa'}{\Omega}\, 2(aa' \beta\beta' yz \,|\, cc' dd' va)(\gamma\gamma' \delta\delta' \,|\, a' bb' u) - A_2(aa' \beta\beta' yz \,|\, cc' dd' uv)$$

$$= \tfrac{2}{3}(aa' \beta\beta' yz \,|\, cc' dd' aa')(\gamma\gamma' \delta\delta' \,|\, bb' uv)$$

$$+\tfrac{4}{3}\underset{\gamma\gamma'}{\Omega}(aa' \beta\beta' \gamma yz)(cc' dd' aa' v)(\gamma' \delta\delta' \,|\, bb' u)$$

$$-A_2(aa' \beta\beta' yz \,|\, cc' dd' uv)$$

$$= \tfrac{2}{3}(aa' \beta\beta' yz \,|aa' cc' dd')(\gamma\gamma' \delta\delta' \,|\, bb' uv)$$

$$+\tfrac{4}{9}\underset{zy}{\Omega}(aa' \beta\beta' \gamma\gamma' z)(cc' dd' aa' v)(\delta\delta' y \,|\, bb' u)$$

$$-A_2(aa' \beta\beta' yz \,|\, cc' dd' uv).$$

The rest of the work is of an analogous character and leads to the identity given in (6).

Substituting these results (1) to (8) in the expression (6) and making use of the result [§ 3 (x)]

$$\lambda_1{}^3+\lambda_1{}^2 A_1+\lambda_1 \tfrac{1}{4}A_2+\tfrac{1}{36}A_3=0, \qquad (8)$$

we find that the expression for the equation of the double lines and "double spaces" factorizes into

$$\{\lambda_1{}^2(aa'\,|\,yz)+\tfrac{1}{2}\lambda_1(aa'\,bb'\,|\,aa'\,yz)+\tfrac{1}{12}(aa'\,bb'\,cc'\,|\,aa'\,\beta\beta'\,yz)\}$$

$$\times\{\lambda_1{}^2(aa'\,|\,uv)+\tfrac{1}{2}\lambda_1(aa'\,\beta\beta'\,|\,aa'\,uv)+\tfrac{1}{12}(aa'\,\beta\beta'\,\gamma\gamma'\,|\,aa'\,bb'\,uv)\}=0.$$

Hence the double line corresponding to the value of the root λ_1 of E_1, where λ_1 satisfies (8), is given by

$$\lambda_1{}^2(aa'\,|\,yz)+\tfrac{1}{2}\lambda_1(aa'\,bb'\,|\,aa'\,yz)+\tfrac{1}{12}(aa'\,bb'\,cc'\,|\,aa'\,\beta\beta'\,yz)=0,$$

with the dual expression for the "double space."

An exactly similar method may be used to consider another special collineation built up from a plane complex and a dual complex of the types

$$(aa'\,a''\,|\,xyz)=0, \quad (\alpha\alpha'\,\alpha''\,|\,uvw)=0,$$

the collineation having the form

$$(aa'\,a''\,|\,\dot{\alpha}'\,\dot{\alpha}''\,x)\,u_{\dot{\alpha}}=0.$$

OPERATIONAL REPRESENTATIONS OF WHITTAKER'S CONFLUENT HYPERGEOMETRIC FUNCTION AND WEBER'S PARABOLIC CYLINDER FUNCTION

By S. Goldstein.

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1.1. Operational representations are given below of functions related to Whittaker's confluent hypergeometric function $W_{k,\,m}(x)$, and Weber's parabolic cylinder function $D_n(x)$*. Various properties of the functions are deduced by the use of theorems in the operational calculus—theorems to a large extent equivalent to known theorems of pure mathematics.

1.2. The operators are the same as those used by Heaviside for the solution of electrical problems. The theory of these operators has been clarified and extended by Bromwich and others, and a general account given by Jeffreys†, in whose tract references will be found. With the primary object of much of the work on the subject—to find the solution of a differential equation, or system of equations, subject to given initial conditions—we shall be only indirectly and incidentally concerned.

Following Jeffreys (*loc. cit.* Chapter 1), we shall consider definite integration as the fundamental notion, so that, if $\phi(p)$ is the operational representation of $f(t)$, then $f(t) = t^n/n!$ when $\phi(p) = p^{-n}$, and if

$$\phi(u) = \sum_{n=0}^{\infty} a_n u^{-n} \tag{1}$$

for sufficiently large u, then, with t real and positive, as throughout this paper,

$$f(t) = \sum_{n=0}^{\infty} \frac{a_n}{n!} t^n. \tag{2}$$

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* Whittaker and Watson, *Modern analysis* (1920), ch. 16. This work will be referred to as M.A.

† *Operational methods in mathematical physics* (Cambridge tract, No. 23, 2nd ed., 1931).

It follows from an application of the Borel sum formula* that

$$\phi(u) = u\int_0^\infty e^{-ut}f(t)\,dt. \tag{3}$$

So long as the integral converges, we may generalize by defining the operational representation of $f(t)$ as $\phi(p)$, where $\phi(u)$ is given by (3), when (1) does not hold. This integral is due to Carson†.

If the integral in (3) is absolutely convergent when u is real and equal to a, then $\phi(\lambda)/\lambda$ is an analytic function of the complex variable λ when $R(\lambda) > a$ and‡

$$f(t) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} e^{\lambda t}\frac{\phi(\lambda)}{\lambda}\,d\lambda \quad (c > a). \tag{4}$$

This integral gives the interpretation of a known operator, and is due to Bromwich§.

The direct deduction of (3) from (4) is not difficult. Let u be real and greater than c; let ϕ be analytic on the right of the path of integration in (4), and let the integral of $\phi(\lambda)/\{\lambda(u-\lambda)\}$ round an infinite semicircle to the right of that path be zero. Then, if we may invert the order of integration||,

$$\begin{aligned}\frac{1}{2\pi i}\int_0^\infty e^{-ut}dt\int_{c-i\infty}^{c+i\infty} e^{\lambda t}\frac{\phi(\lambda)}{\lambda}\,d\lambda &= \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\frac{\phi(\lambda)}{\lambda}\,d\lambda\int_0^\infty e^{-(u-\lambda)t}dt \\ &= \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\frac{\phi(\lambda)}{\lambda(u-\lambda)}\,d\lambda = \frac{\phi(u)}{u}.\end{aligned} \tag{5}$$

When $\phi(\lambda)$ is analytic at infinity, so that (1) holds, (4) is easily seen to be equivalent to (2) (Jeffreys, *loc. cit.*, 22, 23, 25), and (4) could have been taken as the generalization. The order adopted above is perhaps more satisfactory as regards the introduction of the complex variable.

1.3. From now on, we shall write $\phi(p) = f(t)$ to denote that $\phi(p)$ is the operational representation of $f(t)$. Then, if $\phi_1(p) = f_1(t)$ and $\phi_2(p) = f_2(t)$,

$$\phi_1(p)\,\phi_2(p)\,\frac{1}{p} = \int_0^t f_1(\xi)f_2(t-\xi)\,d\xi = \int_0^t f_1(t-\xi)f_2(\xi)\,d\xi. \tag{6}$$

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* Bromwich, *Theory of infinite series* (1908), 267 *et seq.*

† *Electric circuit theory and the operational calculus* (1926), 23.

‡ (4) follows from (3) by an easy transformation of Mellin's inversion formula. See Hardy, *Messenger of Math.*, 50 (1921), 165–171; also Hardy and Titchmarsh, *Quart. J. o Math.*, 1 (1930), 225–226.

§ *Proc. London Math. Soc.* (2), 15 (1916), 401–448.

|| *Cf.* Mellin, *Acta Mathematica*, 25 (1902), 156–162.

This follows from (3) by an application of Borel's multiplication theorem*. It follows from (4) by a simple transformation of a result due to Mellin†.

In connection with the use of the operational calculus to solve the differential equations of mathematical physics, we may note that (6) expresses a well-known superposition principle in physics. The first application of the principle seems to be due to Poisson‡. Other examples are Duhamel's theorem in the theory of the conduction of heat§, Boltzman's theory of elastic afterworking‖, and the Maxwell-Hopkinson theory of dielectric absorption¶.

The formula for generalized integration and differentiation to which Riemann was led** is in accordance with (6).

Lévy has pointed out†† that the integral in (6) is a special case of Volterra's *résultante de la composition* of two permutable functions‡‡.

A generalization of (6) is§§

$$\phi_1(p)\,\phi_2(p)\,\phi_3(p)\dots\phi_n(p)\,p^{-(n-1)}$$

$$=\iiint\dots f_1(\xi_1)f_2(\xi_2)f_3(\xi_3)\dots f_{n-1}(\xi_{n-1})f_n(t-\xi_1-\xi_2-\dots-\xi_{n-1})\,d\xi_1 d\xi_2\dots d\xi_{n-1} \tag{7}$$

over the region for which

$$\xi_1\geqslant 0,\quad \xi_2\geqslant 0,\quad \dots,\quad \xi_{n-1}\geqslant 0;\quad \xi_1+\xi_2+\dots+\xi_{n-1}\leqslant t.$$

1.4. If $\phi(p)=f(t)$ and $\psi(p)=g(t)$, then

$$\phi(u)=u\int_0^\infty e^{-ut}f(t)\,dt,$$

$$\psi(t)=t\int_0^\infty e^{-ut}g(u)\,du,$$

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* Bromwich, *Infinite series*, 280 *et seq.* It was proved in this way by Carson, *loc. cit.*, 41, 44.

† See Hardy, *Messenger of Math.*, 49 (1919), 85-91; 50 (1921), 170.

‡ *Mémoire sur la théorie du magnétisme en mouvement* (*Œuvres Mathématiques*, No. 5), 29, 30.

§ Carslaw, *The conduction of heat* (1921), 17–19.

‖ *Wissenschaftliche Abhandlungen*, 1 (1909), 616–644.

¶ John Hopkinson, *Original papers*, 2 (1901), 10–43.

** *Math. Werke*, 340; *Encyklopädie der Math. Wiss.*, 2, 1 Teil, 2 Hälfte, 771.

†† *Bull. des Sciences Math.*, 50 (1926), 174–192.

‡‡ Volterra and Pérès, *Leçons sur la composition et les fonctions permutables*, 9 and 10.

§§ *Cf.* Hardy, *Messenger of Math.*, 49 (1919), 91.

and, therefore,

$$\int_0^\infty \frac{\phi(u)\,g(u)}{u}\,du = \int_0^\infty g(u)\,du \int_0^\infty e^{-ut} f(t)\,dt$$

$$= \int_0^\infty f(t)\,dt \int_0^\infty e^{-ut} g(u)\,du = \int_0^\infty \frac{f(t)\,\psi(t)}{t}\,dt, \tag{8}$$

if the order of the integrations may be inverted.

The special case of this formula when $\psi = g = 1$ has been given by van der Pol*.

1.5. In the applications below, the conditions attached to (5) and (8) are easily seen to be fulfilled.

Further, in the applications below the only singularity in $\phi(\lambda)$ is at the origin, and, t being positive, the integral of $e^{\lambda t}\phi(\lambda)/\lambda$ round an infinite semicircle to the left of the path of integration in (4) vanishes, so that the path of integration may be deformed into a path consisting of the negative half of the real axis taken twice and an infinitesimal circle round the origin, with $|\arg\lambda| \leqslant \pi$. This contour will be denoted by M.

2.1. Let

$$p = q^2, \tag{9}$$

and consider $q^\mu K_\nu(qr)$, where, if r is complex, $|\arg r| < \frac{1}{4}\pi$, K_ν is a Bessel function, and the notation of Watson's *Theory of Bessel functions* is adopted†. Then

$$q^\mu K_\nu(qr) = \frac{1}{2\pi i}\int_M \frac{e^{\lambda t}}{\lambda}\lambda^{\frac{1}{2}\mu} K_\nu(\lambda^{\frac{1}{2}} r)\,d\lambda. \tag{10}$$

Let $\mu - \nu$ be positive and let ν be non-integral. Then the integral round the infinitesimal circle is zero. Let λ be $e^{\pm i\pi}u^2$ on the negative real axis. Since

$$K_\nu\left(\frac{ur}{i}\right) = \tfrac{1}{2}\pi\,\frac{e^{\frac{1}{2}\nu\pi i} J_{-\nu}(ur) - e^{-\frac{1}{2}\nu\pi i} J_\nu(ur)}{\sin\nu\pi}, \tag{11}$$

(10) reduces to

$$q^\mu K_\nu(qr) = \int_0^\infty e^{-u^2 t} u^{\mu-1}\left[\frac{\sin\frac{1}{2}(\mu+\nu)\pi}{\sin(-\nu\pi)}\,J_\nu(ur) + \frac{\sin\frac{1}{2}(\mu-\nu)\pi}{\sin\nu\pi}\,J_{-\nu}(ur)\right]du. \tag{12}$$

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* *Phil. Mag.* (7), 8 (1929), 864. In this paper theorems in the operational calculus are used to find properties of known functions, mainly Bessel functions.

† This work will be referred to as T.B.F.

If we use the formulae

$$\int_0^\infty e^{-u^2 t} u^{\mu-1} J_\nu(ur)\,du = \frac{\Gamma(\frac{1}{2}\mu+\frac{1}{2}\nu)}{2t^{\frac{1}{2}\mu}\,\Gamma(1+\nu)} \frac{r^\nu}{(4t)^{\frac{1}{2}\nu}} e^{-r^2/4t}\, {}_1F_1\left(\frac{\nu-\mu}{2}+1;\ 1+\nu;\ \frac{r^2}{4t}\right)$$

(T.B.F., 394),

$$\Gamma(\tfrac{1}{2}\mu+\tfrac{1}{2}\nu)\sin\tfrac{1}{2}(\mu+\nu)\pi = \frac{\pi}{\Gamma(1-\frac{1}{2}\mu-\frac{1}{2}\nu)};\ \Gamma(1+\nu)\sin(-\nu\pi) = \frac{\pi}{\Gamma(-\nu)},$$

and $${}_1F_1\left(\tfrac{1}{2}\nu-\tfrac{1}{2}\mu+1;\ 1+\nu;\ \frac{r^2}{4t}\right) = \frac{(4t)^{\frac{1}{2}(\nu+1)}}{r^{\nu+1}} e^{r^2/8t} M_{\frac{1}{2}\mu-\frac{1}{2},\frac{1}{2}\nu}\left(\frac{r^2}{4t}\right)$$

(M.A., 338), we find that

$$q^\mu K_\nu(qr) = \frac{\Gamma(-\nu)}{\Gamma(1-\frac{1}{2}\mu-\frac{1}{2}\nu)} \frac{1}{rt^{\frac{1}{2}(\mu-1)}} e^{-r^2/8t} M_{\frac{1}{2}\mu-\frac{1}{2},\frac{1}{2}\nu}\left(\frac{r^2}{4t}\right)$$

$$+ \frac{\Gamma(\nu)}{\Gamma(1-\frac{1}{2}\mu+\frac{1}{2}\nu)} \frac{1}{rt^{\frac{1}{2}(\mu-1)}} e^{-r^2/8t} M_{\frac{1}{2}\mu-\frac{1}{2},-\frac{1}{2}\nu}\left(\frac{r^2}{4t}\right)$$

$$= \frac{1}{rt^{\frac{1}{2}(\mu-1)}} e^{-r^2/8t} W_{\frac{1}{2}\mu-\frac{1}{2},\frac{1}{2}\nu}\left(\frac{r^2}{4t}\right). \tag{13}$$

If we put $$x = r^2/4t, \quad k = \tfrac{1}{2}\mu-\tfrac{1}{2}, \quad m = \tfrac{1}{2}\nu, \tag{14}$$

then $$(qr)^\mu K_\nu(qr) = (4x)^k e^{-\frac{1}{2}x} W_{k,m}(x), \tag{15}$$

and the restrictions that ν is not integral, and that $\mu-\nu$ is positive, may be removed by analytic continuation.

2.2. Numerous special cases may be written down at once. Many such special cases of the $W_{k,m}$ function are given in M.A., 341, 352, 353, and by Whittaker, *Bull. American Math. Soc.*, 10 (1904), 125–134. Others can easily be found. A few are noted below.

First,

$$W_{m+\frac{1}{2},m}(x) = e^{-\frac{1}{2}x} x^{m+\frac{1}{2}};\ (qr)^{\nu+2} K_\nu(qr) = (2x)^{\nu+1} e^{-x}, \tag{16}$$

and, in general, if s is a positive integer or zero,

$$\left.\begin{aligned} W_{m+s+\frac{1}{2},m}(x) &= e^{-\frac{1}{2}x} x^{m+s+\frac{1}{2}} \left\{1+\sum_{n=1}^{s}(-)^n \right. \\ &\quad \left. \times \frac{s(s-1)\ldots(s-n+1)(2m+s)(2m+s-1)\ldots(2m+s-n+1)}{n!\ x^n}\right\}, \\ (qr)^{\nu+2s+2} K_\nu(qr) &= (2x)^{\nu+2s+1} e^{-x} \left\{1+\sum_{n=1}^{s}(-)^n \right. \\ &\quad \left. \times \frac{s(s-1)\ldots(s-n+1)(\nu+s)(\nu+s-1)\ldots(\nu+s-n+1)}{n!\ x^n}\right\}. \end{aligned}\right\} \tag{17}$$

From (4) it is clear that, with our form of $\phi(p)$,

$$p\phi(p) = \frac{d}{dt} f(t), \tag{18}$$

and (17) can be obtained from (16) by repeated differentiation with respect to t.

A special case of (17) is noteworthy. First remark that

$$W_{s+\frac{1}{2},0}(x) = (-)^s e^{-\frac{1}{2}x} x^{\frac{1}{2}} L_s(x);\ (qr)^{2s+2} K_0(qr) = (-)^s 2^{2s+1} x^{s+1} e^{-x} L_s(x), \tag{19}$$

where $L_s(x)$ is the Laguerre polynomial, defined by

$$L_s(x) = e^x \left(\frac{d}{dx}\right)^s (x^s e^{-x}) \tag{20}$$

[Courant and Hilbert, *Methoden der Mathematischen Physik* (1924), 77. The function $\phi_s(x)$, defined in M.A., 353, ex. 9, is $L_s(x)/s!$]. More generally, if n is a positive integer or zero,

$$\left.\begin{aligned} W_{\frac{1}{2}n+s+\frac{1}{2},\frac{1}{2}n}(x) &= (-)^{n+s} \frac{s!}{(n+s)!} e^{-\frac{1}{2}x} x^{\frac{1}{2}(n+1)} L^n_{n+s}(x), \\ (qr)^{n+2s+2} K_n(qr) &= (-)^{n+s} \frac{s!}{(n+s)!} 2^{n+2s+1} e^{-x} L^n_{n+s}(x), \end{aligned}\right\} \tag{21}$$

where it is easily verified that

$$L^n_{n+s}(x) = \left(\frac{d}{dx}\right)^n L_{n+s}(x). \tag{22}$$

The function in the first of equations (21), divided by x or $x^{\frac{1}{2}}$, occurs in the theory of hydrogen-like atoms in wave mechanics. See, for example, Sommerfeld, *Atombau und Spektrallinien, Wellenmechanischer Ergänzungsband* (1929), 70–77 and 181–184. Sonine's polynomial (M.A., 352, ex. 8) is given by

$$T_n{}^s(x) = \frac{(-)^{n+s} L^n_{n+s}(x)}{[(n+s)!]^2}.$$

The particular case of (6),

$$\frac{\phi(p)}{p} = \int_0^t f(\xi)\, d\xi, \tag{23}$$

is the fundamental notion of the operational calculus. If we apply it to (16), we find that

$$(qr)^\nu K_\nu(qr) = \int_x^\infty e^{-u} (2u)^{\nu-1} du;$$

$$W_{\frac{1}{2}\nu-\frac{1}{2},\frac{1}{2}\nu}(x) = e^{\frac{1}{2}x} x^{-\frac{1}{2}(\nu-1)} \int_x^\infty e^{-u} u^{\nu-1} du. \tag{24}$$

(M.A., 341.)

Another important particular case is

$$W_{0,m}(x) = \left(\frac{x}{\pi}\right)^{\frac{1}{2}} K_m(\tfrac{1}{2}x);\ qr\,K_\nu(qr) = \left(\frac{x}{\pi}\right)^{\frac{1}{2}} e^{-\frac{1}{2}x} K_{\frac{1}{2}\nu}(\tfrac{1}{2}x). \tag{25}$$

Either of these equations is easily proved, and the other inferred.

Finally, we note that, if $D_n(x)$ is Weber's parabolic cylinder function (M.A., 347),

$$\left.\begin{aligned} W_{k,-\frac{1}{4}}(x) &= 2^{-k}(2x)^{\frac{1}{4}} D_{2k-\frac{1}{2}}\{\sqrt{(2x)}\}, \\ (\tfrac{1}{2}\pi)^{\frac{1}{2}}(qr)^{\mu} e^{-qr} &= (2x)^{\frac{1}{2}\mu} e^{-\frac{1}{2}x} D_{\mu-1}\{\sqrt{(2x)}\}. \end{aligned}\right\} \tag{26}$$

2.3. The operator in this last equation is of importance in physical investigations. The equation itself is easily proved directly by considering

$$\frac{1}{2\pi i}\int_{-\infty}^{(0,+)} e^{\lambda t}\lambda^{\frac{1}{2}\mu-1} e^{-\lambda^{\frac{1}{2}}r}\,d\lambda,$$

expanding $e^{-\lambda^{\frac{1}{2}}r}$ in powers of $\lambda^{\frac{1}{2}}$, and integrating term by term. By this process we find

$$\begin{aligned} q^{\mu} e^{-qr} &= t^{-\frac{1}{2}\mu}\,\frac{{}_1F_1\{\frac{1}{2}\mu;\ \frac{1}{2};\ -x\}}{\Gamma(1-\frac{1}{2}\mu)} - rt^{-\frac{1}{2}(\mu+1)}\,\frac{{}_1F_1\{\frac{1}{2}\mu+\frac{1}{2};\ \frac{3}{2};\ -x\}}{\Gamma(\frac{1}{2}-\frac{1}{2}\mu)} \\ &= t^{-\frac{1}{2}\mu} e^{-x}\,\frac{{}_1F_1\{\frac{1}{2}-\frac{1}{2}\mu;\ \frac{1}{2};\ x\}}{\Gamma(1-\frac{1}{2}\mu)} - rt^{-\frac{1}{2}(\mu+1)} e^{-x}\,\frac{{}_1F_1\{1-\frac{1}{2}\mu;\ \frac{3}{2};\ x\}}{\Gamma(\frac{1}{2}-\frac{1}{2}\mu)} \end{aligned}$$

(by Kummer's first formula)

$$= \left(\frac{2}{\pi}\right)^{\frac{1}{2}} (2t)^{-\frac{1}{2}\mu} e^{-\frac{1}{2}x} D_{\mu-1}\{\sqrt{(2x)}\}. \tag{27}$$

(M.A., 347. In the sixth line from the bottom of §16.5 for $-\frac{1}{2}$ read $+\frac{1}{2}$.)

As special cases of (26) note that

$$\left.\begin{aligned} D_0\{\sqrt{(2x)}\} &= e^{-\frac{1}{2}x};\ qe^{-qr} = \frac{1}{(\pi t)^{\frac{1}{2}}} e^{-r^2/4t}, \\ D_{-1}\{\sqrt{(2x)}\} &= \left(\frac{\pi}{2}\right)^{\frac{1}{2}} e^{\frac{1}{2}x}(1-\operatorname{erf}\sqrt{x});\ e^{-qr} = 1-\operatorname{erf}\frac{r}{2t^{\frac{1}{2}}}, \end{aligned}\right\} \tag{28}$$

where

$$\operatorname{erf} x = \frac{2}{\sqrt{\pi}}\int_0^x e^{-u^2}\,du, \tag{29}$$

in agreement with Jeffreys (*loc. cit.*, 112).

The value of $q^{\mu} e^{-qr}$ when μ is a negative integer is easily obtained in terms of the error function from (28) and (66) below.

3.1. We now investigate some of the properties of $W_{k,m}(x)$ by the use of (15). First we find recurrence formulae by using the well-known recurrence formulae for K_ν (T.B.F., 79). Differentiating (15) with respect to r, and multiplying by $2r$, gives

$$\begin{aligned}
&2(qr)^{\mu+1}K_\nu'(qr)+2\mu(qr)^\mu K_\nu(qr)\\
&\quad=2\mu(qr)^\mu K_\nu(qr)-(qr)^{\mu+1}K_{\nu-1}(qr)-(qr)^{\mu+1}K_{\nu+1}(qr)\\
&\quad=(2k+1)\,2^{2k+1}x^k e^{-\frac{1}{2}x}W_{k,m}(x)-2^{2k+1}x^{k+\frac{1}{2}}e^{-\frac{1}{2}x}[W_{k+\frac{1}{2},m-\frac{1}{2}}(x)+W_{k+\frac{1}{2},m+\frac{1}{2}}(x)]\\
&\quad=2^{2k+2}x\frac{d}{dx}\{x^k e^{-\frac{1}{2}x}W_{k,m}(x)\}.
\end{aligned}$$

Hence

$$(1+x)\,W_{k,m}(x)=2xW'_{k,m}(x)+x^{\frac{1}{2}}[W_{k+\frac{1}{2},m-\frac{1}{2}}(x)+W_{k+\frac{1}{2},m+\frac{1}{2}}(x)]. \tag{30}$$

Next apply (18), and multiply by r^2. This gives

$$\begin{aligned}
(qr)^{\mu+2}K_\nu(qr)&=(4x)^{k+1}e^{-\frac{1}{2}x}W_{k+1,m}(x)\\
&=-4^{k+1}x^2\frac{d}{dx}\{x^k e^{-\frac{1}{2}x}W_{k,m}(x)\},
\end{aligned} \tag{31}$$

so that

$$\begin{aligned}
x^k e^{-\frac{1}{2}x}W_{k,m}(x)&=-x^2\frac{d}{dx}\{x^{k-1}e^{-\frac{1}{2}x}W_{k-1,m}(x)\}\\
&=(-)^s\left(x^2\frac{d}{dx}\right)^s\{x^{k-s}e^{-\frac{1}{2}x}W_{k-s,m}(x)\},
\end{aligned} \tag{32}$$

where s is a positive integer. If k is a positive integer, then

$$x^k e^{-\frac{1}{2}x}W_{k,m}(x)=(-)^k\left(x^2\frac{d}{dx}\right)^k\left\{\left(\frac{x}{\pi}\right)^{\frac{1}{2}}e^{-\frac{1}{2}x}K_m(\tfrac{1}{2}x)\right\}. \tag{33}$$

By differentiating out the right side of (31), we find that

$$W_{k+1,m}(x)+xW'_{k,m}(x)+(k-\tfrac{1}{2}x)\,W_{k,m}(x)=0. \tag{34}$$

Finally, from the formula

$$(qr)^{\mu+1}[K_{\nu-1}(qr)-K_{\nu+1}(qr)]+2\nu(qr)^\mu K_\nu(qr)=0,$$

we deduce that

$$x^{\frac{1}{2}}[W_{k+\frac{1}{2},m-\frac{1}{2}}(x)-W_{k+\frac{1}{2},m+\frac{1}{2}}(x)]+2mW_{k,m}(x)=0. \tag{35}$$

Equations (30), (34), and (35) are equivalent to known recurrence formulae (M.A., 352, ex. 3).

3.2. If n is an integer,

$$(qr)^{\mu} K_{n+\frac{1}{2}}(qr) = \left(\frac{\pi}{2}\right)^{\frac{1}{2}} \sum_{s=0}^{n} \frac{(n+s)!}{2^s s!\,(n-s)!}\, e^{-qr}(qr)^{\mu-s-\frac{1}{2}},$$

and, therefore,

$$W_{k,\frac{1}{2}n+\frac{1}{4}}(x) = 2^{-k} \sum_{s=0}^{n} \frac{(n+s)!}{2^s s!\,(n-s)!}\, (2x)^{\frac{1}{4}-\frac{1}{2}s} D_{2k-s-\frac{1}{2}}\{\sqrt{(2x)}\}. \qquad (36)$$

Again, if n is an integer,

$$\begin{aligned}\tfrac{1}{2}x^k e^{-\frac{1}{2}x} W_{k,\frac{1}{2}n}(x) &= (\tfrac{1}{2}qr)^{\mu} K_n(qr)\\ &= \tfrac{1}{2} \sum_{s=0}^{n-1} \frac{(-)^s (n-s-1)!}{s!}\, (\tfrac{1}{2}qr)^{2s+\mu-n}\\ &\quad + (-)^{n+1} \sum_{s=0}^{\infty} \frac{(\frac{1}{2}qr)^{2s+\mu+n}}{s!\,(n+s)!}\, \{\log(\tfrac{1}{2}qr) - \tfrac{1}{2}\psi(s+1) - \tfrac{1}{2}\psi(n+s+1)\}. \qquad (37)\end{aligned}$$

Now, with integration in Bromwich's integral along a contour starting from $-\infty$ on the real axis, encircling the origin once in the positive direction, and returning to $-\infty$,

$$q^{2s} = p^s = \frac{t^{-s}}{\Gamma(1-s)}; \quad (\tfrac{1}{2}qr)^{2s} = \frac{x^s}{\Gamma(1-s)}, \qquad (38)$$

for any s, so that differentiation with respect to s gives*

$$(\tfrac{1}{2}qr)^{2s} \log \tfrac{1}{2}qr = \tfrac{1}{2}\, \frac{x^s}{\Gamma(1-s)}\, \{\log x + \psi(1-s)\}. \qquad (39)$$

Since

$$\Gamma(1-s) = \frac{\pi}{\Gamma(s)\sin s\pi},$$

$$\psi(1-s) = \psi(s) + \pi \cot s\pi,$$

we have, when s is integral,

$$\frac{\psi(1-s)}{\Gamma(1-s)} = (-)^s \Gamma(s),$$

and

$$(\tfrac{1}{2}qr)^{2s} \log(\tfrac{1}{2}qr) = \tfrac{1}{2}\Gamma(s)(-x)^s. \qquad (40)$$

Thus (37) leads to the formula

$$\begin{aligned}e^{-\frac{1}{2}x} W_{k,\frac{1}{2}n}(x) &= \sum_{s=0}^{n-1} (-)^s \frac{(n-s-1)!}{s!}\, \frac{x^{s-\frac{1}{2}n+\frac{1}{2}}}{\Gamma(\frac{1}{2}+\frac{1}{2}n-k-s)}\\ &\quad + (-)^{n+1} \sum_{s=0}^{\infty} \frac{1}{s!\,(n+s)!}\, \frac{x^{s+\frac{1}{2}n+\frac{1}{2}}}{\Gamma(\frac{1}{2}-\frac{1}{2}n-k-s)}\\ &\quad \times \{\log x + \psi(\tfrac{1}{2}-\tfrac{1}{2}n-k-s) - \psi(s+1) - \psi(n+s+1)\}, \qquad (41)\end{aligned}$$

* *Cf.* van der Pol, *loc. cit.*, 873.

when n is a positive integer, unless $\frac{1}{2}+\frac{1}{2}n+k$ is a positive or negative integer, or zero. When $\frac{1}{2}+\frac{1}{2}n+k$ is a positive integer, the function reduces to an elementary function. In any case, when $\frac{1}{2}+\frac{1}{2}n+k+s$ is a positive integer, the corresponding term in the infinite series must be replaced by

$$\frac{1}{s!\,(n+s)!}\,(-)^k(-x)^{s+\frac{1}{2}n+\frac{1}{2}}\,\Gamma(\tfrac{1}{2}+\tfrac{1}{2}n+k+s). \tag{42}$$

When n is zero, the finite series in (41) is to be omitted; and negative values of n need not be considered, since

$$W_{k,-m}(x) = W_{k,m}(x) \quad \text{(because } K_\nu = K_{-\nu}\text{)}.$$

The series (41) can easily be verified for special cases; an interesting example is when $n=0$, $k=-\frac{1}{2}$. Since

$$e^{-\frac{1}{2}x}\,W_{-\frac{1}{2},0}(x) = x^{\frac{1}{2}}\int_x^\infty e^{-u}\,\frac{du}{u}$$

[see (24)], we thus verify the known expansion

$$\int_x^\infty e^{-u}\,\frac{du}{u} = -\gamma-\log x-\sum_{s=1}^{\infty}\frac{(-x)^s}{s\,.\,s!}, \tag{43}$$

where $\gamma=-\psi(1)$, and is Euler's constant.

3.3. Equation (32) is equivalent to

$$\left(\frac{d}{dt}\right)^s\left[t^{-k}e^{-\frac{1}{2}x}\,W_{k,m}(x)\right] = t^{-(k+s)}e^{-\frac{1}{2}x}\,W_{k+s,m}(x), \tag{44}$$

and, if $\lambda<t$, an application of Taylor's theorem gives

$$(t-\lambda)^{-k}\,e^{-r^2/8(t-\lambda)}\,W_{k,m}\left(\frac{r^2}{4t-4\lambda}\right) = \sum_{s=0}^{\infty}\frac{(-\lambda)^s}{s!}\,t^{-(k+s)}\,e^{-\frac{1}{2}x}\,W_{k+s,m}(x). \tag{45}$$

With $$1-\lambda/t = u,$$

this reduces to

$$u^{-k}\,e^{-\frac{1}{2}x/u}\,W_{k,m}\left(\frac{x}{u}\right) = e^{-\frac{1}{2}x}\sum_{s=0}^{\infty}\frac{(u-1)^s}{s!}\,W_{k+s,m}(x), \tag{46}$$

where u is positive.

A known special case of this result may be noted. If $m=0$, $k=\frac{1}{2}$, and $u=1-h$, we get

$$(1-h)^{-1}\,e^{-hx/(1-h)} = x^{-\frac{1}{2}}\,e^{\frac{1}{2}x}\sum_{s=0}^{\infty}\frac{(-h)^s}{s!}\,W_{s+\frac{1}{2},0}(x) = \sum_{s=0}^{\infty}\frac{L_s(x)}{s!}\,h^s. \tag{47}$$

[See M.A., 353, ex. 9, and Courant and Hilbert, *loc. cit.*, 78, equation (68).]

More generally, if $m = \frac{1}{2}n$, $k = \frac{1}{2}(n+1)$, $u = 1-h$, we get*

$$(1-h)^{-(n+1)} e^{-hx/(1-h)}$$

$$= x^{-\frac{1}{2}(n+1)} e^{\frac{1}{2}x} \sum_{s=0}^{\infty} \frac{(-h)^s}{s!} W_{\frac{1}{2}n+s+\frac{1}{2}, \frac{1}{2}n}(x) = (-)^n \sum_{s=0}^{\infty} \frac{h^s}{(n+s)!} L_{n+s}^n(x). \quad (48)$$

Operationally, (45) is an example of the theorem that, if $\phi(p) = f(t)$, then $e^{-\lambda p}\phi(p) = f(t-\lambda)$ for $t > \lambda$.

3.4. Carson's integral (3) gives, with $u = v^2$,

$$(vr)^{\mu} K_{\nu}(vr) = v^2 \int_0^{\infty} e^{-v^2 t} (4x)^k e^{-\frac{1}{2}x} W_{k,m}(x)\, dt,$$

or
$$2^{2k-2} \int_0^{\infty} \exp\left\{-\frac{y^2}{4x} - \tfrac{1}{2}x\right\} x^{k-2} W_{k,m}(x)\, dx = y^{2k-1} K_{2m}(y), \quad (49)$$

after the substitution $vr = y$, provided that $|\arg y| < \frac{1}{4}\pi$, $R(y^2) > 0$. If $|\arg y| = \frac{1}{4}\pi$, or $y = 0$, we must have $k \pm m - \frac{1}{2} > 0$. If this condition is satisfied, and $y = 0$, we obtain

$$\int_0^{\infty} e^{-\frac{1}{2}x} x^{k-2} W_{k,m}(x)\, dx = 0, \quad (50)$$

[*Cf.* (54) below.]

3.5. As an application of (6) take

$$\left.\begin{aligned} \phi_1(p) &= (qr)^{\mu} K_{\nu}(qr), & f_1(t) &= (4x)^k e^{-\frac{1}{2}x} W_{k,m}(x), \\ \phi_2(p) &= (qr)^{2\lambda-\mu+3}, & f_2(t) &= \frac{(4x)^{\lambda-k+1}}{\Gamma(k-\lambda)} \quad (k > \lambda). \end{aligned}\right\}$$

With $u = r^2/4\xi$, (6) then gives

$$\Gamma(k-\lambda)\, x^{k-1} e^{-\frac{1}{2}x} W_{\lambda,m}(x) = \int_x^{\infty} u^{\lambda-1}(u-x)^{k-\lambda-1} e^{-\frac{1}{2}u} W_{k,m}(u)\, du, \quad (51)$$

where $k > \lambda$, and x is positive.

3.6. As a first application of (8), take $\psi = g = 1$. Then, with $u = v^2$, we find

$$\begin{aligned} \int_0^{\infty} x^{k-1} e^{-\frac{1}{2}x} W_{k,m}(x)\, dx &= 2^{-(2k-1)} \int_0^{\infty} (vr)^{\mu} K_{\nu}(vr) \frac{dv}{v} \\ &= 2^{-(2k-1)} \int_0^{\infty} w^{2k} K_{2m}(w)\, dw \\ &= \Gamma(\tfrac{1}{2}+k-m)\, \Gamma(\tfrac{1}{2}+k+m), \end{aligned} \quad (52)$$

if $k \pm m + \frac{1}{2}$ is positive. (T.B.F., 388.)

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* Schrödinger, *Annalen der Physik* (4), 80 (1926), 483–490. *Cf.* (22), above.

As a further example, take*

$$\psi(p) = \left(\frac{r^2}{2p}\right)^{(l-k)} e^{-a^2 r^2/4p}; \qquad g(t) = \left(\frac{r^2 t}{a^2}\right)^{\frac{1}{2}(l-k)} J_{l-k}\{ar\sqrt{t}\}$$

$$(l-k > -1). \quad (53)$$

Then

$$\int_0^\infty x^{l-1} e^{-(a^2+\frac{1}{2})x} W_{k,m}(x)\, dx$$

$$= 2^{-(l+k-1)} \int_0^\infty (vr)^\mu K_\nu(vr) \left(\frac{vr}{a}\right)^{l-k} J_{l-k}(avr) \frac{dv}{v}$$

$$= 2^{-(l+k-1)} \int_0^\infty w^{l+k} K_{2m}(w) J_{l-k}(aw)\, a^{-(l-k)}\, dw$$

$$= \frac{\Gamma(l+m+\frac{1}{2})\,\Gamma(l-m+\frac{1}{2})}{\Gamma(l-k+1)}\, {}_2F_1(l+m+\tfrac{1}{2};\ l-m+\tfrac{1}{2};\ l-k+1;\ -a^2) \quad (54)$$

(T.B.F. 410).

The necessary conditions for convergence are $l \pm m + \frac{1}{2} > 0$, and $|I(a)| < 1$, since then $R(a^2+1) > 0$. The condition $l-k > -1$ may now be removed. In particular, with $k = l+s$, where s is a positive integer or zero, we find

$$\int_0^\infty x^{l-1} e^{-(a^2+\frac{1}{2})x} W_{l+s,m}(x)\, dx = (-)^s\, 2^{-(2l+s-1)} \int_0^\infty w^{2l+s} K_{2m}(w) J_s(aw)\, a^s\, dw$$

$$= (-a^2)^s \frac{\Gamma(s+l+m+\frac{1}{2})\,\Gamma(s+l-m+\frac{1}{2})}{\Gamma(s+1)}$$

$$\times {}_2F_1(s+l+m+\tfrac{1}{2},\ s+l-m+\tfrac{1}{2};\ s+1;\ -a^2)$$

$$(l \pm m + \tfrac{1}{2} > 0,\ |I(a)| < 1). \quad (55)$$

If, further, we take $l-m+\frac{1}{2} = 1$, we have the formula

$$\int_0^\infty x^{m-\frac{1}{2}} e^{-(a^2+\frac{1}{2})x} W_{s+m+\frac{1}{2},m}(x)\, dx = (-a^2)^s\, \Gamma(s+2m+1)(1+a^2)^{-(s+2m+1)}$$

$$(m > -\tfrac{1}{2}). \quad (56)$$

4.1. The special case in which $m = -\frac{1}{4}$ will give us formulae involving the parabolic cylinder function; but it is, in general, simpler to consider this case separately.

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* *Cf.* Jeffreys, *loc. cit.*, 95; van der Pol, *loc. cit.*, 878.

If we put $$y = \sqrt{(2x)} = r/\sqrt{(2t)}, \tag{57}$$

we have* from (26),

$$(qr)^{\mu} e^{-qr} = \left(\frac{2}{\pi}\right)^{\frac{1}{2}} y^{\mu} e^{-\frac{1}{4}y^2} D_{\mu-1}(y). \tag{58}$$

First, Bromwich's integral gives

$$\left(\frac{2}{\pi}\right)^{\frac{1}{2}} y^{\mu} e^{-\frac{1}{4}y^2} D_{\mu-1}(y) = \frac{1}{2\pi i} \int_M \frac{e^{\lambda t}}{\lambda} (r\lambda)^{\frac{1}{2}\mu} e^{-\lambda^{\frac{1}{2}} r}\, d\lambda,$$

whence, on putting $\lambda = e^{\pm i\pi} u^2/(2t)$ on the negative real axis, we find that

$$e^{-\frac{1}{4}y^2} D_{\mu-1}(y) = \left(\frac{2}{\pi}\right)^{\frac{1}{2}} \int_0^{\infty} e^{-\frac{1}{2}u^2} u^{m-1} \sin(\tfrac{1}{2}m\pi - uy)\, du, \quad (m>0), \tag{59}$$

a result ascribed to Adamoff for integral values of m (M.A., 353, ex. 16).

4.2. Recurrence formulae can be found in the same way as before. Differentiation of (58) with respect to r, after division by r^{μ}, leads to the formula

$$\frac{d}{dy}\left[e^{-\frac{1}{4}y^2} D_{\mu-1}(y)\right] = -e^{-\frac{1}{4}y^2} D_{\mu}(y), \tag{60}$$

whence $$D'_{\mu-1}(y) - \tfrac{1}{2}y D_{\mu-1}(y) + D_{\mu}(y) = 0 \tag{61}$$

(a known formula); and, when n is integral,

$$e^{-\frac{1}{4}y^2} D_{\mu}(y) = (-)^n \frac{d^n}{dy^n}\left[e^{-\frac{1}{4}y^2} D_{\mu-n}(y)\right], \tag{62}$$

so that $$e^{-\frac{1}{4}y^2} D_n(y) = (-)^n \frac{d^n}{dy^n}\left(e^{-\frac{1}{2}y^2}\right). \tag{63}$$

(Hermite's formula.)

The application of (18) leads to the result

$$\frac{d}{dt}\left\{(2t)^{-\frac{1}{2}\mu} e^{-\frac{1}{4}y^2} D_{\mu-1}(y)\right\} = (2t)^{-\frac{1}{2}\mu-1} e^{-\frac{1}{4}y^2} D_{\mu+1}(y), \tag{64}$$

whence $$y^{\mu-1} e^{-\frac{1}{4}y^2} D_{\mu+1}(y) = -\frac{d}{dy}\left[y^{\mu} e^{-\frac{1}{4}y^2} D_{\mu-1}(y)\right], \tag{65}$$

which, in conjunction with (60), gives the known formula

$$D_{\mu+1}(y) = y D_{\mu}(y) - \mu D_{\mu-1}(y). \tag{66}$$

* For the special case in which μ is a positive integer, *cf.* Doetsch, *Math. Zeitschrift*, 32 (1930), 595, equation (9).

4.3. By a use of Taylor's theorem or the formula $e^{-\lambda p}\phi(p) = f(t-\lambda)$ for $t > \lambda$, similar to that in § 3.3, combined with (64), we have

$$\{2(t-\lambda)\}^{-\frac{1}{2}\mu} e^{-r^2/8(t-\lambda)} D_{\mu-1}\left\{\frac{r}{\sqrt{(2t-2\lambda)}}\right\} = \sum_{n=0}^{\infty} \frac{(-\lambda)^n}{n!} (2t)^{-\frac{1}{2}\mu-n} e^{-\frac{1}{4}y^2} D_{\mu+2n-1}(y)$$

for $\lambda < t$. If we put $(1-\lambda/t)^{\frac{1}{2}} = u$,

with u positive, this last equation reduces to*

$$u^{-\mu} e^{-\frac{1}{4}y^2/u^2} D_{\mu-1}\left(\frac{y}{u}\right) = e^{-\frac{1}{4}y^2} \sum_{n=0}^{\infty} \frac{(u^2-1)^n}{2^n n!} D_{\mu+2n-1}(y). \tag{67}$$

We may also use Taylor's theorem in conjunction with (62). We thus obtain the equation

$$e^{-\frac{1}{4}(y+\eta)^2} D_\mu(y+\eta) = e^{-\frac{1}{4}y^2} \sum_{n=0}^{\infty} (-)^n \frac{\eta^n}{n!} D_{\mu+n}(y), \tag{68}$$

an equation given by Hermite for $\mu = 0$ as the definition of the Hermite polynomial.

Operationally, the equation is found by writing

$$q^\mu e^{-q(r+\rho)} = e^{-qr} \sum_{n=0}^{\infty} (-)^n \frac{q^{\mu+n}\rho^n}{n!}$$

and interpreting term by term.

4.4. From Carson's integral (3) we have

$$\left(\frac{2}{\pi}\right)^{\frac{1}{2}} \int_0^\infty y^{\mu-1} D_{\mu+1}(y) \exp\left\{-\tfrac{1}{2}\frac{\eta^2}{y^2} - \tfrac{1}{4}y^2\right\} dy = \eta^\mu e^{-\eta}, \tag{69}$$

provided that $|\arg \eta| < \frac{1}{4}\pi$.

4.5. In (6) take

$$\left.\begin{aligned} \phi_1(p) &= (qr)^\mu e^{-qr}, \quad f_1(t) = \left(\frac{2}{\pi}\right)^{\frac{1}{2}} y^\mu e^{-\frac{1}{4}y^2} D_{\mu-1}(y), \\ \phi_2(p) &= (qr)^{\nu-\mu+2}, \quad f_2(t) = \frac{2^{\frac{1}{2}\nu-\frac{1}{2}\mu+1} y^{\nu-\mu+2}}{\Gamma(\frac{1}{2}\mu-\frac{1}{2}\nu)} \quad (\mu > \nu). \end{aligned}\right\}$$

Then

$$2^{\frac{1}{2}(\mu-\nu-2)} \Gamma(\tfrac{1}{2}\mu-\tfrac{1}{2}\nu)\, y^{\mu-1} e^{-\frac{1}{4}y^2} D_\nu(y) = \int_y^\infty u^\nu (u^2-y^2)^{\frac{1}{2}(\mu-\nu-2)} e^{-\frac{1}{4}u^2} D_\mu(u)\, du. \tag{70}$$

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* For the special case $\mu = 1$, *cf.* Doetsch, *loc. cit.*, 590, equation (7).

More generally, by taking

$$\phi_1(p) = (qr)^{\mu+1} e^{-qr}, \quad \phi_2(p) = (qr)^{\nu+1} e^{-q\alpha r},$$

we find that*

$$y^{\mu-1} e^{-\frac{1}{4}y^2(1+\alpha)^2} D_{\mu+\nu-1}\{y(1+\alpha)\}$$

$$= \left(\frac{2}{\pi}\right)^{\frac{1}{2}} \int_y^\infty u^{\mu+\nu-1} \exp\left\{-\tfrac{1}{4}u^2\left(1+\frac{a^2y^2}{u^2-y^2}\right)\right\}$$

$$\times (u^2-y^2)^{-\frac{1}{2}(\nu+1)} D_\mu(u)\, D_\nu\left(\frac{auy}{\sqrt{(u^2-y^2)}}\right) du. \quad (71)$$

In particular, if $\nu = 0$, $\alpha = 1$,

$$y^{\mu-1} e^{-y^2} D_{\mu-1}(2y) = \left(\frac{\pi}{2}\right)^{\frac{1}{2}} \int_y^\infty u^{\mu-1} \exp\left\{-\tfrac{1}{4}u^2\left(\frac{u^2+y^2}{u^2-y^2}\right)\right\} (u^2-y^2)^{-\frac{1}{2}} D_\mu(u)\, du. \quad (72)$$

4.6. With

$$\left.\begin{aligned} &\phi(p) = \left(\frac{\pi}{2}\right)^{\frac{1}{2}} (qr)^\mu e^{-qr}, && f(t) = y^\mu e^{-\frac{1}{4}y^2} D_{\mu-1}(y), \\ &\psi(p) = \left(\frac{r^2}{2p}\right)^{\frac{1}{2}(\nu-\mu)} e^{(\frac{1}{2}-\alpha)r^2/2p}, && g(t) = \left(\frac{r^2 t}{2\alpha-1}\right)^{\frac{1}{4}(\nu-\mu}} J_{\frac{1}{2}(\nu-\mu)}\{rt^{\frac{1}{2}}(2\alpha-1)^{\frac{1}{2}}\} \\ &&& (\nu-\mu+2>0), \end{aligned}\right\}$$

we have, from (8), with $\nu > 0$, $R(\alpha) > 0$, $|\arg a| < \frac{1}{2}\pi$,

$$\int_0^\infty y^{\nu-1} e^{(\frac{1}{4}-\alpha)y^2} D_{\mu-1}(y)\, dy$$

$$= \left(\frac{\pi}{2}\right)^{\frac{1}{2}} \int_0^\infty (vr)^\mu e^{-vr} \left(\frac{r^2v^2}{2a-1}\right)^{\frac{1}{4}(\nu-\mu)} J_{\frac{1}{2}(\nu-\mu)}\{rv(2a-1)^{\frac{1}{2}}\} \frac{dv}{v}$$

$$= \left(\frac{\pi}{2}\right)^{\frac{1}{2}} (2a-1)^{-\frac{1}{4}(\nu-\mu)} \int_0^\infty e^{-w} w^{\frac{1}{2}\nu+\frac{1}{2}\mu-1} J_{\frac{1}{2}(\nu-\mu)}\{w(2a-1)^{\frac{1}{2}}\} dw$$

$$= \frac{\pi^{\frac{1}{2}} 2^{\frac{1}{2}\mu-\nu-\frac{1}{2}} \Gamma(\nu)}{a^{\frac{1}{2}\nu} \Gamma(\frac{1}{2}\nu-\frac{1}{2}\mu+1)} {}_2F_1\left(\frac{\nu}{2}, \frac{1-\mu}{2}; \frac{\nu-\mu}{2}+1; 1-\frac{1}{2a}\right). \quad (73)$$

(T.B.F., 385.) The result agrees with that given by Watson, *Proc. London Math. Soc.* (2), 8 (1910), 397–400. The result can be written in various forms by transformations of the hypergeometric functions, and can also be expressed in terms of Legendre functions (T.B.F., 387).

* When μ and ν are positive integers, *cf.* Doetsch, *loc. cit.*, 595, equation (J).

By this means we arrive at the formula

$$\int_0^\infty y^{\nu-1} e^{(\frac{1}{4}-\alpha)y^2} D_{\mu-1}(y)\,dy = \left(\frac{\pi}{2}\right)^{\frac{1}{2}} \frac{\Gamma(\nu)\, P^{-\frac{1}{2}(\nu-\mu)}_{\frac{1}{2}\nu+\frac{1}{2}\mu-1}\{1/\sqrt{(2\alpha)}\}}{(2\alpha)^{\frac{1}{4}\nu+\frac{1}{4}\mu}(2\alpha-1)^{\frac{1}{4}\nu-\frac{1}{4}\mu}}, \qquad (74)$$

which generalizes a result due to Curzon, *Proc. London Math. Soc.* (2), 12 (1913), 236–259.

The restriction $\nu-\mu+2>0$ may now be removed. In particular, if $\mu=\nu+2n$, we have

$$\int_0^\infty y^{\nu-1} e^{(\frac{1}{4}-\alpha)y^2} D_{\nu+2n-1}(y)\,dy$$

$$= (-)^n (\tfrac{1}{2}\pi)^{\frac{1}{2}} (2\alpha-1)^{\frac{1}{2}n} \int_0^\infty e^{-w} w^{\nu+n-1} J_n\{w(2\alpha-1)^{\frac{1}{2}}\}\,dw$$

$$= \frac{(1-2\alpha)^n (\frac{1}{2}\pi)^{\frac{1}{2}} \Gamma(2n+\nu)}{2^n (2\alpha)^{n+\frac{1}{2}\nu}\,\Gamma(n+1)}\, {}_2F_1\left(n+\tfrac{1}{2}\nu,\ \tfrac{1}{2}-\tfrac{1}{2}\nu;\ n+1,\ 1-\frac{1}{2\alpha}\right) \qquad (75)$$

$$(\nu>0,\ |\arg\alpha|<\tfrac{1}{2}\pi).$$

With $\nu=1$ this gives

$$\int_0^\infty e^{(\frac{1}{4}-\alpha)y^2} D_{2n}(y)\,dy = (\tfrac{1}{2}\pi)^{\frac{1}{2}} \frac{(2n)!}{2^n . n!} (2\alpha)^{-\frac{1}{2}} \left(\frac{1}{2\alpha}-1\right)^n. \qquad (76)$$

[*Cf.* Doetsch, *loc. cit.*, 591, eq. (A).] With $\nu=2$ we get similarly

$$\int_0^\infty y e^{(\frac{1}{4}-\alpha)y^2} D_{2n+1}(y)\,dy = (\tfrac{1}{2}\pi)^{\frac{1}{2}} \frac{(2n+1)!}{2^n . n!} (2\alpha)^{-\frac{3}{2}} \left(\frac{1}{2\alpha}-1\right)^n. \qquad (77)$$

Equations (76) and (77) follow from (60) on integrating with respect to y from 0 to ∞ after multiplication by $e^{\frac{1}{4}y^2} D_{\mu+2n-1}(y)$ and taking $u=(2\alpha)^{-\frac{1}{2}}$, $\mu=1$ or 2 respectively (M.A., 351).

Two particular cases of (73) may also be noted. We have

$$\int_0^\infty y^\nu e^{(\frac{1}{4}-\alpha)y^2} D_{-\nu}(y)\,dy = (\tfrac{1}{2}\pi)^{\frac{1}{2}} (2\alpha-1)^{-\frac{1}{2}\nu} \int_0^\infty e^{-w} J_\nu\{w(2\alpha-1)^{\frac{1}{2}}\}\,dw$$

$$= (\tfrac{1}{2}\pi)^{\frac{1}{2}} \frac{\{\sqrt{(2\alpha)}-1\}^\nu}{(2\alpha-1)^\nu \sqrt{(2\alpha)}} \quad (\nu>-1,\ |\arg\alpha|<\tfrac{1}{2}\pi) \qquad (78)$$

(T.B.F., 386), and

$$\int_0^\infty y^\nu e^{(\frac{1}{4}-\alpha)y^2} D_{-(\nu+2)}(y)\,dy = (\tfrac{1}{2}\pi)^{\frac{1}{2}} (2\alpha-1)^{-\frac{1}{2}(\nu+1)} \int_0^\infty e^{-w} J_{(\nu+1)}\{w(2\alpha-1)^{\frac{1}{2}}\} \frac{dw}{w}$$

$$= (\tfrac{1}{2}\pi)^{\frac{1}{2}} \frac{\{\sqrt{(2\alpha)}-1\}^{\nu+1}}{(\nu+1)(2\alpha-1)^{\nu+1}} \quad (\nu>-1,\ |\arg\alpha|<\tfrac{1}{2}\pi). \qquad (79)$$

(T.B.F., 386.)

5.1. The integrals on the left in (54), (55), (56), (73)–(79) are of Carson's type, and new operational representations can be derived from them, from which new equations may be deduced. Thus from (54) it follows that, with $l \pm m + \frac{1}{2} > 0$,

$$t^{l-1} W_{k,m}(t) = \frac{\Gamma(l+m+\frac{1}{2})\,\Gamma(l-m+\frac{1}{2})}{\Gamma(l-k+1)}\, p \times {}_2F_1(l+m+\tfrac{1}{2},\ l-m+\tfrac{1}{2};\ l-k+1;\ \tfrac{1}{2}-p), \quad (80)$$

while, if we write $-p$ instead of $\frac{1}{2}-p$ in the hypergeometric function on the right, we must multiply the left by $e^{-\frac{1}{2}t}$.

Some of the simpler cases may be noted. Thus from (56) we have, when s is a positive integer or zero*,

$$t^{m-\frac{1}{2}} e^{\frac{1}{2}t} W_{s+m+\frac{1}{2},m}(t) = \Gamma(2m+s+1)\, p^{-2m}\left(\frac{1}{p}-1\right)^s \quad (m > -\tfrac{1}{2}). \quad (81)$$

For $m = 0$ this reduces to

$$L_s(t) = s!\left(1-\frac{1}{p}\right)^s,$$

a formula, obtainable by elementary means from (20), which was given by van der Pol, *loc. cit.*, 892. [In *Phil. Mag.*, 13 (1932), 537–577, van der Pol and Niessen include some formulae involving Laguerre polynomials. They give, for example, the special case of (83) below.]

From (81) it follows that

$$t^{m-\frac{1}{2}} e^{\frac{1}{2}t} \sum_{s=0}^{\infty} \frac{W_{s+m+\frac{1}{2},m}(t)}{\Gamma(2m+s+1)\,\Gamma(s+1)} = \frac{1}{e}\, p^{-2m} e^{1/p} = \frac{1}{e}\, t^m I_{2m}(2t^{\frac{1}{2}})$$

(Jeffreys, *loc. cit.*, 95), or

$$\frac{1}{e}\, t^{\frac{1}{2}} e^{-\frac{1}{2}t} I_{2m}(2t^{\frac{1}{2}}) = \sum_{s=0}^{\infty} \frac{W_{s+m+\frac{1}{2},m}(t)}{\Gamma(2m+s+1)\,\Gamma(s+1)}. \quad (82)$$

---

* The application of Bromwich's integral to this formula gives us simply a special case of Whittaker's definition of $W_{k,m}$ (M.A., 339). We have

$$W_{k,m}(t) = W_{k,-m}(t) = -\frac{1}{2\pi i}\,\Gamma(k+\tfrac{1}{2}+m)\, e^{-\frac{1}{2}t} t^k \int_{\infty}^{(0+)} (-u)^{-k-\frac{1}{2}-m}\left(1+\frac{u}{t}\right)^{k-\frac{1}{2}-m} e^{-u}\, du,$$

where $u = -t$ is outside the contour. On putting $\lambda = -u/t$, $k = s+m+\frac{1}{2}$, we get the Bromwich integral equivalent to (81). But, if $k-\frac{1}{2} \pm m$ is not zero or a positive integer, Whittaker's integral will not lead to an operational representation of $W_{k,m}$, since, after transformation into an integral of Bromwich's type, the integrand will have a singularity on the right of the path of integration, as defined in (4).

Similarly

$$et^{\frac{1}{2}}e^{-\frac{1}{2}t}J_{2m}(2t^{\frac{1}{2}}) = \sum_{s=0}^{\infty}(-)^s \frac{W_{s+m+\frac{1}{2},m}(t)}{\Gamma(2m+s+1)\,\Gamma(s+1)}. \tag{83}$$

The justification of the processes involved, by integrating an infinite series term by term along the contour M (p. 106), is simple, and the expansions hold for all real values of m.

5.2. These formulae lead to the consideration of expansions of the form

$$f(t) = \sum_{s=0}^{\infty} a_s W_{s+m+\frac{1}{2},m}(t). \tag{84}$$

It follows from the differential equation in the usual way that*

$$\int_0^{\infty} W_{s+m+\frac{1}{2},m}(t)\, W_{r+m+\frac{1}{2},m}(t)\, \frac{dt}{t} = 0 \quad (r \neq s,\ m > -\tfrac{1}{2}). \tag{85}$$

To find the value of the integral when $r = s$, we use (34), and the recurrence formula

$$tW'_{k,m}(t) - (k-\tfrac{1}{2}t)\, W_{k,m}(t) = (k-\tfrac{1}{2}+m)(k-\tfrac{1}{2}-m)\, W_{k-1,m}(t), \tag{86}$$

which follows from (30), (34), and (35), and is given explicitly in M.A., 352, ex. 3. Then, with $k = s+m+\frac{1}{2}$, and $m > -\frac{1}{2}$, we have

$$\begin{aligned}
I_s &= \int_0^{\infty} \{W_{k,m}(t)\}^2 \frac{dt}{t} \\
&= -\int_0^{\infty} W_{k,m}(t) \left\{ W'_{k-1,m}(t) + \left(\frac{k-1}{t} - \tfrac{1}{2}\right) W_{k-1,m}(t) \right\} dt \\
&= -\Big[W_{k,m}(t)\, W_{k-1,m}(t) \Big]_0^{\infty} + \int_0^{\infty} W_{k,m}(t)\, W_{k-1,m}(t)\, \frac{dt}{t} \\
&\qquad + \int_0^{\infty} W_{k-1,m}(t) \left\{ W'_{k,m}(t) - \left(\frac{k}{t} - \tfrac{1}{2}\right) W_{k,m}(t) \right\} dt \\
&= (k-\tfrac{1}{2}+m)(k-\tfrac{1}{2}-m) \int_0^{\infty} \{W_{k-1,m}(t)\}^2 \frac{dt}{t} \\
&= s(2m+s)\, I_{s-1},
\end{aligned}$$

and

$$I_0 = \int_0^{\infty} e^{-t} t^{2m}\, dt = \Gamma(2m+1),$$

* See Whittaker, *Bull. Amer. Math. Soc.*, 10 (1904), 133.

so that

$$I_s = \int_0^\infty \{W_{s+m+\frac{1}{2},\,m}(t)\}^2 \frac{dt}{t} = \Gamma(2m+s+1)\,\Gamma(s+1). \tag{87}$$

[When $2m$ is integral, (87) and also (100)–(103) below are known. See the general formula given by Schrödinger, *loc. cit.*, 486, equation (115), or, for example, Sommerfeld, *loc. cit.*, 185, 186, 76, and 77.]

Hence, if we assume that the expansion (84) exists, and that it is legitimate to integrate term by term between the limits 0 and ∞ after multiplication by $t^{-1} W_{s+m+\frac{1}{2},\,m}(t)$, the coefficients are given by

$$a_s = \frac{1}{\Gamma(2m+s+1)\,\Gamma(s+1)} \int_0^\infty f(t)\, W_{s+m+\frac{1}{2},\,m}(t)\, \frac{dt}{t}. \tag{88}$$

5.3. It follows that (82) and (83) give the formulae

$$\int_0^\infty t^{-\frac{1}{2}} e^{-\frac{1}{2}t} I_{2m}(2t^{\frac{1}{2}})\, W_{s+m+\frac{1}{2},\,m}(t)\, dt = e \quad (m > -\tfrac{1}{2}), \tag{89}$$

$$\int_0^\infty t^{-\frac{1}{2}} e^{-\frac{1}{2}t} J_{2m}(2t^{\frac{1}{2}})\, W_{s+m+\frac{1}{2},\,m}(t)\, dt = (-)^s/e \quad (m > -\tfrac{1}{2}). \tag{90}$$

These results are verified and generalized by expanding the Bessel functions in power series, and integrating term by term, using (54) with $\alpha^2 = 0$. Thus

$$\int_0^\infty t^{l-1} e^{-\frac{1}{2}t} I_\nu(2at^{\frac{1}{2}})\, W_{k,\,m}(t)\, dt$$
$$= \frac{\Gamma(\frac{1}{2}\nu+l+m+\frac{1}{2})\,\Gamma(\frac{1}{2}\nu+l-m+\frac{1}{2})}{\Gamma(\nu+1)\,\Gamma(\frac{1}{2}\nu+l-k+1)}\, a^\nu$$
$$\times {}_2F_2(\tfrac{1}{2}\nu+l+m+\tfrac{1}{2},\ \tfrac{1}{2}\nu+l-m+\tfrac{1}{2};\ \nu+1,\ \tfrac{1}{2}\nu+l-k+1;\ a^2) \tag{91}$$
$$(l+\tfrac{1}{2}\nu-m+\tfrac{1}{2} > 0).$$

We have, then, two methods of evaluating integrals of the form

$$\int_0^\infty f(t)\, W_{k,\,m}(t)\, dt. \tag{92}$$

We may expand $e^{\frac{1}{2}t} f(t)$ in series, and integrate term by term, using (54) with $\alpha^2 = 0$. If, however, $k = s+m+\frac{1}{2}$, and $m > -\frac{1}{2}$, we have, by (88),

$$\int_0^\infty f(t)\, W_{s+m+\frac{1}{2},\,m}(t)\, dt = s!\, b_s, \tag{93}$$

where

$$t\, f(t) = \sum_{s=0}^\infty \frac{b_s}{\Gamma(2m+s+1)}\, W_{s+m+\frac{1}{2},\,m}(t), \tag{94}$$

so that

$$t^{m+\frac{1}{2}} e^{\frac{1}{2}t} f(t) = \sum_{s=0}^{\infty} \frac{b_s}{\Gamma(2m+s+1)} t^{m-\frac{1}{2}} e^{\frac{1}{2}t} W_{s+m+\frac{1}{2},m}(t)$$

$$= p^{-2m} \sum_{s=0}^{\infty} b_s \left(\frac{1}{p} - 1\right)^s$$

from (81). Hence, if

$$t^{m+\frac{1}{2}} e^{\frac{1}{2}t} f(t) = \phi(p), \tag{95}$$

then

$$p^{2m} \phi(p) = \sum_{s=0}^{\infty} b_s \left(\frac{1}{p} - 1\right)^s, \tag{96}$$

or b_s is the coefficient of ξ^s in the expansion of

$$(1+\xi)^{-2m} \phi\left(\frac{1}{1+\xi}\right), \tag{97}$$

in powers of ξ.

5.31. A simple example may be given in illustration, though we may remark that this, and a good many other integrals required in wave-mechanics, are special cases of (54) or (55). Consider

$$S = -\left(\frac{n}{2}\right)^5 \int_0^{\infty} t^4 L_{n+1}^3(t) e^{-\frac{1}{2}(n+1)t} dt$$

$$= (-)^n \left(\frac{n}{2}\right)^5 \frac{(n+1)!}{(n-2)!} \int_0^{\infty} t^2 e^{-\frac{1}{2}nt} W_{n,\frac{3}{2}}(t)\, dt.$$

(Sommerfeld, *loc. cit.*, 94.)

Here

$$t^{m+\frac{1}{2}} e^{\frac{1}{2}t} f(t) = t^4 e^{-\frac{1}{2}(n-1)t}; \quad \phi(p) = \frac{24p}{(p+\gamma)^5},$$

where

$$\gamma = \tfrac{1}{2}(n-1), \quad 1+\gamma = \tfrac{1}{2}(n+1).$$

The integral is, therefore, $(n-2)!\ b_{n-2}$, where b_{n-2} is the coefficient of ξ^{n-2} in the expansion of

$$\frac{24(1+\xi)}{(1+\gamma)^5} \left\{1 + \frac{\gamma\xi}{1+\gamma}\right\}^{-5}.$$

Hence

$$S = 2n^7 (n+1)! \left(\frac{n-1}{n+1}\right)^{n-2}.$$

5.32. If $\phi(p)$ is the operational representation of $f(t)$, then the operational representation of $t^r f(t)$ is

$$p\left(-\frac{d}{dp}\right)^r \left\{\frac{\phi(p)}{p}\right\}^*. \tag{98}$$

* Van der Pol, *loc. cit.*, 863.

Hence, by Leibniz's theorem, if l is a positive integer, or zero,

$$t^{m+l-\frac{1}{2}} e^{\frac{1}{2}t} W_{s+m+\frac{1}{2},m}(t)$$

$$= p^{-(2m+l)} \left\{ \Gamma(2m+s+l+1) \left(\frac{1}{p}-1\right)^s \right.$$

$$+ \frac{l.s}{1!} \Gamma(2m+s+l) \left(\frac{1}{p}-1\right)^{s-1}$$

$$+ \frac{l(l-1)s(s-1)}{2!} \Gamma(2m+s+l-1) \left(1-\frac{1}{p}\right)^{s-2}$$

$$\left. + \frac{l(l-1)(l-2)s(s-1)(s-2)}{3!} \Gamma(2m+s+l-2) \left(1-\frac{1}{p}\right)^{s-3} + \ldots \right\}. \tag{99}$$

Thus, if r, s, $l+m-n$ are positive integers or zero, and $l+m+n+1>0$, then

$$\frac{1}{s!} \int_0^\infty t^{l-1} W_{r+n+\frac{1}{2},n}(t) W_{s+m+\frac{1}{2},m}(t)\, dt \tag{100}$$

is the coefficient of ξ^s in the expansion of

$$(1+\xi)^{l-m+n} \left\{ \Gamma(l+m+n+r+1) \xi^r + \frac{(l+m-n)r}{1!} \Gamma(l+m+n+r) \xi^{r-1} \right.$$

$$\left. + \frac{(l+m-n)(l+m-n-1)r(r-1)}{2!} \Gamma(l+m+n+r-1) \xi^{r-2} + \ldots \right\}. \tag{101}$$

and the restrictions $m>-\frac{1}{2}$, $n>-\frac{1}{2}$ may be removed. In particular, if s and l are positive integers or zero, and $l+2m+1>0$, then

$$\frac{1}{s!} \int_0^\infty t^{l-1} \{W_{s+m+\frac{1}{2},m}(t)\}^2 dt$$

$$= \Gamma(l+2m+s+1) + \frac{l^2 s}{1!} \Gamma(l+2m+s) + \frac{l^2(l-1)^2}{(2!)^2} s(s-1) \Gamma(l+2m+s-1)$$

$$+ \frac{l^2(l-1)^2(l-2)^2}{(3!)^2} s(s-1)(s-2) \Gamma(l+2m+s-2) + \ldots. \tag{102}$$

When $l=1$, the result is most easily obtained by writing the recurrence formula (86) in the form

$$W_{s+m+\frac{1}{2},m}(t)$$

$$= \frac{2s+2m+1}{t} W_{s+m+\frac{1}{2},m}(t) - 2W'_{s+m+\frac{1}{2},m}(t) + \frac{s(s+2m)}{t} W_{s+m-\frac{1}{2},m}(t).$$

Hence

$$\frac{1}{s!}\int_0^\infty \{W_{s+m+\frac{1}{2},m}(t)\}^2\,dt = \frac{(2s+2m+1)}{s!}\int_0^\infty \{W_{s+m+\frac{1}{2},m}(t)\}^2\,\frac{d}{dt}$$

$$= (2s+2m+1)\,\Gamma(2m+s+1) \tag{103}$$

for $m > -\frac{1}{2}$, and the result is extended to hold for $m > -1$ by analytic continuation.

6.1. Simple cases of formulae for the parabolic cylinder functions may be briefly set out. From (76) and (77) we get, with n a positive integer or zero,

$$\left.\begin{aligned} &t^{-\frac{1}{2}}e^{\frac{1}{2}t}D_{2n}\{\sqrt{(2t)}\} = \sqrt{\pi}\,\frac{(2n)!}{2^n n!}\,p^{\frac{1}{2}}\left(\frac{1}{p}-1\right)^n,\\ \text{and}\qquad &e^{\frac{1}{2}t}D_{2n+1}\{\sqrt{(2t)}\} = \sqrt{(\tfrac{1}{2}\pi)}\,\frac{(2n+1)!}{2^n n!}\,p^{-\frac{1}{2}}\left(\frac{1}{p}-1\right)^n. \end{aligned}\right\} \tag{104}$$

Hence

$$e^{\frac{1}{2}t}t^{-\frac{1}{2}}\sum_{n=0}^{\infty}\frac{2^n}{(2n)!}D_{2n}\{\sqrt{(2t)}\} = \frac{1}{e}\sqrt{\pi}\,p^{\frac{1}{2}}e^{1/p}$$

$$= \frac{1}{e}\sqrt{\pi}\,t^{-\frac{1}{4}}I_{-\frac{1}{2}}(2t^{\frac{1}{2}}) = \frac{1}{e}\,t^{-\frac{1}{2}}\cosh(2t^{\frac{1}{2}}),$$

or

$$\sum_{n=0}^{\infty}\frac{2^n}{(2n)!}D_{2n}(x) = e^{-\frac{1}{4}x^2-1}\cosh(2^{\frac{1}{2}}x). \tag{105}$$

Similarly

$$\sum_{n=0}^{\infty}\frac{(-2)^n}{(2n)!}D_{2n}(x) = e^{-\frac{1}{4}x^2+1}\cos(2^{\frac{1}{2}}x), \tag{106}$$

$$\sum_{n=0}^{\infty}\frac{2^n D_{2n+1}(x)}{(2n+1)!} = 2^{-\frac{1}{2}}e^{-\frac{1}{4}x^2-1}\sinh(2^{\frac{1}{2}}x), \tag{107}$$

$$\sum_{n=0}^{\infty}\frac{(-2)^n D_{2n+1}(x)}{(2n+1)!} = 2^{-\frac{1}{2}}e^{-\frac{1}{4}x^2+1}\sin(2^{\frac{1}{2}}x). \tag{108}$$

(*Cf.* Doetsch, *loc. cit.*, 592, 593, and van der Pol and Niessen, *loc. cit.*, 584.) Hence

$$\frac{1}{e}\left(\frac{2}{\pi}\right)^{\frac{1}{2}}\int_0^\infty e^{-\frac{1}{4}x^2}\cosh(2^{\frac{1}{2}}x)\,D_{2n}(x)\,dx$$

$$= (-)^n e\left(\frac{2}{\pi}\right)^{\frac{1}{2}}\int_0^\infty e^{-\frac{1}{4}x^2}\cos(2^{\frac{1}{2}}x)\,D_{2n}(x)\,dx$$

$$= \frac{1}{e\sqrt{\pi}}\int_0^\infty e^{-\frac{1}{4}x^2}\sinh(2^{\frac{1}{2}}x)\,D_{2n+1}(x)\,dx$$

$$= \frac{(-)^n e}{\sqrt{\pi}}\int_0^\infty e^{-\frac{1}{4}x^2}\sin(2^{\frac{1}{2}}x)\,D_{2n+1}(x)\,dx = 2^n. \tag{109}$$

6.2. The deductions from (78) and (79) are more abstruse. We have

$$(2t)^{\frac{1}{2}(\nu-1)}e^{-\frac{1}{2}t}D_{-\nu}\{\sqrt{(2t)}\} = \left(\frac{\pi}{2}\right)^{\frac{1}{2}}\frac{\{\sqrt{(1+p)}-1\}^{\nu}}{p^{\nu-1}\sqrt{(1+p)}} \tag{110}$$

and

$$(2t)^{\frac{1}{2}(\nu-1)}e^{-\frac{1}{2}t}D_{-(\nu+2)}\{\sqrt{(2t)}\} = \left(\frac{\pi}{2}\right)^{\frac{1}{2}}\frac{\{\sqrt{(1+p)}-1\}^{\nu+1}}{(\nu+1)p^{\nu}}. \tag{111}$$

An application of (6) gives us

$$\int_0^t (2\xi)^{\frac{1}{2}(\nu-1)}[2(t-\xi)]^{\frac{1}{2}(\mu-1)}e^{-\frac{1}{2}t}D_{-\nu}\{\sqrt{(2\xi)}\}D_{-(\mu+2)}\{\sqrt{(2t-2\xi)}\}d\xi$$

$$= \frac{\pi}{2}\frac{\{\sqrt{(1+p)}-1\}^{\mu+\nu+1}}{(\mu+1)p^{\mu+\nu}\sqrt{(1+p)}} = \left(\frac{\pi}{2}\right)^{\frac{1}{2}}\frac{1}{\mu+1}(2t)^{\frac{1}{2}(\mu+\nu)}e^{-\frac{1}{2}t}D_{-(\mu+\nu+1)}\{\sqrt{(2t)}\},$$

or

$$\int_0^{\frac{1}{2}\pi} D_{-\nu}(x\sin\theta)D_{-(\mu+2)}(x\cos\theta)\sin^{\nu}\theta\cos^{\mu}\theta\,d\theta$$

$$= \left(\frac{\pi}{2}\right)^{\frac{1}{2}}\frac{1}{\mu+1}D_{-(\mu+\nu+1)}(x) \quad (\mu>-1,\ \nu>-1). \tag{112}$$

THE POLYTOPES WITH REGULAR-PRISMATIC VERTEX FIGURES

PART 2*.

By H. S. M. COXETER.

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Preface.

The idea of the fundamental region of a group is familiar. (A list of references will be found at the end of this paper.) The *orthoscheme*, which Schläfli† associates with a regular polytope, is really a fundamental region for the group of symmetries of the polytope. Dr. J. A. Todd‡ has recently used this fact in order to obtain *abstract definitions* for the groups of symmetries of the regular polytopes.

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* Part 1 of this paper appears in the *Phil. Trans. Royal Soc.* (A), 229 (1930), 329-425. The paragraphing of this part follows on that of Part 1.

† "Theorie der vielfachen Kontinuität", *N. Denkschr. Schweiz. Ges. Natw.*, 38 (1901).

‡ "The groups of symmetries of the regular polytopes", *Proc. Camb. Phil. Soc.*, 27 (1931), 212–231.

On 24 November, 1930, Dr. G. de B. Robinson urged me to seek a fundamental region for the "pure Archimedean" polytope n_{21}. By fitting together three special orthoschemes, I found the required fundamental region, namely a simplex all of whose dihedral angles were either $\frac{1}{2}\pi$ or $\frac{1}{3}\pi$. This result led to an empirical generalization, and to an extension of the Schläfli symbol $\{k_1, k_2, \ldots, k_{m-1}\}$. Afterwards I proved the general result, which can be stated as follows :

Every group of real orthogonal substitutions on m variables, having as fundamental region a simplex all of whose dihedral angles are submultiples of π, is either the whole group of symmetries of some m-dimensional uniform polytope, or a sub-group thereof.

An abstract definition for such a group, and in particular for the group of automorphisms of the twenty-seven lines on a cubic surface or of the twenty-eight bitangents of a plane quartic, can be written down at once.

A preliminary account of this work appears in the *Journal London Math. Soc.*, 6 (1931), 132–136. In the last line but eight of page 134, the words "central projections of" should be inserted after the word "vertices".

I should like to express here my thanks to Dr. Todd and Dr. Robinson for their inspiration and encouragement.

14. *An extension of the Schläfli symbol.*

14.1. At the end of §5.3 we observed that the vertex figure of the polytope $t_n\{k_1, k_2, \ldots, k_{m-1}\}$ is a generalized prism whose two constituents are the vertex figures of

$$\{k_n, k_{n-1}, \ldots, k_1\} \quad \text{and} \quad \{k_{n+1}, k_{n+2}, \ldots, k_{m-1}\}.$$

This fact suggests the new notation

$$t_n\{k_1, k_2, \ldots, k_{m-1}\} = \begin{Bmatrix} k_n, k_{n-1}, \ldots, k_1 \\ k_{n+1}, k_{n+2}, \ldots, k_{m-1} \end{Bmatrix},$$

which is justified by the identities

$$t_0\{k_1, k_2, \ldots, k_{m-1}\} = \{k_1, k_2, \ldots, k_{m-1}\}$$

and

$$t_{m-1}\{k_1, k_2, \ldots, k_{m-1}\} = \{k_{m-1}, k_{m-2}, \ldots, k_1\}.$$

We thus define

$$\begin{Bmatrix} i_0, i_1, \ldots, i_{p-1} \\ j_0, j_1, \ldots, j_{q-1} \end{Bmatrix} = \left[\{i_1, \ldots, i_{p-1}\} 2\cos\frac{\pi}{i_0}, \quad \{j_1, \ldots, j_{q-1}\} 2\cos\frac{\pi}{j_0}\right]^{+1}$$

(in the notation of §7.1).

It follows from this definition that $\begin{Bmatrix} i_0, i_1, \ldots, i_{p-1} \\ j_0, j_1, \ldots, j_{q-1} \end{Bmatrix}$ has in general two kinds of bounding figure:

$$\begin{Bmatrix} i_0, i_1, \ldots, i_{p-2} \\ j_0, j_1, \ldots, j_{q-1} \end{Bmatrix} \quad \text{and} \quad \begin{Bmatrix} i_0, i_1, \ldots, i_{p-1} \\ j_0, j_1, \ldots, j_{q-2} \end{Bmatrix}.$$

Thence it follows that the general element (apart from vertices) is

$$\begin{Bmatrix} i_0, i_1, \ldots, i_{p'-1} \\ j_0, j_1, \ldots, j_{q'-1} \end{Bmatrix} \quad (0 \leqslant p' \leqslant p,\ 0 \leqslant q' \leqslant q).$$

This result is perfectly analogous to the fact that the general element of $\{k_1, k_2, \ldots, k_{m-1}\}$ is $\{k_1, k_2, \ldots, k_{m'-1}\}$ $(1 \leqslant m' \leqslant m)$, and it is easily seen to agree with (5.23).

14.2. Since

$$\{k_1, k_2, \ldots, k_{m-1}\} = \left(\{k_2, \ldots, k_{m-1}\} 2 \cos \frac{\pi}{k_1}\right)^{+1},$$

it is natural to define

$$\begin{Bmatrix} k_1, & i_0, i_1, \ldots, i_{p-1} \\ & j_0, j_1, \ldots, j_{q-1} \end{Bmatrix} = \left(\begin{Bmatrix} i_0, i_1, \ldots, i_{p-1} \\ j_0, j_1, \ldots, j_{q-1} \end{Bmatrix} 2 \cos \frac{\pi}{k_1}\right)^{+1}.$$

The general element (apart from vertices and edges) is now

$$\begin{Bmatrix} k_1, & i_0, i_1, \ldots, i_{p'-1} \\ & j_0, j_1, \ldots, j_{q'-1} \end{Bmatrix} \quad (0 \leqslant p' \leqslant p,\ 0 \leqslant q' \leqslant q).$$

Similarly, we can define inductively

$$\begin{Bmatrix} k_1, k_2, \ldots, k_n, & i_0, i_1, \ldots, i_{p-1} \\ & j_0, j_1, \ldots, j_{q-1} \end{Bmatrix} = \left(\begin{Bmatrix} k_2, \ldots, k_n, & i_0, i_1, \ldots, i_{p-1} \\ & j_0, j_1, \ldots, j_{q-1} \end{Bmatrix} 2 \cos \frac{\pi}{k_1}\right)^{+1}.$$

The general element of not more than n dimensions is simply the general element of $\{k_1, k_2, \ldots, k_n\}$, while the general element of more than n dimensions is

$$\begin{Bmatrix} k_1, k_2, \ldots, k_n, & i_0, i_1, \ldots, i_{p'-1} \\ & j_0, j_1, \ldots, j_{q'-1} \end{Bmatrix} \quad (0 \leqslant p' \leqslant p,\ 0 \leqslant q' \leqslant q).$$

Since $\{i_1, \ldots, i_{p-1}\}$ is p-dimensional, $\begin{Bmatrix} i_0, i_1, \ldots, i_{p-1} \\ j_0, j_1, \ldots, j_{q-1} \end{Bmatrix}$ is $(p+q+1)$-dimensional, whence $\begin{Bmatrix} k_1, k_2, \ldots, k_n, & i_0, i_1, \ldots, i_{p-1} \\ & j_0, j_1, \ldots, j_{q-1} \end{Bmatrix}$ is $(n+p+q+1)$-dimensional. Thus the number of dimensions is one more than the number of digits involved in the symbol.

Note that

$$\left\{k_1, k_2, \ldots, k_n, \begin{matrix} j_0, j_1, \ldots, j_{q-1} \\ i_0, i_1, \ldots, i_{p-1} \end{matrix}\right\} = \left\{k_1, k_2, \ldots, k_n, \begin{matrix} i_0, i_1, \ldots, i_{p-1} \\ j_0, j_1, \ldots, j_{q-1} \end{matrix}\right\}.$$

14.3. Let us now illustrate this notation by giving numerical values to the i's, j's, and k's.

$\left\{\begin{matrix}3\\3\end{matrix}\right\} = t_1\alpha_3 = \beta_3 = \{3, 4\}$, the octahedron, whose vertex figure is $[\alpha_1, \alpha_1] = \beta_2$;

$\left\{\begin{matrix}3\\4\end{matrix}\right\} = t_1\beta_3$, the cuboctahedron ,, ,, ,, $[\alpha_1, \beta_1]$;

$\left\{\begin{matrix}3\\5\end{matrix}\right\} = t_1\{3, 5\}$, the icosidodecahedron ,, ,, ,, $[\alpha_1, \alpha_1\tau]$;

$\left\{\begin{matrix}3\\6\end{matrix}\right\} = t_1\{3, 6\}$,, ,, ,, $[\alpha_1, \alpha_1\sqrt{3}]$;

$\left\{\begin{matrix}4\\4\end{matrix}\right\} = t_1\delta_3 = \delta_3 = \{4, 4\}$, "squared paper" ,, ,, ,, $[\beta_1, \beta_1] = \beta_2\sqrt{2}$.

$\left\{3, \begin{matrix}3\\3\end{matrix}\right\} = \{3, 3, 4\} = \beta_4$,, ,, ,, $\left\{\begin{matrix}3\\3\end{matrix}\right\} = \{3, 4\} = \beta_3$;

$\left\{4, \begin{matrix}3\\3\end{matrix}\right\} = \{4, 3, 4\} = \delta_4$,, ,, ,, $\left\{\begin{matrix}3\\3\end{matrix}\right\}\sqrt{2} = \beta_3\sqrt{2}$;

$\left\{3, \begin{matrix}3\\4\end{matrix}\right\} = h\delta_4 = \alpha_3 h$,, ,, ,, $\left\{\begin{matrix}3\\4\end{matrix}\right\} = t_1\beta_3 = e\alpha_3$;

$\left\{\begin{matrix}3\\3, 3\end{matrix}\right\} = t_1\alpha_4$,, ,, ,, $[\alpha_1, \alpha_2]$;

$\left\{\begin{matrix}3\\3, 4\end{matrix}\right\} = t_1\beta_4 = \{3, 4, 3\}$,, ,, ,, $[\alpha_1, \beta_2] = \gamma_3$;

$\left\{\begin{matrix}4\\3, 3\end{matrix}\right\} = t_1\gamma_4$,, ,, ,, $[\beta_1, \alpha_2]$;

$\left\{\begin{matrix}4\\3, 4\end{matrix}\right\} = t_1\delta_4$,, ,, ,, $[\beta_1, \beta_2]$;

$\left\{\begin{matrix}3\\4, 3\end{matrix}\right\} = t_1\{3, 4, 3\}$,, ,, ,, $[\alpha_1, \alpha_2\sqrt{2}]$;

$\left\{\begin{matrix}3\\3, 5\end{matrix}\right\} = t_1\{3, 3, 5\}$,, ,, ,, $[\alpha_1, \{5\}]$;

$\left\{\begin{matrix}5\\3, 3\end{matrix}\right\} = t_1\{5, 3, 3\}$,, ,, ,, $[\alpha_1\tau, \alpha_2]$.

$$\left\{3,\ 3,\ \begin{matrix}3\\3\end{matrix}\right\} = \{3,\ 3,\ 3,\ 4\} = \beta_5;$$

$$\left\{4,\ 3,\ \begin{matrix}3\\3\end{matrix}\right\} = \{4,\ 3,\ 3,\ 4\} = \delta_5;$$

$$\left\{3,\ \begin{matrix}3\\3,\ 3\end{matrix}\right\} = h\gamma_5;$$

$$\left\{3,\ \begin{matrix}3\\3,\ 4\end{matrix}\right\} = \{3,\ 3,\ 4,\ 3\} = h\delta_5;$$

$$\left\{\begin{matrix}3\\3,\ 3,\ 3\end{matrix}\right\} = t_1\alpha_5; \qquad \left\{\begin{matrix}3,\ 3\\3,\ 3\end{matrix}\right\} = t_2\alpha_5;$$

$$\left\{\begin{matrix}3\\3,\ 3,\ 4\end{matrix}\right\} = t_1\beta_5; \qquad \left\{\begin{matrix}3,\ 3\\3,\ 4\end{matrix}\right\} = t_2\beta_5;$$

$$\left\{\begin{matrix}4\\3,\ 3,\ 3\end{matrix}\right\} = t_1\gamma_5; \quad \left\{\begin{matrix}4\\3,\ 3,\ 4\end{matrix}\right\} = t_1\delta_5;$$

$$\left\{\begin{matrix}3\\3,\ 4,\ 3\end{matrix}\right\} = t_1\{3,\ 3,\ 4,\ 3\} = \{3,\ 4,\ 3,\ 3\} = t_2\delta_5 = \left\{\begin{matrix}3,\ 4\\3,\ 4\end{matrix}\right\};$$

$$\left\{\begin{matrix}3\\4,\ 3,\ 3\end{matrix}\right\} = t_1\{3,\ 4,\ 3,\ 3\}; \qquad \left\{\begin{matrix}3,\ 3\\4,\ 3\end{matrix}\right\} = t_2\{3,\ 3,\ 4,\ 3\}.$$

$$\left\{3^{m-3},\ \begin{matrix}3\\3\end{matrix}\right\}^* = \beta_m; \quad \left\{3,\ \begin{matrix}3\\3^{m-3}\end{matrix}\right\} = h\gamma_m; \quad \left\{4,\ 3^{m-4},\ \begin{matrix}3\\3\end{matrix}\right\} = \delta_m; \quad \left\{3,\ \begin{matrix}3\\3^{m-4},\ 4\end{matrix}\right\} = h\delta_m.$$

$$\left\{\begin{matrix}3^n\\3^{m-n-1}\end{matrix}\right\} = t_n\alpha_m; \quad \left\{\begin{matrix}3^n\\3^{m-n-2},\ 4\end{matrix}\right\} = t_n\beta_m; \quad \left\{\begin{matrix}3^{n-1},\ 4\\3^{m-n-1}\end{matrix}\right\} = t_n\gamma_m; \quad \left\{\begin{matrix}3^{n-1},\ 4\\3^{m-n-2},\ 4\end{matrix}\right\} = t_n\delta_m.$$

$$\left\{3^n,\ \begin{matrix}3^p\\3^q\end{matrix}\right\} = n_{pq}.$$

14.4. The notation can be further extended. By §5.8 with l for n, if Π_m has an $(l+1)$-th vertex figure, then

$$t_l\Pi_m = [\Pi_l',\ \Pi_{m-l-1,\,l+1}]^{+1}.$$

Now let $$\Pi_m = \left\{k_1,\ k_2,\ \ldots,\ k_n,\ \begin{matrix}i_0,\ i_1,\ \ldots,\ i_{p-1}\\j_0,\ j_1,\ \ldots,\ j_{q-1}\end{matrix}\right\}.$$

In this case, if $l \leqslant n$,

Π_l' is the vertex figure of $\{k_l,\ k_{l-1},\ \ldots,\ k_1\}$

---

* 3^n stands for 3, 3, .. , 3, with n 3's.

and

$\Pi_{m-l-1,\,l+1}$ is the vertex figure of $\left\{ k_{l+1}, k_{l+2}, \ldots, k_n, \begin{matrix} i_0, i_1, \ldots, i_{p-1} \\ j_0, j_1, \ldots, j_{q-1} \end{matrix} \right\}$.

Accordingly, it is perfectly analogous to write

$$t_l \left\{ k_1, k_2, \ldots, k_n, \begin{matrix} i_0, i_1, \ldots, i_{p-1} \\ j_0, j_1, \ldots, j_{q-1} \end{matrix} \right\} = \left\{ \begin{matrix} k_l, k_{l-1}, \ldots, k_1 \\ k_{l+1}, k_{l+2}, \ldots, k_n, \end{matrix} \begin{matrix} i_0, i_1, \ldots, i_{p-1} \\ j_0, j_1, \ldots, j_{q-1} \end{matrix} \right\} \quad (14.41)$$

$$(l \leqslant n).$$

The three kinds of bounding figure are obtained by omitting the k_1 or the i_{p-1} or the j_{q-1}, respectively.

In particular,

$$t_{n-1} \left\{ k_1, k_2, \ldots, k_n, \begin{matrix} i_0, i_1, \ldots, i_{p-1} \\ j_0, j_1, \ldots, j_{q-1} \end{matrix} \right\} = \left\{ \begin{matrix} k_{n-1}, k_{n-2}, \ldots, k_1 \\ k_n, \end{matrix} \begin{matrix} i_0, i_1, \ldots, i_{p-1} \\ j_0, j_1, \ldots, j_{q-1} \end{matrix} \right\},$$

and, finally,

$$t_n \left\{ k_1, k_2, \ldots, k_n, \begin{matrix} i_0, i_1, \ldots, i_{p-1} \\ j_0, j_1, \ldots, j_{q-1} \end{matrix} \right\} = \left\{ \begin{matrix} k_n, k_{n-1}, \ldots, k_1 \\ i_0, i_1, \ldots, i_{p-1} \\ j_0, j_1, \ldots, j_{q-1} \end{matrix} \right\}.$$

This last symbol means

$$\left[\{k_{n-1}, \ldots, k_1\} 2 \cos\frac{\pi}{k_n}, \left[\{i_1, \ldots, i_{p-1}\} 2 \cos\frac{\pi}{i_0}, \ \{j_1, \ldots, j_{q-1}\} 2 \cos\frac{\pi}{j_0} \right] \right]^{+1}.$$

By (4.22), the inner square brackets can be removed, and the rows of our three-rowed symbol are permutable. Writing

$$h_r = k_{n-r},$$

we now have

$$t_n \left\{ h_{n-1}, h_{n-2}, \ldots, h_0, \begin{matrix} i_0, i_1, \ldots, i_{p-1} \\ j_0, j_1, \ldots, j_{q-1} \end{matrix} \right\} = \left\{ \begin{matrix} h_0, h_1, \ldots, h_{n-1} \\ i_0, i_1, \ldots, i_{p-1} \\ j_0, j_1, \ldots, j_{q-1} \end{matrix} \right\} \quad (14.42)$$

$$= t_p \left\{ i_{p-1}, i_{p-2}, \ldots, i_0, \begin{matrix} j_0, j_1, \ldots, j_{q-1} \\ h_0, h_1, \ldots, h_{n-1} \end{matrix} \right\}$$

$$= t_q \left\{ j_{q-1}, j_{q-2}, \ldots, j_0, \begin{matrix} h_0, h_1, \ldots, h_{n-1} \\ i_0, i_1, \ldots, i_{p-1} \end{matrix} \right\}.$$

When every h, i, j is equal to 3, these results reduce to (7.35) and (7.36). Thus, in the notation of (12.11),

$$\left\{\begin{matrix} 3^n \\ 3^p \\ 3^q \end{matrix}\right\} = O_{npq}. \tag{14.43}$$

14.5. Since the vertex figure and general element of

$$\left\{\begin{matrix} h_0, h_1, \ldots, h_{n-1} \\ i_0, i_1, \ldots, i_{p-1} \\ j_0, j_1, \ldots, j_{q-1} \end{matrix}\right\} \tag{14.51}$$

are respectively

(14.52)

$$\left[\{h_1, \ldots, h_{n-1}\}\, 2\cos\frac{\pi}{h_0},\ \{i_1, \ldots, i_{p-1}\}\, 2\cos\frac{\pi}{i_0},\ \{j_1, \ldots, j_{q-1}\}\, 2\cos\frac{\pi}{j_0}\right]$$

and

$$\left\{\begin{matrix} h_0, h_1, \ldots, h_{n'-1} \\ i_0, i_1, \ldots, i_{p'-1} \\ j_0, j_1, \ldots, j_{q'-1} \end{matrix}\right\} \quad (0 \leqslant n' \leqslant n,\ 0 \leqslant p' \leqslant p,\ 0 \leqslant q' \leqslant q),$$

the following existence conditions are necessary:

$$\left\{\begin{matrix} h_0, h_1, \ldots, h_{n-2} \\ i_0, i_1, \ldots, i_{p-1} \\ j_0, j_1, \ldots, j_{q-1} \end{matrix}\right\}, \quad \left\{\begin{matrix} h_0, h_1, \ldots, h_{n-1} \\ i_0, i_1, \ldots, i_{p-2} \\ j_0, j_1, \ldots, j_{q-1} \end{matrix}\right\}, \quad \left\{\begin{matrix} h_0, h_1, \ldots, h_{n-1} \\ i_0, i_1, \ldots, i_{p-1} \\ j_0, j_1, \ldots, j_{q-2} \end{matrix}\right\}$$

must all be finite, and the sum of the squared circumradii of the three constituents of the prism (14.52) must not exceed unity. By (2.93), the latter condition is equivalent to

$$\frac{\Delta_{n-1}(h_2, \ldots, h_{n-1})}{\Delta_n(h_1, \ldots, h_{n-1})}\cos^2\frac{\pi}{h_0} + \frac{\Delta_{p-1}(i_2, \ldots, i_{p-1})}{\Delta_p(i_1, \ldots, i_{p-1})}\cos^2\frac{\pi}{i_0} + \frac{\Delta_{q-1}(j_2, \ldots, j_{q-1})}{\Delta_q(j_1, \ldots, j_{q-1})}\cos^2\frac{\pi}{j_0} \leqslant 1,$$

or (subtracting both sides from 3)

$$\frac{\Delta_{n+1}(h_0, h_1, \ldots, h_{n-1})}{\Delta_n(h_1, \ldots, h_{n-1})} + \frac{\Delta_{p+1}(i_0, i_1, \ldots, i_{p-1})}{\Delta_p(i_1, \ldots, i_{p-1})} + \frac{\Delta_{q+1}(j_0, j_1, \ldots, j_{q-1})}{\Delta_q(j_1, \ldots, j_{q-1})} \geqslant 2, \tag{14.53}$$

in virtue of (2.89). As usual, equality indicates degeneracy.

Here we have tacitly assumed that $npq > 0$. The existence of

$$\left\{ \begin{matrix} i_0, i_1, \ldots, i_{p-1} \\ j_0, j_1, \ldots, j_{q-1} \end{matrix} \right\}$$

depends solely on the existence of the regular polytope

$$\{i_{p-1}, i_{p-2}, \ldots, i_0, j_0, j_1, \ldots, j_{q-1}\},$$

since the former polytope is the t_p truncation of the latter.

14.6. In Chapter 17 we shall see that the vertices of

$$\left\{ \begin{matrix} h_{n-1}, h_{n-2}, \ldots, h_0, & i_0, i_1, \ldots, i_{p-1} \\ & j_0, j_1, \ldots, j_{q-1} \end{matrix} \right\} \tag{14.61}$$

are the centres of the bounding $\left\{ \begin{matrix} h_0, h_1, \ldots, h_{n-2} \\ i_0, i_1, \ldots, i_{p-1} \\ j_0, j_1, \ldots, j_{q-1} \end{matrix} \right\}$'s of (14.51), so that the existence of (14.61) will follow from the existence of (14.51). But this fact need not be used here; we shall simply find all possible h's, i's, and j's for which (14.51) exists, and then observe that the corresponding polytopes (14.61) are familiar. By (14.42), there cannot be further values of the h's, i's, j's for which (14.61) exists.

When $h_0 = \ldots = h_{n-1} = i_0 = \ldots = i_{p-1} = j_0 = \ldots = j_{q-1} = 3$, (14.53) becomes

$$\tfrac{1}{2}\left(\frac{n+2}{n+1} + \frac{p+2}{p+1} + \frac{q+2}{q+1}\right) \geqslant 2,$$

or

$$npq \leqslant n+p+q+2, \tag{14.62}$$

which is the same as the existence condition (7.32) for O_{npq}.

Assuming that $npq > 0$ (as we may, by the remark at the end of §14.5), the only digit in the symbol $\left\{ \begin{matrix} 3^n \\ 3^p \\ 3^q \end{matrix} \right\}$ that can be increased is the last in a row. For, by (14.42),

$$\left\{ \begin{matrix} 3^{n-1}, 4 \\ 3 \\ 3 \end{matrix} \right\} = t_n \left\{ \begin{matrix} 4, 3^{n-1}, 3 \\ 3 \end{matrix} \right\} = t_n \delta_{n+3} = t_2 \delta_{n+3}.$$

This being degenerate, we cannot introduce further digits into any row.

$\left\{\begin{matrix} 3^{n-1}, h \\ i \\ j \end{matrix}\right\}$ is impossible if $h > 4$ or $i > 3$, since

$$\frac{\Delta_{n+1}(3, 3, \ldots, 3, h)}{\Delta_n(3, \ldots, 3, h)} + \frac{\Delta_2(i)}{\Delta_1} + \frac{\Delta_2(j)}{\Delta_1}$$

$$= \frac{\{(n+1)/2^n\} - (n/2^{n-1})\cos^2\pi/h}{(n/2^{n-1}) - \{(n-1)/2^{n-2}\}\cos^2\pi/h} + \frac{\sin^2\pi/i}{1} + \frac{\sin^2\pi/j}{1}$$

$$= \frac{1}{2(n-1)}\left(n - \frac{1}{1-(n-1)\cos 2\pi/h}\right) + \sin^2\frac{\pi}{i} + \sin^2\frac{\pi}{j},$$

which decreases when h or i increases.

Thus, if $npq > 0$, there are only two families of polytopes of the kind we are investigating:

(i) The three semi-reciprocals

$$\left\{3^n, \begin{matrix} 3^p \\ 3^q \end{matrix}\right\} = n_{pq}, \quad \left\{3^p, \begin{matrix} 3^q \\ 3^n \end{matrix}\right\} = p_{qn}, \quad \left\{3^q, \begin{matrix} 3^n \\ 3^p \end{matrix}\right\} = q_{np}$$

and their common truncation

$$\left\{\begin{matrix} 3^n \\ 3^p \\ 3^q \end{matrix}\right\} = O_{npq},$$

with the existence condition $npq \leqslant n+p+q+2$.

(ii) The three semi-reciprocals

$$\left\{4, 3^{n-1}, \begin{matrix} 3 \\ 3 \end{matrix}\right\} = \delta_{n+3}, \quad \left\{3, \begin{matrix} 3 \\ 3^{n-1}, 4 \end{matrix}\right\} = h\delta_{n+3} = \left\{3, \begin{matrix} 3^{n-1}, 4 \\ 3 \end{matrix}\right\}$$

and their common truncation

$$\left\{\begin{matrix} 3^{n-1}, 4 \\ 3 \\ 3 \end{matrix}\right\} = t_2\delta_{n+3}.$$

14.7. The simplest examples of three-rowed symbols are

$$\left\{\begin{matrix} 3 \\ 3 \\ 3 \end{matrix}\right\} = \{3, 4, 3\}, \quad \left\{\begin{matrix} 3 \\ 3 \\ 4 \end{matrix}\right\} = t_1\delta_4,$$

$$\left\{\begin{matrix} 3 \\ 3 \\ 3, 3 \end{matrix}\right\} = t_2\beta_5, \quad \left\{\begin{matrix} 3 \\ 3 \\ 3, 4 \end{matrix}\right\} = \{3, 4, 3, 3\}.$$

14.8. Since $\left\{\begin{matrix}3\\3\end{matrix}\right\} = \{3, 4\}$, whenever a row of an extended Schläfli symbol ends with "3, 4", this combination can be replaced by a pair of 3's, one over the other. In this manner, the identities

$$\left\{3, \begin{matrix}3\\3, 4\end{matrix}\right\} = \{3, 3, 4, 3\} \quad \text{and} \quad \left\{\begin{matrix}3\\3\\3, 4\end{matrix}\right\} = \{3, 4, 3, 3\}$$

lead respectively to

$$\left\{\begin{matrix}3\\3, 3\\3\end{matrix}\right\} = \{3, 3, 4, 3\} \quad \text{and} \quad \left\{\begin{matrix}3\\3\\3\\3\end{matrix}\right\} = \{3, 4, 3, 3\}.$$

The symbols

$$\left\{\begin{matrix}j\\h,\\3^{m-5}, \begin{matrix}i\\k\end{matrix}\end{matrix}\right\}, \quad \left\{\begin{matrix}k\\i,\\3^{m-5}, \begin{matrix}j\\h\end{matrix}\end{matrix}\right\}, \quad \left\{\begin{matrix}h\\j,\\3^{m-5}, \begin{matrix}k\\i\end{matrix}\end{matrix}\right\}, \quad \left\{\begin{matrix}i\\k,\\3^{m-5}, \begin{matrix}h\\j\end{matrix}\end{matrix}\right\},$$

wherein $h = i = j = k = 3$, can be associated with Du Val's cycle of four semi-reciprocal $h\delta_m$'s. (These are semi-reciprocal in the sense that the reciprocal of each possesses the vertices of the other three together.) In each member of this cycle, the centres of the β_{m-1}'s are the vertices of the opposite member, while the centres of the $h\gamma_{m-1}$'s are the vertices of the two adjacent members.

14.9. It may have seemed pedantic to consider general values for the numbers involved in the symbols (14.51) and (14.61), when never more than two of these numbers can actually exceed 3. It is therefore worth while to remark that, in a Minkowskian or hyper-Minkowskian space (with a certain number of "time-like" dimensions), the h's, i's, and j's can be as great as we please, the restriction (14.53) being withdrawn.

In particular, Du Val has investigated the "pure Archimedean" polytopes n_{21} with $n > 5$.

Such considerations, however, are outside the scope of the present work.

15. *Spherical simplexes whose dihedral angles are submultiples of* π.

15.1. An ordinary spherical triangle can be regarded as the intersection of a sphere with three independent planes through its centre. The

angles of the spherical triangle are just the angles between pairs of the planes. If the sphere is of unit radius, these three angles suffice to determine the spherical triangle in both shape and size. But they must not be too small. In fact, if the angles are (2 3), (3 1), (1 2), the area of the spherical triangle is

$$(2\ 3)+(3\ 1)+(1\ 2)-\pi.$$

This function of the angles must consequently be positive. The limit of a sequence of spherical triangles of diminishing angles on spheres of suitably increasing radii is a *plane* triangle, for which

$$(2\ 3)+(3\ 1)+(1\ 2)-\pi=0.$$

These notions can easily be extended to m dimensions. A *spherical simplex* is defined as one of the 2^m parts into which an $(m-1)$-sphere is divided by m independent primes through its centre. Of the two supplementary angles between a pair of the primes, that one which is inside the spherical simplex is called a *dihedral angle*. If the $(m-1)$-sphere is of unit radius, the spherical simplex is completely determined by its $\frac{1}{2}m(m-1)$ dihedral angles. As in three dimensions, these dihedral angles must not be too small. But when $m>3$, the content of a spherical simplex is no longer a simple function of the angles. We accordingly seek a more tractable criterion.

Let the m primes be called 1, 2, ..., m; and let* $(r\ s)$ denote the dihedral angle between the primes r and s, so that $(s\ r)=(r\ s)$. Using Cartesian coordinates, with the origin at the centre of the $(m-1)$-sphere, let the prime r have the equation

$$\sum_{i=1}^{m} a_{ri}x_i=0,$$

where

$$\sum_{i=1}^{m} a_{ri}^2=1 \quad (r=1,\ 2,\ \ldots,\ m).$$

We can suppose the signs of the a's adjusted so that the spherical simplex is just the aggregate of points satisfying

$$\sum_{i=1}^{m} x_i^2=1, \qquad \sum_{i=1}^{m} a_{ri}x_i\geqslant 0.$$

It follows that

$$\sum_{i=1}^{m} a_{ri}a_{is}=-\cos(r\ s).$$

* Not to be confused with the (ij) of §9.2.

The m primes being independent,

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} & \dots & a_{1m} \\ a_{21} & a_{22} & a_{23} & \dots & a_{2m} \\ a_{31} & a_{32} & a_{33} & \dots & a_{3m} \\ \dots & \dots & \dots & & \dots \\ a_{m1} & a_{m2} & a_{m3} & \dots & a_{mm} \end{vmatrix} \neq 0.$$

Squaring this inequality we have*

$$(15.11) \quad \begin{vmatrix} 1 & -\cos(1\,2) & -\cos(1\,3) & \dots & -\cos(1\,m) \\ -\cos(2\,1) & 1 & -\cos(2\,3) & \dots & -\cos(2\,m) \\ -\cos(3\,1) & -\cos(3\,2) & 1 & \dots & -\cos(3\,m) \\ \dots & \dots & \dots & \dots & \dots \\ -\cos(m\,1) & -\cos(m\,2) & -\cos(m\,3) & \dots & 1 \end{vmatrix} > 0.$$

15.2. The section of our spherical simplex by the prime through the origin perpendicular to all the primes $1, 2, \dots, i-1, i+1, \dots, m$ is a spherical simplex of one fewer dimensions, whose dihedral angles are

$$(r\ s) \quad (r \neq i \neq s).$$

Hence the above determinant must remain positive when any number of rows are removed, along with the corresponding columns. The inequalities obtained in this manner provide the required criteria for the existence of a spherical simplex of given dihedral angles.

Note that, provided that no obtuse angles are admitted, the value of our determinant diminishes when any one of the angles is diminished. The limit of a sequence of spherical simplexes of diminishing dihedral angles on $(m-1)$-spheres of suitably increasing radii is a *Euclidean* simplex, for which the determinant vanishes. But the simpler determinants, derived by omitting corresponding rows and columns, remain definitely positive.

Let us now enumerate the spherical and Euclidean simplexes all of whose dihedral angles are submultiples of π. (This restriction implies that no dihedral angle shall be obtuse.)

15.3. If the primes $1, 2, \dots, m$ fall into two sets, say $1, 2, \dots, i$ and $i+1, i+2, \dots, m$, such that every prime of the former set is perpendicular to every prime of the latter, then our determinant breaks up into two

* This result is due to Schläfli (*loc. cit.* in Preface).

factors, whence the existence of the spherical simplex depends on the existence of two simpler simplexes, viz., that whose dihedral angles are $(r\ s)$ with $r \leqslant i$ and $s \leqslant i$, and that whose dihedral angles are $(r\ s)$ with $r > i$ and $s > i$.

In particular, there is a spherical simplex *all* of whose dihedral angles are right, namely the spherical simplex bounded by

$$x_1 = 0, \quad x_2 = 0, \quad \ldots, \quad x_m = 0.$$

15.4. Apart from one trivial case, it is impossible to have a *closed chain* of acute dihedral angles, such as

$$(1\ 2), \quad (2\ 3), \quad \ldots, \quad (i{-}1\ i), \quad (i\ 1).$$

For, to take the most favourable possibility, suppose that

$$m = i, \quad (1\ 2) = (2\ 3) = \ldots = (i{-}1\ i) = (i\ 1) = \tfrac{1}{3}\pi,$$

and

$$(r\ s) = \tfrac{1}{2}\pi \ (1 < |r{-}s| < i{-}1).$$

Our determinant becomes

$$\begin{vmatrix} 1 & -\frac{1}{2} & 0 & 0 & \ldots & 0 & -\frac{1}{2} \\ -\frac{1}{2} & 1 & -\frac{1}{2} & 0 & \ldots & 0 & 0 \\ 0 & -\frac{1}{2} & 1 & -\frac{1}{2} & \ldots & 0 & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ -\frac{1}{2} & 0 & 0 & 0 & \ldots & -\frac{1}{2} & 1 \end{vmatrix} = 0 \quad (i > 2).$$

Therefore the simplex in this case is Euclidean; and any further diminution of the angles, or insertion of extra angles, will render it non-existent.

15.5. Apart from one trivial case, it is impossible to have more than three acute $(r\ s)$'s with a common r (or s). For, to take the most favourable possibility, suppose that

$$m = 5, \quad (1\ 2) = (1\ 3) = (1\ 4) = (1\ 5) = \tfrac{1}{3}\pi \ \text{ and } \ (r\ s) = \tfrac{1}{2}\pi \ \ (r > 1, \ s > 1).$$

Our determinant becomes

$$\begin{vmatrix} 1 & -\frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & 1 & 0 & 0 & 0 \\ -\frac{1}{2} & 0 & 1 & 0 & 0 \\ -\frac{1}{2} & 0 & 0 & 1 & 0 \\ -\frac{1}{2} & 0 & & 0 & 1 \end{vmatrix} = 0.$$

Therefore the simplex in this case is Euclidean; and any further diminution of the angles, or insertion of new angles, will render it non-existent.

15.6. Apart from one trivial case, it is impossible to have a double occurrence of three acute $(r\,s)$'s with a common r. For, to take the most favourable possibility, suppose that

$$(1\ 3) = (2\ 3) = (3\ 4) = (4\ 5) = \ldots = (m-4\ m-3) = (m-3\ m-2)$$
$$= (m-2\ m-1) = (m-2\ m) = \tfrac{1}{3}\pi,$$

with right angles for all the rest. In this case, too, the determinant vanishes. Therefore the simplex is again Euclidean; and any further diminution of angles, or insertion of extra angles, will render it non-existent.

15.7. The only type of simplex which remains to be considered is that whose acute dihedral angles form three open chains all emanating from one bounding prime, thus:

$$(1\ 2), \quad (2\ 3), \quad \ldots, \quad (n\ n+1);$$
$$(1\ n+2), \quad (n+2\ n+3), \quad \ldots, \quad (n+p\ n+p+1);$$
$$(1\ n+p+2), \quad (n+p+2\ n+p+3), \quad \ldots, \quad (n+p+q\ n+p+q+1).$$
$$(m = n+p+q+1.)$$

Let us rename these angles as follows:

$$\pi/h_0, \quad \pi/h_1, \quad \ldots, \quad \pi/h_{n-1};$$
$$\pi/i_0, \quad \pi/i_1, \quad \ldots, \quad \pi/i_{p-1};$$
$$\pi/j_0, \quad \pi/j_1, \quad \ldots, \quad \pi/j_{q-1}.$$

Then our simplex can conveniently be denoted by the symbol

$$(15.71) \qquad \begin{pmatrix} h_0, & h_1, & \ldots, & h_{n-1} \\ i_0, & i_1, & \ldots, & i_{p-1} \\ j_0, & j_1, & \ldots, & j_{q-1} \end{pmatrix}.$$

15.8. In order to clarify this notation, we may rename the m bounding primes:

$$(15.81) \qquad 0,\ 1,\ 2,\ \ldots,\ n,\ 1',\ 2',\ \ldots,\ p',\ 1'',\ 2'',\ \ldots,\ q''.$$

Then π/h_0, π/i_0, π/j_0 are the angles between 0 and 1, 1', 1'' respectively,

π/h_r is the angle between r and $r+1$, π/i_r between r'' and $(r+1)'$, and π/j_r between r'' and $(r+1)''$. Every other angle is a right angle.

We might have regarded our acute dihedral angles as forming only two chains, one emanating from the middle of the other. But this aspect destroys the symmetry which exists between the h's, i's, and j's, obscuring the fact that the three rows of the symbol (15.71) can be permuted bodily.

If $n = 0$, so that there are no h's, we are left with a single chain of acute dihedral angles; thus

$$(15.82) \qquad \begin{pmatrix} i_0, i_1, \ldots, i_{p-1} \\ j_0, j_1, \ldots, j_{q-1} \end{pmatrix} = (i_{p-1}, i_{p-2}, \ldots, i_0, j_0, j_1, \ldots, j_{q-1}).$$

A simplex of this type, say*

$$(k_1, k_2, \ldots, k_{m-1}),$$

is what Schläfli calls an *orthoscheme*. In this case, returning to the notation of § 15.1, we have

$$(r\ r+1) = \pi/k_r,$$

but all $(r\ s)$'s for which r and s differ by more than 1 are right angles. So (15.11) becomes

$$\Delta_m(k_1, k_2, \ldots, k_{m-1}) > 0,$$

in the notation of § 3.5.

The simplest orthoscheme is (k): an arc of length π/k. The next simplest is (k_1, k_2): a right-angled spherical triangle, of angles π/k_1 and π/k_2. The simplest other simplex of type (15.71) is $\begin{pmatrix} h \\ i \\ j \end{pmatrix}$: a hyperspherical tetrahedron with three right dihedral angles at one vertex, the remaining three dihedral angles being π/h, π/i, π/j.

Analogous symbols for the special simplexes discussed in § 15.5 and § 15.6 are respectively

$$\begin{pmatrix} 3 \\ 3 \\ 3 \\ 3 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 3, & 3, & \ldots, & 3, & 3 \\ 3, & & & & 3 \end{pmatrix}.$$

The special kind of simplex discussed in § 15.4 is unique, in that the number

* Not to be confused with the $(x_1, x_2, \ldots, x_n)$ of § 3.6.

of acute dihedral angles is equal to (instead of less than) the number of bounding primes. In this case the brackets must close up to form a complete circle; so the appropriate symbol is

$$\textcircled{3^m} \quad (m = i).$$

The enumeration of orthoschemes (with integral k's) involves exactly the same work as the enumeration of regular polytopes, as undertaken in § 3.5; so we need not repeat it here. The connection will appear later. Since obviously

$$(k_{m-1}, k_{m-2}, \ldots, k_1) = (k_1, k_2, \ldots, k_{m-1}),$$

the result is as follows:

m	Spherical orthoschemes	Euclidean orthoschemes
2	(k)	
3	(3, 3), (3, 4), (3, 5)	(4, 4), (3, 6)
4	(3, 3, 3), (3, 3, 4), (3, 3, 5), (3, 4, 3)	(4, 3, 4)
5	(3, 3, 3, 3), (3, 3, 3, 4)	(4, 3, 3, 4), (3, 3, 4, 3)
>5	(3^{m-1}), $(3^{m-2}, 4)$	$(4, 3^{m-3}, 4)$

For the sake of completeness, we might add: for $m = 1$, the very simple simplex

$$(\ )$$

which has, and is, a single vertex; and, for $m = 2$ (Euclidean), the straight segment

$$(\infty),$$

so called because it subtends a zero angle at infinity.

15.9. It is easily proved by induction that the proper determinant for

$$\begin{pmatrix} h_0, & h_1, & \ldots, & h_{n-1} \\ i_0, & i_1, & \ldots, & i_{p-1} \\ j_0, & j_1, & \ldots, & j_{q-1} \end{pmatrix}$$

is equal to

$$\left(\frac{\Delta_{n+1}(h_0, h_1, \ldots, h_{n-1})}{\Delta_n(h_1, \ldots, h_{n-1})} + \frac{\Delta_{p+1}(i_0, i_1, \ldots, i_{p-1})}{\Delta_p(i_1, \ldots, i_{p-1})} + \frac{\Delta_{q+1}(j_0, j_1, \ldots, j_{q-1})}{\Delta_q(j_1, \ldots, j_{q-1})} - 2\right)$$
$$\times \Delta_n(h_1, \ldots, h_{n-1})\, \Delta_p(i_1, \ldots, i_{p-1})\, \Delta_q(j_1, \ldots, j_{q-1}).$$

We saw in §15.2 that the existence of a spherical simplex depends, not only on (15.11), but also on the existence of a spherical simplex of one fewer dimensions, derived by suppressing any one of the original bounding primes. If we suppress the prime 0 (in the notation of §15.8), we obtain the orthoscheme

$$(h_1, \ldots, h_{n-1}, 2, i_1, \ldots, i_{p-1}, 2, j_1, \ldots, j_{q-1}),$$

whose existence depends solely on the joint existence of the three orthoschemes

$$(h_1, \ldots, h_{n-1}), \quad (i_1, \ldots, i_{p-1}), \quad (j_1, \ldots, j_{q-1}).$$

But the suppression of the prime r $(r > 0)$ leads to the simplex

$$\begin{pmatrix} h_0, h_1, \ldots, h_{r-2}, 2, h_{r+1}, \ldots, h_{n-1} \\ i_0, i_1, \ldots, i_{p-1} \\ j_0, j_1, \ldots, j_{q-1} \end{pmatrix},$$

whose existence depends solely on the joint existence of

$$\begin{pmatrix} h_0, h_1, \ldots, h_{r-2} \\ i_0, i_1, \ldots, i_{p-1} \\ j_0, j_1, \ldots, j_{q-1} \end{pmatrix} \quad \text{and} \quad (h_{r+1}, \ldots, h_{n-1}).$$

Similarly for the suppression of r' or r''.

Hence the simplex (15.71) certainly exists if

$$\begin{pmatrix} h_0, h_1, \ldots, h_{n-2} \\ i_0, i_1, \ldots, i_{p-1} \\ j_0, j_1, \ldots, j_{q-1} \end{pmatrix}, \quad \begin{pmatrix} h_0, h_1, \ldots, h_{n-1} \\ i_0, i_1, \ldots, i_{p-2} \\ j_0, j_1, \ldots, j_{q-1} \end{pmatrix}, \quad \begin{pmatrix} h_0, h_1, \ldots, h_{n-1} \\ i_0, i_1, \ldots, i_{p-1} \\ j_0, j_1, \ldots, j_{q-2} \end{pmatrix}$$

all exist and are definitely spherical (not Euclidean), and if further the inequality (14.53) is satisfied. In this inequality we have "$>$" for a definitely spherical simplex, and "$=$" for a Euclidean one.

When all the h's, i's, and j's are equal to 3, (14.53) becomes (14.62). Thus the simplex

$$\begin{pmatrix} 3^n \\ 3^p \\ 3^q \end{pmatrix}$$

is spherical when $$npq < n+p+q+2,$$

and Euclidean when

$$npq = n+p+q+2.$$

The particular cases are as follows:

$m = n+p+q+1$	Spherical	Euclidean
$p+q+1$	$\begin{pmatrix} 3^p \\ 3^q \end{pmatrix} = (3^{p,q})$	
$n+3$	$\begin{pmatrix} 3^n \\ 3 \\ 3 \end{pmatrix}$	
6	$\begin{pmatrix} 3, 3 \\ 3, 3 \\ 3 \end{pmatrix}$	
7	$\begin{pmatrix} 3, 3, 3 \\ 3, 3 \\ 3 \end{pmatrix}$	$\begin{pmatrix} 3, 3 \\ 3, 3 \\ 3, 3 \end{pmatrix}$
8	$\begin{pmatrix} 3, 3, 3, 3 \\ 3, 3 \\ 3 \end{pmatrix}$	$\begin{pmatrix} 3, 3, 3 \\ 3, 3, 3 \\ 3 \end{pmatrix}$
9		$\begin{pmatrix} 3, 3, 3, 3, 3 \\ 3, 3 \\ 3 \end{pmatrix}$

The next simplest possibility is

$$\begin{pmatrix} 3^{n-1}, 4 \\ 3^p \\ 3^q \end{pmatrix}.$$

In this case, since $\Delta_{n+1}(3^{n-1}, 4) = 1/2^n$ (as we saw in §3.5), we have $\frac{1}{2}\left(1+\frac{p+2}{p+1}+\frac{q+2}{q+1}\right) \geqslant 2$, or $pq \leqslant 1$. Since the orthoschemes have already been considered, we can assume that $pq > 0$. Therefore we must have

$$p = q = 1.$$

The simplex $$\begin{pmatrix} 3^{n-1}, 4 \\ 3 \\ 3 \end{pmatrix},$$

being Euclidean, is the last possibility.

Recapitulating, the only possible simplexes whose dihedral angles are submultiples of π are the following:

$( )$ $(m = 1,$ spherical or Euclidean),

(k) $(m = 2,$ spherical),

(∞) $(m = 2,$ Euclidean),

$\begin{pmatrix} 3^n \\ 3^p \\ 3^q \end{pmatrix}$ $(npq \leqslant n+p+q+2, \quad m = n+p+q+1,$ spherical or Euclidean),

$(3^{m-2}, 4)$ (spherical),

$(4, 3^{m-3}, 4)$ (Euclidean),

$(3, 5)$ $(m = 3,$ spherical),

$(3, 6)$ $(m = 3,$ Euclidean),

$(3, 3, 5), \quad (3, 4, 3)$ $(m = 4,$ spherical),

$(3, 3, 4, 3), \quad \begin{pmatrix} 3 \\ 3 \\ 3 \\ 3 \end{pmatrix}$ $(m = 5,$ Euclidean),

$\begin{pmatrix} 3^{m-4}, 4 \\ 3 \\ 3 \end{pmatrix}, \quad \begin{pmatrix} 3, & & 3 \\ & 3^{m-5}, & \\ 3, & & 3 \end{pmatrix}$ (Euclidean),

$\textcircled{3^m}$ (Euclidean),

and an endless variety of new spherical simplexes derivable from pairs of known spherical simplexes in the manner described in § 15.3.

16. *Groups whose fundamental regions are simplexes.*

16.1. Let $$R_1, \ R_2, \ \ldots, \ R_m$$

denote the reflections in the m primes of § 15.1. These operations clearly generate a *group* of congruent transformations or orthogonal substitutions. Since a reflection is a *negative* operation (*i.e.* a transformation whose matrix has a negative determinant), the group contains both positive and negative operations; accordingly it is said to be *extended*.

It is well known that the product $R_r R_s$ is a rotation through angle $2(r\ s)$ about the secundum of intersection of the two primes r and s. Since

$$R_s^2 = 1 \quad \text{and} \quad (R_r R_s)(R_s R_t) = R_r R_t,$$

all these products are expressible in terms of $m-1$ of them, provided that these $m-1$ involve all the R's. A suitable set of products is*

$$S_r = R_{r+1} R_1 \quad (r = 1, 2, \ldots, m-1).$$

These rotations also generate a group. Since a rotation is a positive operation, every operation of this group must be *positive*; accordingly the group is said to be *unextended*.

Since the operation R_1, of period 2, belongs to the former group but not to the latter, and since

$$R_s = S_{s-1} R_1 \quad (s = 2, 3, \ldots, m),$$

it follows that the unextended group is a sub-group of index 2 in the extended group.

The operation R_s transforms the simplex of §15.1 into a new simplex, having the prime s in common with the original one. The operations

$$R_r R_s, \quad (R_r R_s)^2, \quad (R_r R_s)^3, \quad \ldots$$

transform the original simplex into a cycle of new ones, all meeting in the secundum of intersection of r and s. *If every dihedral angle of the simplex is a sub-multiple of* π, these simplexes will not overlap. In fact we shall have

$$(R_r R_s)^{\pi/(r\,s)} = 1.$$

In this case, the simplex is called a *fundamental region* for either of the groups.

16.2. We shall let g_m denote the order of the extended group, so that $\frac{1}{2}g_m$ is the order of the unextended group. g_m may be finite or infinite; we shall soon see that it is finite or infinite according as the fundamental region is spherical or Euclidean.

The operations of the extended group satisfy the relations

$$(16.21) \qquad \begin{cases} R_s^2 = 1 & (s = 1, 2, \ldots, m), \\ (R_r R_s)^{\pi/(r\,s)} = 1 & (r, s = 1, 2, \ldots, m;\ r \neq s), \end{cases}$$

* This S_r is not to be confused with the S_m of §8.9.

which are equivalent to

$$R_1^2 = 1, \quad (S_r R_1)^2 = 1 \quad (r = 1, 2, \ldots, m-1) \tag{16.22}$$

and

$$\begin{cases} S_{r-1}^{\pi/(r\,1)} = 1 \quad (r = 2, 3, \ldots, m), \\ (S_{r-1} S_{s-1}^{-1})^{\pi/(r\,s)} = 1 \quad (r, s = 2, 3, \ldots, m;\ r \neq s). \end{cases} \tag{16.23}$$

The g_m operations of the extended group transform the fundamental region into a net of g_m simplexes, fitting together so as to fill the whole $(m-1)$-space (spherical or Euclidean) at least once. An obvious extension of an argument used by Burnside* proves that the net fills the space exactly once, and that the equations (16.21) constitute an *abstract definition* for the extended group.

Since the extended group can be derived from the unextended group by the insertion of R_1, which is related to the S's by (16.22), it follows that the equations (16.23) constitute an abstract definition for the unextended group.

Since the g_m simplexes, each of finite content, fit together to fill the $(m-1)$-space just once, it follows that g_m is finite or infinite according as the space is spherical or Euclidean.

16.3. The $\frac{1}{2}g_m$ operations of the unextended group transform the fundamental region into one half of the net of simplexes, namely into a set of $\frac{1}{2}g_m$ simplexes of which no two have a common bounding prime. Any negative operation of the extended group (*e.g.* R_1) transforms this half of the net of simplexes into the other half. It is useful to regard every simplex as being "shaded" or "non-shaded" according to the half-net to which it belongs. A beautiful account of the case when $m = 3$ is given by F. Klein in Chapter 1 of his *Lectures on the icosahedron*†.

16.4. Let us now consider the particular simplexes that can serve as fundamental regions. Take first the kind of simplex discussed in § 15.3, viz. that for which

$$(r\,s) = \tfrac{1}{2}\pi \quad \text{whenever} \quad r \leqslant i < s.$$

The relation $(R_r R_s)^2 = 1$ simply means that R_r and R_s commute. Hence, substituting in (16.21), we see that the extended group is in this case the

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* W. Burnside, *Theory of groups of finite order* (2nd ed., 1911), 399 (§ 291).

† Second edition in English, 1913.

direct product of the groups generated by

$$R_1,\ R_2,\ \ldots,\ R_i,$$

and by

$$R_{i+1},\ R_{i+2},\ \ldots,\ R_m.$$

In particular, the group corresponding to the simplex all of whose dihedral angles are right angles is the direct product of m groups each of order 2.

If

$$(r\,1) = \tfrac{1}{2}\pi \quad (r = 2,\ 3,\ \ldots,\ m),$$

(16.23) becomes

$$\begin{cases} S_{r-1}^2 = 1 & (r = 2,\ 3,\ \ldots,\ m), \\ (S_{r-1}\,S_{s-1})^{\pi/(r\,s)} = 1 & (r,\ s = 2,\ 3,\ \ldots,\ m,\ r \neq s). \end{cases}$$

These equations being of the same form as (16.21), it follows that the unextended group corresponding to a simplex for which one bounding prime is perpendicular to all the others, is simply isomorphic with the extended group corresponding to the simplex (of one fewer dimensions) which these other primes cut out on the special one.

16.5. Corresponding to the simplex (3^m) discussed in §15.4, we have the extended group defined by

$$(16.51) \qquad \begin{cases} R_s^{\,2} = 1 & (s = 1,\ 2,\ \ldots,\ m), \\ (R_r\,R_s)^2 = 1 & (1 < |r-s| < m-1), \\ (R_r\,R_s)^3 = 1 & (r-s = -1 \text{ or } m-1), \end{cases}$$

and the unextended group defined by

$$(16.52) \qquad \begin{cases} S_1^{\,3} = 1, \quad S_{m-1}^3 = 1, \\ S_u^{\,2} = 1 \quad (u = 2,\ 3,\ \ldots,\ m-2), \\ (S_u\,S_v^{-1})^2 = 1 \quad (|u-v| > 1), \\ (S_u\,S_{u+1}^{-1})^3 = 1 \quad (u = 1,\ 2,\ \ldots,\ m-2). \end{cases}$$

Since the simplex is Euclidean, both these groups are of infinite order. It is convenient to give them the respective symbols

$$\boxed{3^m} \quad \text{and} \quad \boxed{3^m}'.$$

E.g., the infinite group discussed in Burnside's §299 is our

$$\boxed{3^3}',$$

its fundamental region being the plane equilateral triangle (3^3).

16.6. Corresponding to the simplex $\begin{pmatrix} 3 \\ 3 \\ 3 \\ 3 \end{pmatrix}$ discussed in §15.5, we have the extended group defined by

$$(16.61)\quad \begin{cases} R_1^2 = R_2^2 = R_3^2 = R_4^2 = R_5^2 = 1, \\ (R_2R_3)^2 = (R_2R_4)^2 = (R_2R_5)^2 = (R_3R_4)^2 = (R_3R_5)^2 = (R_4R_5)^2 = 1, \\ (R_1R_2)^3 = (R_1R_3)^3 = (R_1R_4)^3 = (R_1R_5)^3 = 1, \end{cases}$$

and the unextended group defined by

$$(16.62)\quad \begin{cases} S_1^3 = S_2^3 = S_3^3 = S_4^3 = 1, \\ (S_1S_2^{-1})^2 = (S_1S_3^{-1})^2 = (S_1S_4^{-1})^2 \\ \qquad = (S_2S_3^{-1})^2 = (S_2S_4^{-1})^2 = (S_3S_4^{-1})^2 = 1. \end{cases}$$

Since the simplex is Euclidean, both these groups are of infinite order. It is convenient to give them the respective symbols

$$\begin{bmatrix} 3 \\ 3 \\ 3 \\ 3 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 3 \\ 3 \\ 3 \\ 3 \end{bmatrix}' .$$

Similarly, we obtain two infinite groups

$$(16.63)\qquad \begin{bmatrix} 3, & & 3 \\ & 3^{m-5}, & \\ 3, & & 3 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 3, & & 3 \\ & 3^{m-5}, & \\ 3, & & 3 \end{bmatrix}'$$

from the Euclidean simplex discussed in §15.6.

16.7. Having mentioned all the "trivial" cases, let us turn our attention to the two groups which have the fundamental region (15.71). We shall call the extended and unextended groups

$$(16.71)\qquad \begin{bmatrix} h_0, & h_1, & \ldots, & h_{n-1} \\ i_0, & i_1, & \ldots, & i_{p-1} \\ j_0, & j_1, & \ldots, & j_{q-1} \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} h_0, & h_1, & \ldots, & h_{n-1} \\ i_0, & i_1, & \ldots, & i_{p-1} \\ j_0, & j_1, & \ldots, & j_{q-1} \end{bmatrix}'$$

respectively.

The following change of notation is convenient:

$$(16.72)\qquad \begin{cases} O = R_1, & \\ N_r = R_{r+2} & (r = 0, 1, \ldots, n-1), \\ P_r = R_{r+n+2} & (r = 0, 1, \ldots, p-1)^*, \\ Q_r = R_{r+n+p+2} & (r = 0, 1, \ldots, q-1); \end{cases}$$

$$(16.73)\qquad \begin{cases} N_r' = N_r O = S_{r+1} & (r = 0, 1, \ldots, n-1), \\ P_r' = P_r O = S_{r+n+1} & (r = 0, 1, \ldots, p-1), \\ Q_r' = Q_r O = S_{r+n+p+1} & (r = 0, 1, \ldots, q-1). \end{cases}$$

The extended group is defined by

$$(16.74)\qquad \begin{cases} O^2 = 1, & \\ N_r^2 = 1 & (r = 0, 1, \ldots, n-1), \\ P_r^2 = 1 & (r = 0, 1, \ldots, p-1), \\ Q_r^2 = 1 & (r = 0, 1, \ldots, q-1), \\ (ON_0)^{h_0} = (N_{r-1}N_r)^{h_r} = 1 & (r = 1, 2, \ldots, n-1), \\ (OP_0)^{i_0} = (P_{r-1}P_r)^{i_r} = 1 & (r = 1, 2, \ldots, p-1), \\ (OQ_0)^{j_0} = (Q_{r-1}Q_r)^{j_r} = 1 & (r = 1, 2, \ldots, q-1), \\ (ON_r)^2 = (OP_r)^2 = (OQ_r)^2 = 1 & (r > 0), \\ (N_rN_s)^2 = (P_rP_s)^2 = (Q_rQ_s)^2 = 1 & (|r-s| > 1), \\ (P_rQ_s)^2 = (Q_rN_s)^2 = (N_rP_s)^2 = 1; & \end{cases}$$

and the unextended group by

$$(16.75)\qquad \begin{cases} N_0'^{h_0} = N_r'^2 = (N_{r-1}'N_r')^{h_r} = 1 & (r = 1, 2, \ldots, n-1), \\ P_0'^{i_0} = P_r'^2 = (P_{r-1}'P_r')^{i_r} = 1 & (r = 1, 2, \ldots, p-1), \\ Q_0'^{j_0} = Q_r'^2 = (Q_{r-1}'Q_r')^{j_r} = 1 & (r = 1, 2, \ldots, q-1), \\ (N_r'N_s')^2 = (P_r'P_s')^2 = (Q_r'Q_s')^2 = 1 & (|r-s| > 1), \\ (P_r'Q_s')^2 = (Q_r'N_s')^2 = (N_r'P_s')^2 = 1 & (r+s > 0), \\ (P_0'Q_0'^{-1})^2 = (Q_0'N_0'^{-1})^2 = (N_0'P_0'^{-1})^2 = 1. & \end{cases}$$

* This P_r is not to be confused with the P_m of § 8.3.

16.8. When $p = q = 0$, we have two groups whose fundamental region is the orthoscheme $(h_0, h_1, \ldots, h_{n-1})$, namely:

$$[h_0, h_1, \ldots, h_{n-1}],$$

defined by

$$(16.81)\qquad \begin{cases} O^2 = N_r{}^2 = 1 & (r = 0, 1, \ldots, n-1), \\ (ON_0)^{h_0} = (N_{r-1}N_r)^{h_r} = 1 & (r = 1, 2, \ldots, n-1), \\ (ON_r)^2 = 1 & (r > 0), \\ (N_rN_s)^2 = 1 & (|r-s| > 1); \end{cases}$$

and

$$[h_0, h_1, \ldots, h_{n-1}]',$$

defined by

$$(16.82)\qquad \begin{cases} N_0'{}^{h_0} = N_r'{}^2 = (N_{r-1}'N_r')^{h_r} = 1 & (r = 1, 2, \ldots, n-1), \\ (N_r'N_s')^2 = 1 & (|r-s| > 1). \end{cases}$$

(16.81) and (16.82) are due to Todd*, who obtained them as abstract definitions for the extended and unextended groups of the regular polytope $\{h_0, h_1, \ldots, h_{n-1}\}$. The groups given by Burnside in his §296† are respectively:

I, $[n]'$;

II, $[2, n]'$, which is the same as $[n]$;

III, $[3, 3]'$;

IV, $[3, 4]'$;

V, $[3, 5]'$.

Further, the infinite groups discussed in his §§ 300, 301 are respectively

$$[4, 4]' \quad \text{and} \quad [3, 6]'.$$

The extended groups

$$[3, 3], \quad [3, 4], \quad [3, 5]$$

are mentioned by Klein‡. Todd has considered the groups

$$[3^{m-1}], \quad [3^{m-2}, 4], \quad [3, 3, 5], \quad [3, 4, 3],$$

$$[3^{m-1}]', \quad [3^{m-2}, 4]', \quad [3, 3, 5]', \quad [3, 4, 3]'$$

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* *Loc. cit.* in Preface.

† *Theory of groups*, 408.

‡ *Lectures on the icosahedron*, 24.

in detail, generating each group, save the last of all, by means of *two* operations. $[3^{m-1}]$ and $[3^{m-1}]'$ are easily recognizable as the symmetric and alternating groups of degree $m+1$. $[3^{m-2}, 4]$ and $[3^{m-2}, 4]'$ are discussed by Dr. A. Young in the fifth of his papers on *Substitutional analysis**; his "(AB) subgroup" can be identified with our $\begin{bmatrix} 3^{m-3} \\ 3 \\ 3 \end{bmatrix}$. L. E. Dickson† gives an abstract definition for $\begin{bmatrix} 3, 3 \\ 3 \\ 3 \end{bmatrix}'$ closely resembling ours.

16.9. The rest of our finite groups, namely

$$\begin{bmatrix} 3, 3 \\ 3, 3 \\ 3 \end{bmatrix}, \quad \begin{bmatrix} 3, 3, 3 \\ 3, 3 \\ 3 \end{bmatrix}, \quad \begin{bmatrix} 3, 3, 3, 3 \\ 3, 3 \\ 3 \end{bmatrix},$$

$$\begin{bmatrix} 3, 3 \\ 3, 3 \\ 3 \end{bmatrix}', \quad \begin{bmatrix} 3, 3, 3 \\ 3, 3 \\ 3 \end{bmatrix}', \quad \begin{bmatrix} 3, 3, 3, 3 \\ 3, 3 \\ 3 \end{bmatrix}',$$

are less familiar. We shall find the orders of all of them, and identify the first and fifth with important geometrical groups, viz., the group of automorphisms of the twenty-seven lines on the general cubic surface, and the group of automorphisms of the twenty-eight bitangents of the general plane quartic curve.

17. *How each of the groups is related to a uniform polytope.*

17.1. When, as in §5.1, $t_n \Pi_m$ is regarded as the part of space which is inside both Π_m and Π_m', the elements $t_n \Pi_s$ $(s > n)$ arise as actual truncations of the s-dimensional elements of Π_m, and the elements $t_{n-m+s} \Pi_{s, m-s}$ $(s \geqslant m-n)$ as actual truncations of the s-dimensional elements of Π_m'. But, if $r+s = m-1$, the s-dimensional elements of Π_m' *correspond* to the r-dimensional elements of Π_m, in the sense that their centres are collinear

* *Proc. London Math. Soc.* (2), 31 (1930), 273.

† *Linear groups* (Leipzig, 1901), 293.

with the common centre of Π_m and Π_m'. Also the vertices of $t_n \Pi_m$ correspond to the Π_n's of Π_m. Thus, for every element Π_r of Π_m, there is a corresponding element, say Π^r, of $t_n \Pi_m$, viz.:

$$\Pi^r = t_{n-r-1} \Pi_{m-r-1, r+1} \quad (r < n),$$

$$\Pi^n = \alpha_0,$$

$$\Pi^s = t_n \Pi_s \qquad (s > n).$$

If $\Pi_m = \left\{ h_{n-1}, h_{n-2}, \ldots, h_0, \begin{matrix} i_0, i_1, \ldots, i_{p-1} \\ j_0, j_1, \ldots, j_{q-1} \end{matrix} \right\}$, we have

$$\Pi^r = t_{n-r-1} \left\{ h_{n-r-2}, h_{n-r-3}, \ldots, h_0, \begin{matrix} i_0, i_1, \ldots, i_{p-1} \\ j_0, j_1, \ldots, j_{q-1} \end{matrix} \right\}$$

$$= \begin{Bmatrix} h_0, h_1, \ldots, h_{n-r-2} \\ i_0, i_1, \ldots, i_{p-1} \\ j_0, j_1, \ldots, j_{q-1} \end{Bmatrix}, \quad (r < n) \tag{17.11}$$

and

$$\Pi^s = t_n \left\{ h_{n-1}, h_{n-2}, \ldots, h_0, \begin{matrix} i_0, i_1, \ldots, i_{p'-1} \\ j_0, j_1, \ldots, j_{q'-1} \end{matrix} \right\}$$

$$= \begin{Bmatrix} h_0, h_1, \ldots, h_{n-1} \\ i_0, i_1, \ldots, i_{p'-1} \\ j_0, j_1, \ldots, j_{q'-1} \end{Bmatrix} \quad (s = n+p'+q'+1; \quad p' \geqslant 0, \quad q' \geqslant 0). \tag{17.12}$$

Putting $q' = q$, we see that the elements

$$\begin{Bmatrix} h_0, h_1, \ldots, h_{n-1} \\ i_0, i_1, \ldots, i_{p'-1} \\ j_0, j_1, \ldots, j_{q-1} \end{Bmatrix} \quad \text{of} \quad \begin{Bmatrix} h_0, h_1, \ldots, h_{n-1} \\ i_0, i_1, \ldots, i_{p-1} \\ j_0, j_1, \ldots, j_{q-1} \end{Bmatrix}$$

correspond to the elements

$$\left\{ h_{n-1}, h_{n-2}, \ldots, h_0, \begin{matrix} i_0, i_1, \ldots, i_{p'-1} \\ j_0, j_1, \ldots, j_{q-1} \end{matrix} \right\} \quad \text{of} \quad \left\{ h_{n-1}, h_{n-2}, \ldots, h_0, \begin{matrix} i_0, i_1, \ldots, i_{p-1} \\ j_0, j_1, \ldots, j_{q-1} \end{matrix} \right\},$$

and [by (17.11) with the h's and i's interchanged] to the $(p-p'-1)$-

dimensional elements

$$\{i_{p-1}, i_{p-2}, \ldots, i_{p'+2}\} \quad \text{of} \quad \left\{ \begin{matrix} & & j_0, j_1, \ldots, j_{q-1} \\ i_{p-1}, i_{p-2}, \ldots, i_0, & & \\ & & h_0, h_1, \ldots, h_{n-1} \end{matrix} \right\},$$

and [by (17.12) with the h's and j's interchanged] to the elements

$$\left\{ \begin{matrix} & & h_0, h_1, \ldots, h_{n-1} \\ j_{q-1}, j_{q-2}, \ldots, j_0, & & \\ & & i_0, i_1, \ldots, i_{p'-1} \end{matrix} \right\} \quad \text{of} \quad \left\{ \begin{matrix} & & h_0, h_1, \ldots, h_{n-1} \\ j_{q-1}, j_{q-2}, \ldots, j_0, & & \\ & & i_0, i_1, \ldots, i_{p-1} \end{matrix} \right\}.$$

When every h, i, j is equal to 3, this means that the elements $O_{np'q}$ of O_{npq} correspond to the elements $n_{p'q}$ of n_{pq}, and to the elements $\alpha_{p-p'-1}$ of p_{qn}, and to the elements $q_{np'}$ of q_{np}. We have thus incidentally proved the generalized "semi-reciprocation theorem" enunciated at the end of §7.8.

Clearly, also, the *vertices* of $\left\{ \begin{matrix} h_0, h_1, \ldots, h_{n-1} \\ i_0, i_1, \ldots, i_{p-1} \\ j_0, j_1, \ldots, j_{q-1} \end{matrix} \right\}$ correspond to the elements

$$\{h_{n-1}, h_{n-2}, \ldots, h_1\}, \quad \{i_{p-1}, i_{p-2}, \ldots, i_1\}, \quad \{j_{q-1}, j_{q-2}, \ldots, j_1\}$$

of the other three polytopes, respectively.

17.2. Let W_0 be a vertex of $\left\{ \begin{matrix} h_0, h_1, \ldots, h_{n-1} \\ i_0, i_1, \ldots, i_{p-1} \\ j_0, j_1, \ldots, j_{q-1} \end{matrix} \right\}$, and let $W_{n'pq}, W_{np'q}, W_{npq'}$ be the centres of "adjacent" elements

$$\left\{ \begin{matrix} h_0, h_1, \ldots, h_{n'-1} \\ i_0, i_1, \ldots, i_{p-1} \\ j_0, j_1, \ldots, j_{q-1} \end{matrix} \right\}, \quad \left\{ \begin{matrix} h_0, h_1, \ldots, h_{n-1} \\ i_0, i_1, \ldots, i_{p'-1} \\ j_0, j_1, \ldots, j_{q-1} \end{matrix} \right\}, \quad \left\{ \begin{matrix} h_0, h_1, \ldots, h_{n-1} \\ i_0, i_1, \ldots, i_{p-1} \\ j_0, j_1, \ldots, j_{q'-1} \end{matrix} \right\}$$

respectively. By saying that these elements are to be "adjacent", we mean that W_0 must be a common vertex of the elements whose centres are W_{0pq}, W_{n0q}, W_{np0}, and that the elements whose centres are $W_{n'pq}$, $W_{np'q}$, $W_{npq'}$ must respectively belong to those whose centres are $W_{(n'+1)pq}$, $W_{n(p'+1)q}$, $W_{np(q'+1)}$.

Having defined

$$W_0, \quad W_{n'pq}, \quad W_{np'q}, \quad W_{npq'}$$
$$(0 \leqslant n' < n, \quad 0 \leqslant p' < p, \quad 0 \leqslant q' < q),$$

let

$$X_n, \quad X_{n-n'-1}, \quad X_{p'q}, \quad X_{pq'}$$

denote the centres of the corresponding elements of

$$\left\{h_{n-1}, h_{n-2}, \ldots, h_0, \begin{matrix} i_0, i_1, \ldots, i_{p-1} \\ j_0, j_1, \ldots, j_{q-1} \end{matrix}\right\}.$$

The conditions of adjacency now state that the element

$$\{h_{n-1}, h_{n-2}, \ldots, h_{n-r+1}\}$$

whose centre is X_r (or the vertex X_0, if $r = 0$) belongs to the element $\{h_{n-1}, h_{n-2}, \ldots, h_{n-r}\}$ whose centre is X_{r+1} $(r < n)$, that the element $\{h_{n-1}, h_{n-2}, \ldots, h_1\}$ whose centre is X_n belongs both to the element $\{h_{n-1}, h_{n-2}, \ldots, h_0, j_0, j_1, \ldots, j_{q-1}\}$ whose centre is X_{0q} and to the element $\{h_{n-1}, h_{n-2}, \ldots, h_0, i_0, i_1, \ldots, i_{p-1}\}$ whose centre is X_{p0}, and that the elements

$$\left\{h_{n-1}, h_{n-2}, \ldots, h_0, \begin{matrix} i_0, i_1, \ldots, i_{p'-1} \\ j_0, j_1, \ldots, j_{q-1} \end{matrix}\right\}, \quad \left\{h_{n-1}, h_{n-2}, \ldots, h_0, \begin{matrix} i_0, i_1, \ldots, i_{p-1} \\ j_0, j_1, \ldots, j_{q'-1} \end{matrix}\right\},$$

whose centres are $X_{p'q}$, $X_{pq'}$, belong respectively to the analogous elements whose centres are $X_{(p'+1)q}$, $X_{p(q'+1)}$.

We next have to define certain primes passing through the common centre (W_{npq} or X_{pq}) of our $(n+p+q+1)$-dimensional polytopes. Each such prime is determined by $n+p+q$ further points lying on it, which points may be taken equally well as W's or as X's:

Prime	Determining W's	Determining X's
0	All save W_0	All save X_n
r_h $(0 \leqslant r < n)$	All save W_{rpq}	All save X_{n-r-1}
r_i $(0 \leqslant r < p)$	All save W_{nrq}	All save X_{rq}
r_j $(0 \leqslant r < q)$	All save W_{npr}	All save X_{pr}

The $n+p+q+1$ primes

$$(17.21) \qquad 0_h, 0_h, 1_h, \ldots, (n-1)_h, \quad 0_i, 1_i, \ldots, (p-1)_i, \quad 0_j, 1_j, \ldots, (q-1)_j$$

bound a spherical simplex, whose vertices are central projections of the W's (or X's) on an $(n+p+q)$-sphere concentric with the polytopes. This simplex is called a *fundamental simplex* for each of our four polytopes (viz. the three semi-reciprocals and their common truncation).

When the polytopes are degenerate, the definition is simpler, since the W's actually coincide with the X's, thus determining a *Euclidean*

fundamental simplex. In this case we can regard 0, r_h, r_i, r_j as primes of the $(n+p+q)$-space filled by the polytopes.

17.3. In order to identify the fundamental simplex with (15.71), we must calculate its dihedral angles. These are simply the angles between pairs of our primes, and may conveniently be called

$$(0\,r_h),\quad (0\,r_i),\quad (0\,r_j),\quad (r_h\,s_h),\quad (r_i\,s_i),\quad (r_j\,s_j),\quad (r_i\,s_j),\quad (r_j\,s_h),\quad (r_h\,s_i).$$

Let us first suppose that $q = 0$.

In this case the W's are the centres of certain elements of

$$\left\{\begin{matrix} h_0,\ h_1,\ \ldots,\ h_{n-1} \\ i_0,\ i_1,\ \ldots,\ i_{p-1} \end{matrix}\right\},$$

while the X's are the centres of elements, one of every kind, of the regular polytope

$$\{h_{n-1},\ h_{n-2},\ \ldots,\ h_0,\ i_0,\ i_1,\ \ldots,\ i_{p-1}\}.$$

In fact, X_0 is a vertex, X_1 is the centre of an edge through X_0, X_2 is the centre of a plane face through this edge, and so on. The process continues as far as X_{n+p}, the centre of a bounding figure, provided that we write $X_{n+p'+1}$ for $X_{p'0}$.

If we put

$$h_r = k_{n-r},\quad i_r = k_{n+r+1},\quad n+p+1 = m, \tag{17.31}$$

then the points X_0, X_1, ..., X_{m-1} are the centres of elements of

$$\{k_1,\ k_2,\ \ldots,\ k_{m-1}\},$$

and it is natural to complete the sequence by letting X_m denote the centre of the whole polytope. Since the circumscribing sphere-analogue of any element is a section of the circumscribing sphere-analogue of any higher element containing that element, all the lines X_0X_1, X_1X_2, ..., $X_{m-1}X_m$ are mutually perpendicular. Hence, if r denotes the prime determined by all the X's save X_{r-1}, and $(r\ s)$ denotes the angle between the primes r and s, it follows that $(r\ s)$ is a right angle whenever r and s differ by more than 1. It remains to prove that

$$(r\ r+1) = \pi/k_r \quad (r = 1,\ 2,\ \ldots,\ m-1). \tag{17.32}$$

[The remaining angle, $(m\ m+1)$, is irrelevant to our present purpose.]

When $m = 2$, we have a regular k_1-gon, centre X_2. X_0 is one end of the side whose centre is X_1. The angle $X_1 X_2 X_0$ is clearly

$$(1\,2) = \pi/k_1.$$

We can therefore use induction, and assume the corresponding result for all regular $(m-1)$-dimensional polytopes. Since the analogous X's for the bounding figure $\{k_1, k_2, \ldots, k_{m-2}\}$ whose centre is X_{m-1} are precisely

$$X_0,\ X_1,\ \ldots,\ X_{m-1},$$

we have

$$(r\,r+1) = \pi/k_r \quad (r = 1, 2, \ldots, m-2). \tag{17.33}$$

Further, if $X_0', X_1', \ldots, X_{m-1}'$ are the analogous X's for the vertex figure $\{k_2, k_3, \ldots, k_{m-1}\}$ at the vertex X_0, then X_{r-1}' and X_r are collinear with X_0 $(r = 1, 2, \ldots, m)$. Therefore

$$(r\,r+1) = \pi/k_r \quad (r = 2, 3, \ldots, m-1). \tag{17.34}$$

(17.33) and (17.34) together give (17.32).

Reverting to the other notation, we have

$$(0\,r_h) = (n+1\ n-r), \quad (0\,r_i) = (n+1\ n+r+2),$$

$(r_h\,s_h) = (n-r\ n-s)$, $(r_i\,s_i) = (n+r+2\ n+s+2)$, $(r_h\,s_i) = (n-r\ n+s+2)$, and therefore, if $q = 0$,

$$\begin{cases} (0\,r_h) = (0\,r_i) = \tfrac{1}{2}\pi & (r > 0), \\ (r_h\,s_h) = (r_i\,s_i) = \tfrac{1}{2}\pi & (|r-s| > 1), \\ (r_h\,s_i) = \tfrac{1}{2}\pi, & \\ (0\,0_h) = \pi/h_0, \quad (0\,0_i) = \pi/i_0, & \\ \left((r-1)_h r_h\right) = \pi/h_r, \quad \left((r-1)_i r_i\right) = \pi/i_r & (r > 0). \end{cases} \tag{17.35}$$

Interchanging q and n, and j and h; if $n = 0$,

$$\begin{cases} (0\,r_i) = (0\,r_j) = \tfrac{1}{2}\pi & (r > 0), \\ (r_i\,s_i) = (r_j\,s_j) = \tfrac{1}{2}\pi & (|r-s| > 1), \\ (r_i\,s_j) = \tfrac{1}{2}\pi, & \\ (0\,0_i) = \pi/i_0, \quad (0\,0_j) = \pi/j_0, & \\ \left((r-1)_i r_i\right) = \pi/i_r, \quad \left((r-1)_j r_j\right) = \pi/j_r & (r > 0). \end{cases} \tag{17.36}$$

Similarly, if $p = 0$,

$$(17.37)\quad \begin{cases} (0\,r_j) = (0\,r_h) = \frac{1}{2}\pi & (r > 0), \\ (r_j\,s_j) = (r_h\,s_h) = \frac{1}{2}\pi & (|r-s| > 1), \\ (r_j\,s_h) = \frac{1}{2}\pi, & \\ (0\,0_j) = \pi/j_0, \quad (0\,0_h) = \pi/h_0, & \\ \big((r-1)_j\,r_j\big) = \pi/j_r, \quad \big((r-1)_h\,r_h\big) = \pi/h_r & (r > 0). \end{cases}$$

Let X'_{n-1}, $X'_{n-n'-2}$, $X'_{p'q}$, $X'_{pq'}$ be the points associated with

$$\left\{ h_{n-2}, h_{n-3}, \ldots, h_0, \begin{matrix} i_0, i_1, \ldots, i_{p-1} \\ j_0, j_1, \ldots, j_{q-1} \end{matrix} \right\}$$

in the same way as the points X_n, $X_{n-n'-1}$, $X_{p'q}$, $X_{pq'}$ are associated with

$$\left\{ h_{n-1}, h_{n-2}, \ldots, h_0, \begin{matrix} i_0, i_1, \ldots, i_{p-1} \\ j_0, j_1, \ldots, j_{q-1} \end{matrix} \right\}.$$

If the former polytope is chosen to be the actual vertex figure of the latter at the vertex X_0, then X'_{r-1} and X_r are collinear with X_0 ($r = 1, 2, \ldots, n$). So also are $X'_{p'q}$ and $X_{p'q}$, and $X'_{pq'}$ and $X_{pq'}$. Therefore the angles involved in (17.36) are independent of n. Thus the restriction "$n = 0$" can be removed. Similarly, the restrictions "$p = 0$" and "$q = 0$" can be removed from (17.37) and (17.35) respectively. (17.35), (17.36) and (17.37) together give *all* the dihedral angles of the fundamental simplex, viz.:

$$(17.38)\quad \begin{cases} (0\,r_h) = (0\,r_i) = (0\,r_j) = \frac{1}{2}\pi & (r > 0), \\ (r_h\,s_h) = (r_i\,s_i) = (r_j\,s_j) = \frac{1}{2}\pi & (|r-s| > 1), \\ (r_i\,s_j) = (r_j\,s_h) = (r_h\,s_i) = \frac{1}{2}\pi, & \\ (0\,0_h) = \pi/h_0, \quad (0\,0_i) = \pi/i_0, \quad (0\,0_j) = \pi/j_0, & \\ \big((r-1)_h\,r_h\big) = \pi/h_r, \ \big((r-1)_i\,r_i\big) = \pi/i_r, \ \big((r-1)_j\,r_j\big) = \pi/j_r & (r > 0). \end{cases}$$

Thus the primes (17.21) can be identified with the primes (15.81), and the fundamental simplex with the simplex (15.71).

17.4. As in §16.7, let O, N_r, P_r, Q_r denote the reflections in the primes 0, r_h, r_i, r_j, respectively. With the help of the assumption made in §2.4, we can prove by induction that these reflections are symmetries of $\left\{ h_{n-1}, h_{n-2}, \ldots, h_0, \begin{matrix} i_0, i_1, \ldots, i_{p-1} \\ j_0, j_1, \ldots, j_{q-1} \end{matrix} \right\}$, and so also of the related polytopes.

To do this we first prove, by induction with respect to m, the corresponding result for $\{k_1, k_2, \ldots, k_{m-1}\}$, then observe that every symmetry of $\{i_{p-1}, i_{p-2}, \ldots, i_0, j_0, j_1, \ldots, j_{q-1}\}$ is also a symmetry of $\left\{\begin{matrix} i_0, i_1, \ldots, i_{p-1} \\ j_0, j_1, \ldots, j_{q-1} \end{matrix}\right\}$, and finally use induction with respect to n.

Apart from two exceptional cases, the same method serves to prove that these reflections actually generate the whole group of symmetries of $\left\{h_{n-1}, h_{n-2}, \ldots, h_0, \begin{matrix} i_0, i_1, \ldots, i_{p-1} \\ j_0, j_1, \ldots, j_{q-1} \end{matrix}\right\}$, thus identifying this group of symmetries with the group

$$\left[\begin{matrix} h_0, h_1, \ldots, h_{n-1} \\ i_0, i_1, \ldots, i_{p-1} \\ j_0, j_1, \ldots, j_{q-1} \end{matrix}\right] \tag{17.41}$$

whose abstract definition is (16.74).

The first exceptional case is when

$$\{i_0, i_1, \ldots, i_{p-1}\} = \{j_0, j_1, \ldots, j_{q-1}\},$$

since then $\left\{\begin{matrix} i_0, i_1, \ldots, i_{p-1} \\ j_0, j_1, \ldots, j_{q-1} \end{matrix}\right\}$ has twice as many symmetries as

$$\{i_{p-1}, i_{p-2}, \ldots, i_0, j_0, j_1, \ldots, j_{q-1}\}.$$

(See §5.5.) The second exceptional case is when

$$i_0 = j_0, i_1 = 4 = j_1 \quad \text{and} \quad i_2 = i_3 = \ldots = 3 = j_2 = j_3 = \ldots,$$

since then $\left\{\begin{matrix} i_0, i_1, \ldots, i_{p-1} \\ j_0, j_1, \ldots, j_{q-1} \end{matrix}\right\}$ has $\binom{p+q}{p}$ times as many symmetries as $\{i_{p-1}, i_{p-2}, \ldots, i_0, j_0, j_1, \ldots, j_{q-1}\}$*.

Thus, whenever the group of symmetries of

$$\left\{h_{n-1}, h_{n-2}, \ldots, h_0, \begin{matrix} i_0, i_1, \ldots, i_{p-1} \\ j_0, j_1, \ldots, j_{q-1} \end{matrix}\right\}$$

---

* The only actual examples of this second exception are:

$p = 1,\ q = 2,\ i_0 = 3,\ n = 0$ or $1,\ h_0 = 3$;

$p = 2,\ q = 2,\ \ = 3,\ n = 0.$

is not identical with (17.41), it contains the latter as a sub-group. The same can, of course, be said for the group of symmetries of $\begin{Bmatrix} h_0, h_1, \ldots, h_{n-1} \\ i_0, i_1, \ldots, i_{p-1} \\ j_0, j_1, \ldots, j_{q-1} \end{Bmatrix}$, since this polytope is a truncation of the other.

17.5. When every h, i, j is equal to 3, the results are as follows. If n, p, q are all different, or if $p = q = 0$, the groups of symmetries of n_{pq}, of p_{qn}, of q_{np}, and of O_{npq}, are one and the same group, viz. $\begin{bmatrix} 3^n \\ 3^p \\ 3^q \end{bmatrix}$. If $n \neq p = q \neq 0$, the group of symmetries of p_{qn}, *i.e.* of p_{pn}, is $\begin{bmatrix} 3^n \\ 3^p \\ 3^p \end{bmatrix}$, and is a sub-group of index 2 in the group of symmetries of n_{pp} or of O_{npp}. Finally, if $n = p = q \neq 0$, the group of symmetries of n_{pq}, *i.e.* of n_{nn}, contains $\begin{bmatrix} 3^n \\ 3^n \\ 3^n \end{bmatrix}$ as a sub-group of index 2, and is itself a sub-group of index 3 in the group of symmetries of O_{nnn}.

By (12.23), it follows that the order of the group $\begin{bmatrix} 3^n \\ 3^p \\ 3^q \end{bmatrix}$ is always

$$(n+1)!\ (p+1)!\ (q+1)!\ [npq]. \tag{17.51}$$

For the values of $[npq]$, see §12.6. The order of the unextended group $\begin{bmatrix} 3^n \\ 3^p \\ 3^q \end{bmatrix}'$ is just half as great. Tabulating results in the finite cases, we have:

n	p	q	Order of $\begin{bmatrix} 3^n \\ 3^p \\ 3^q \end{bmatrix}$	Order of $\begin{bmatrix} 3^n \\ 3^p \\ 3^q \end{bmatrix}'$
0	p	q	$(p+q+2)!$	$\frac{1}{2}(p+q+2)!$
n	1	1	$2^{n+2}(n+3)!$	$2^{n+1}(n+3)!$
2	2	1	51840	25920
3	2	1	2903040	1451520
4	2	1	696729600	348364800

In particular, as we saw in § 13.1, $\begin{bmatrix} 3, 3 \\ 3, 3 \\ 3, \end{bmatrix}$ is the group of automorphisms of the 27 lines on a general cubic surface. $\begin{bmatrix} 3, 3 \\ 3, 3 \\ 3, \end{bmatrix}'$, being a self-conjugate sub-group of index 2, must therefore be identical with the simple group* $A(4, 3)$.

17.6. For a one-rowed symbol there are no "exceptional cases": $[k_1, k_2, \ldots, k_{m-1}]$, which is the same as $[k_{m-1}, k_{m-2}, \ldots, k_1]$, is precisely the group of symmetries of $\{k_1, k_2, \ldots, k_{m-1}\}$; while $[k_1, k_2, \ldots, k_{m-1}]'$ is the group of *positive* symmetries. The finite cases are as follows:

$[3^{m-1}]$, of order $(m+1)!$ (the symmetric group);

$[3^{m-1}]'$, of order $\frac{1}{2}(m+1)!$ (the alternating group);

$[3^{m-2}, 4]$, of order $2^m m!$; $[3^{m-2}, 4]'$, of order $2^{m-1} m!$;

$[k]$, of order $2k$; $[k]'$, of order k (the cyclic group);

$[3, 5]$, of order 120; $[3, 5]'$, of order 60 (the icosahedral group);

$[3, 3, 5]$, of order 14400; $[3, 3, 5]'$, of order 7200;

$[3, 4, 3]$, of order 1152; $[3, 4, 3]'$, of order 576.

These groups are not all distinct. In fact, the following two simple isomorphisms are well known:

$$[3, 3] \sim [3, 4]', \quad [3, 3, 3]' \sim [3, 5]'.$$

17.7. If a certain finite polytope $\Pi^{(1)}_{m_1}$ has a fundamental simplex determined by the centres of m_1 particular elements $\Pi^{(1)}_{r_1}$, and similarly for $\Pi^{(2)}_{m_2}$, then the fundamental simplex of the generalized prism

$$[\Pi^{(1)}_{m_1}, \ \Pi^{(2)}_{m_2}]$$

is defined as being determined by the centres of the m_1+m_2 elements

$$[\Pi^{(1)}_{r_1}, \ \Pi^{(2)}_{m_2}] \quad \text{and} \quad [\Pi^{(1)}_{m_1}, \ \Pi^{(2)}_{r_2}].$$

It is easily seen that the essential features of a fundamental simplex are maintained, since the reflections in its bounding primes are symmetries of the whole prism.

---

* L. E. Dickson, *Linear groups* (1901), 306–307.

If the vertices of the prism are given by coordinates of the form

$$(x_1, \ldots, x_p;\ x_{p+1}, \ldots, x_q)$$

as in § 4.1, then the prime determined by the centres of all but one of the elements $[\Pi^{(1)}_{r_1}, \Pi^{(2)}_{m_2}]$ and of all the elements $[\Pi^{(1)}_{m_1}, \Pi^{(2)}_{r_2}]$ has an equation of the form

$$F(x_1, \ldots, x_p) = 0.$$

Likewise, the prime determined by the centres of all the elements $[\Pi^{(1)}_{r_1}, \Pi^{(2)}_{m_2}]$ and of all but one of the elements $[\Pi^{(1)}_{m_1}, \Pi^{(2)}_{r_2}]$ has an equation of the form

$$G(x_{p+1}, \ldots, x_q) = 0.$$

Since these two primes are perpendicular, the fundamental simplex is of the type considered in § 15.3, the "two simpler simplexes" being the fundamental simplexes of $\Pi^{(1)}_{m_1}$ and $\Pi^{(2)}_{m_2}$.

In this connection it only remains to observe that the extended group which has this simplex for a fundamental region is the direct product of the groups similarly related to the simpler simplexes; and that the group of symmetries of the prism either is, or contains as a sub-group, the direct product of the groups of symmetries of the constituents.

17.8. We saw in § 14.8 that, if $j_{q-2} = 3$ and $j_{q-1} = 4$, the row

$$j_0, j_1, \ldots, j_{q-1}$$

of an extended Schläfli symbol can be replaced by

$$j_0, j_1, \ldots, j_{q-3}, \begin{matrix} 3 \\ 3 \end{matrix}$$

without altering the polytope represented. Let us see how this transformation affects the corresponding simplex.

The simplex represented by the transformed symbol is bounded by primes $r_h, r_i, 0$ and $0_j, 1_j, \ldots, (q-3)_j$, along with two extra primes, say $(q-2)_j$ and $(q-2)_j'$, each inclined at an angle $\frac{1}{3}\pi$ to $(q-3)_j$, but perpendicular to one another and to all the other primes. Let $(q-1)_j$ denote the bisector of the angle $\left((q-2)_j(q-2)_j'\right)$. Then, since $(q-2)_j$ and $(q-2)_j'$ are similarly situated with respect to the rest of the primes, the new prime $(q-1)_j$ divides the simplex into two equal halves, simplexes whose bounding primes are obtained from those of the whole simplex by replacing $(q-2)_j'$ or $(q-2)_j$ by $(q-1)_j$. Of the half which involves $(q-2)_j$, the bounding prime $(q-1)_j$ is perpendicular to all the others except $(q-2)_j$.

The angle $\left((q-2)_j\,(q-1)_j\right) = \frac{1}{2}\left((q-2)_j\,(q-2)_j'\right) = \frac{1}{4}\pi$. Thus the effect of changing $\begin{matrix}3\\3\end{matrix}$ into 3, 4 is to halve the simplex, and so to double the order of the group for which the simplex is a fundamental region. In fact, a new operation of period 2, viz. the reflection in $(q-1)_j$, is introduced into the group.

The group of symmetries of δ_m or $\{4,\ 3^{m-3},\ 4\}$ is $[4,\ 3^{m-3},\ 4]$ or $\left[\begin{matrix}3,\ 4\\ 3^{m-4},\ 4\end{matrix}\right]$. Therefore

$$\left[\begin{matrix}3\\3\\3^{m-4},\ 4\end{matrix}\right]$$

is a sub-group of index 2 in this group, and

$$\left[\begin{matrix}3\\3\\3^{m-5},\ \begin{matrix}3\\3\end{matrix}\end{matrix}\right]$$

is a sub-group of index 4. This last symbol is given in a more symmetrical form in (16.63). When $m = 5$, it reduces to

$$\left[\begin{matrix}3\\3\\3\\3\end{matrix}\right].$$

17.9. Consider the point whose coordinates consist of r repetitions of $1-(r/m)$ followed by $m-r$ repetitions of $-(r/m)$. For all values of r, this point lies in the $(m-1)$-space

$$x_1+x_2+\ldots+x_m = 0. \tag{17.91}$$

The point is the origin both when $r = 0$ and when $r = m$. The m points obtained by giving r the values $0, 1, \ldots, m-1$ are the vertices of a Euclidean simplex whose bounding primes are

$$x_1-x_2 = 0,\quad x_2-x_3 = 0,\quad \ldots,\quad x_{m-1}-x_m = 0,\quad x_1-x_m = 1. \tag{17.92}$$

Since these, regarded as primes in m dimensions, are all perpendicular to (17.91), we can obtain the angles between them by the usual rule. In fact, calling them $1, 2, \ldots, m$, we have

$$(1\ 2) = (2\ 3) = \ldots = (m-1\ m) = (m\ 1) = \tfrac{1}{3}\pi \quad \text{and} \quad (r\ s) = \tfrac{1}{2}\pi$$
$$(1 < |r-s| < m-1).$$

Thus this simplex is of the kind considered in § 15.4.

The reflections in the first $m-1$ of the primes (17.92) are simply the transpositions of consecutive pairs of coordinates. These generate the symmetric group on $x_1, x_2, \ldots, x_m$, which is the group of symmetries of α_{m-1}. Since this is a sub-group of index 2 in the group of symmetries of $e\alpha_{m-1}$ (see §6.9), and since the remaining reflection, viz. that in $x_1 - x_m = 1$, changes the origin into the point $(1; 0, 0, \ldots, 0; -1)$, it follows that $\boxed{3^m}$ (defined in §16.5) is a sub-group of index 2 in the group of symmetries of $\alpha_{m-1}h$.

Having now considered every possibility, we can assert that *every group of real orthogonal substitutions on m variables, having as fundamental region a simplex all of whose dihedral angles are submultiples of π, is either the whole group of symmetries of some m-dimensional uniform polytope, or a sub-group thereof.*

18. *The twenty-seven lines and the twenty-eight bitangents.*

18.1. The most important of our *extended* groups is

$$\begin{bmatrix} 3, 3 \\ 3, 3 \\ 3 \end{bmatrix},$$

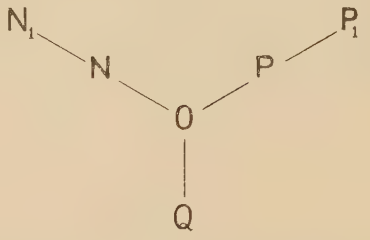
since this, being the group of symmetries of 2_{21} or $(PA)_6$, is also the group of automorphisms of the lines on a general cubic surface.

By (16.74) it has the *abstract definition*

$$(18.11)\quad \begin{cases} O^2 = 1, \\ N^2 = N_1{}^2 = 1, \\ P^2 = P_1{}^2 = 1, \\ Q^2 = 1, \\ (ON)^3 = (NN_1)^3 = 1, \\ (OP)^3 = (PP_1)^3 = 1, \\ (OQ)^3 = 1, \\ (ON_1)^2 = (OP_1)^2 = 1, \\ (PQ)^2 = (P_1Q)^2 = (QN)^2 = (QN_1)^2 \\ \qquad = (NP)^2 = (N_1P)^2 = (NP_1)^2 = (N_1P_1)^2 = 1. \end{cases}$$

For simplicity we have written N for N_0, P for P_0, Q for Q_0.

This abstract definition is conveniently represented by the following diagram:



The six generating operations are each of period 2; all pairs of them not directly linked in the diagram are permutable, and the products of linked pairs are of period 3. Analogous diagrams can be made for all our abstract definitions of extended groups (each link, in the general case, being marked with the period of the corresponding product).

18.2. In the notation of § 9.4, the α_4

$$b_3 b_4 b_5 b_6 c_{12}$$

belongs to both the β_5's

$$b_2 b_3 b_4 b_5 b_6 c_{12} c_{13} c_{14} c_{15} c_{16} \quad \text{and} \quad b_1 b_3 b_4 b_5 b_6 c_{12} c_{32} c_{42} c_{52} c_{62},$$

and is therefore of *type* 2_{01} (see § 7.5). On the other hand, the α_4

$$b_2 b_3 b_4 b_5 b_6$$

belongs to the former of these β_5's and also to the α_5 $(= 2_{20})$

$$b_1 b_2 b_3 b_4 b_5 b_6,$$

so that it is of type 2_{10}.

It follows that the requirements of § 17.2 are satisfied if we take

X_0	to be the vertex					b_6,
X_1	,,	,,	centre of the edge			$b_5 b_6$,
X_2	,,	,,	,,	,,	α_2	$b_4 b_5 b_6$,
X_{01}	,,	,,	,,	,,	α_4	$b_3 b_4 b_5 b_6 c_{12}$,
X_{11}	,,	,,	,,	,,	β_5	$b_2 b_3 b_4 b_5 b_6 c_{12} c_{13} c_{14} c_{15} c_{16}$,
X_{20}	,,	,,	,,	,,	α_5	$b_1 b_2 b_3 b_4 b_5 b_6$.

These six points determine the six primes

$$1_h,\ 0_h,\ 0,\ 0_i,\ 1_i,\ 0_j$$

(by omitting them in turn, and joining the remaining five to the centre X_{21}). By § 17 . 4, the generating operations

$$N_1,\ N,\ O,\ P,\ P_1,\ Q$$

are the reflections in these primes respectively.

18 . 3. Consider the transpositions (12)*, (23), (34), (45), (56), and the bifid substitution [456 . 123], defined in (9 . 44).

(56)	alters	X_0	but leaves the other	X's	invariant,		
(45)	,,	X_1	,,	,,	,,	,,	,
(34)	,,	X_2	,,	,,	,,	,,	,
(23)	,,	X_{01}	,,	,,	,,	,,	,
(12)	,,	X_{11}	,,	,,	,,	,,	,
[456 . 123]	,,	X_{20}	,,	,,	,,	,,	.

Therefore

$$(18.31)\quad \begin{cases} N_1 = (56), \quad N = (45), \quad O = (34), \quad P = (23), \quad P_1 = (12), \\ \text{and} \qquad\qquad Q = [456 . 123]. \end{cases}$$

In terms of the lines on the cubic surface (Schläfli's notation), the operation (12) consists in interchanging the two halves of the double-six

$$\begin{pmatrix} a_1 b_1 c_{23} c_{24} c_{25} c_{26} \\ a_2 b_2 c_{13} c_{14} c_{15} c_{16} \end{pmatrix},$$

and the operation [456 . 123] consists in interchanging the two halves of the double-six

$$\begin{pmatrix} a_4\ a_5\ a_6\ c_{23} c_{31} c_{12} \\ c_{56} c_{64} c_{45} b_1\ b_2\ b_3 \end{pmatrix}.$$

18 . 4. Clearly the generating operations

$$N_1,\ N,\ O,\ P,\ P_1$$

can be taken to be any open chain of transpositions, and then Q can be either of the two bifid substitutions which separate the numbers involved in N and N_1 from the numbers involved in P and P_1. In order to employ

* Not to be confused with the (1 2) of § 15 . 1.

the notation of § 13 . 5, it is natural to choose the chain

$$(23),\ (36),\ (61),\ (14),\ (45).$$

Q must now be either [236 . 145] or [145 . 236]. Taking the latter value, we have

$$Q = [145 . 236] = (13)\,[345 . 126](13).$$

But, as we observed in § 13 . 9,

$$[345 . 126] = H_4.$$

Moreover,

$$(13) = (61)(36)(61).$$

Hence, by § 13 . 5,

$$N_1 = H_6 H_8 H_6, \quad N = H_2, \quad O = H_0, \quad P = H_7, \quad P_1 = H_3 H_1 H_3,$$

and

$$Q = H_0 H_2 H_0 H_4 H_0 H_2 H_0.$$

On substituting in (18.11) we obtain an abstract definition for the group in terms of the H's, and so ultimately in terms of the *two* operations ω and H_0. But the new abstract definition is excessively complicated; in fact, the definition in terms of *six* operations is altogether preferable.

18 . 5. The most important of our *unextended* groups is

$$\begin{bmatrix} 3, 3, 3 \\ 3, 3 \\ 3 \end{bmatrix}',$$

since this, being (by § 17 . 5) the group of *positive* symmetries of 3_{21} or $(PA)_7$, is also (by § 11 . 5) the group of automorphisms of the bitangents of a plane quartic of genus 3.

By (16 . 75) it has the *abstract definition*

$$(18.51)\quad \begin{cases} N'^3 = N_1'^2 = N_2'^2 = (N'N_1')^3 = (N_1'N_2')^3 = 1, \\ P'^3 = P_1'^2 = (P'P_1')^3 = 1, \\ Q'^3 = 1, \\ (N'N_2')^2 = 1, \\ (P_1'Q')^2 = (Q'N_1')^2 = (Q'N_2')^2 = (N_1'P')^2 \\ \quad = (N_2'P')^2 = (N'P_1')^2 = (N_1'P_1')^2 = (N_2'P_1')^2 = 1, \\ (P'Q'^{-1})^2 = (Q'N')^2 = (N'P')^2 = 1. \end{cases}$$

For simplicity we have written N' for $N_0'^{-1}$, P' for P_0', Q' for Q_0'.

18.6. In the notation of § 9.3, the α_5

$$c_{12}\,C_{38}\,C_{48}\,C_{58}\,C_{68}\,C_{78}$$

belongs to both the β_6's

$$c_{12}\,c_{13}\,c_{14}\,c_{15}\,c_{16}\,c_{17}\,C_{28}\,C_{38}\,C_{48}\,C_{58}\,C_{68}\,C_{78}$$

and

$$c_{12}\,c_{32}\,c_{42}\,c_{52}\,c_{62}\,c_{72}\,C_{18}\,C_{38}\,C_{48}\,C_{58}\,C_{68}\,C_{78},$$

and is therefore of type 3_{01}. On the other hand, the α_5

$$C_{28}\,C_{38}\,C_{48}\,C_{58}\,C_{68}\,C_{78}$$

belongs to the former of the β_6's and also to the α_6 $(= 3_{20})$

$$C_{18}\,C_{28}\,C_{38}\,C_{48}\,C_{58}\,C_{68}\,C_{78},$$

so that it is of type 3_{10}.

It follows that the requirements of § 17.2 are satisfied if we take

X_0	to be the vertex	C_{78},
X_1	,, ,, centre of the edge	$C_{68}\,C_{78}$,
X_2	,, ,, ,, ,, α_2	$C_{58}\,C_{68}\,C_{78}$,
X_3	,, ,, ,, ,, α_3	$C_{48}\,C_{58}\,C_{68}\,C_{78}$,
X_{01}	,, ,, ,, ,, α_5	$c_{12}\,C_{38}\,C_{48}\,C_{58}\,C_{68}\,C_{78}$,
X_{11}	,, ,, ,, ,, β_6	$c_{12}\,c_{13}\,c_{14}\,c_{15}\,c_{16}\,c_{17}\,C_{28}\,C_{38}\,C_{48}\,C_{58}\,C_{68}\,C_{78}$,
X_{20}	,, ,, ,, ,, α_6	$C_{18}\,C_{28}\,C_{38}\,C_{48}\,C_{58}\,C_{68}\,C_{78}$.

These seven points determine the seven primes

$$2_h,\ 1_h,\ 0_h,\ 0,\ 0_i,\ 1_i,\ 0_j$$

(by omitting them in turn, and joining the remaining six to the centre X_{21}). By § 17.4, the generating operations of the corresponding extended group, viz.

$$N_2,\ N_1,\ N_0,\ O,\ P_0,\ P_1,\ Q_0,$$

are the reflections in these primes respectively. Finally, by (16.73),

$$N_2' = N_2 O = O N_2, \quad N_1' = N_1 O = O N_1, \quad N' = (N_0 O)^{-1} = O N_0,$$

$$P' = P_0 O, \quad P_1' = P_1 O, \quad Q' = Q_0 O.$$

18.7. Consider the transpositions (12), (23), (34), (45), (56), (67), and the bifid reflection [4567.1238], defined in §9.3.

(67)	alters	X_0	but leaves the other	X's	invariant,		
(56)	,,	X_1	,,	,,	,,	,,	,
(45)	,,	X_2	,,	,,	,,	,,	,
(34)	,,	X_3	,,	,,	,,	,,	,
(23)	,,	X_{01}	,,	,,	,,	,,	,
(12)	,,	X_{11}	,,	,,	,,	,,	,
[4567.1238]	,,	X_{20}	,,	,,	,,	,,	.

Therefore

$$N_2 = (67), \quad N_1 = (56), \quad N_0 = (45), \quad O = (34), \quad P_0 = (23),$$
$$P_1 = (12), \quad \text{and} \quad Q_0 = [4567.1238].$$

Finally,

$$(18.71) \quad \begin{cases} N_2' = (34)(67), \quad N_1' = (34)(56), \quad N' = (345), \quad P' = (234), \\ P_1' = (12)(34), \quad \text{and} \quad Q' = [4567.1238](34). \end{cases}$$

18.8. Let us now express the generating operations

$$N_2', \ N_1', \ N', \ P', \ P_1', \ Q'$$

in terms of the cyclic permutation and bifid substitution of §11.5. We shall call the latter K_0, so that*

$$K_0 = [1357.2468]\mathrm{ST}.$$

It is convenient also to let

$$K_n = (1234567)^{-n} K_0 (1234567)^n,$$

so that $K_1 = [2461.3578]\mathrm{ST}$, and so on. (ST is permutable with every operation.)

From (9.35) we derive fourteen relations such as

$$K_0 K_1 K_0 = (18)\mathrm{ST} = K_1 K_0 K_1.$$

* The expression at the end of §11.5 lacks the S, in error.

Six of these lead to expressions for the N''s and P''s. In order to obtain Q', we observe that

$$[4567 \,.\, 1238] = (26)\, K_4 \mathrm{ST}(26),$$

and

$$(26) = (28)(68)(28).$$

The results are as follows:

$$N_2' = K_2 K_4 K_2 K_0 K_5 K_0, \quad N_1' = K_4 K_2 K_6 K_4, \quad N' = K_2 K_4 K_2 K_3 K_5 K_3,$$

$$P' = K_1 K_3 K_1 K_2 K_4 K_2, \quad P_1' = K_2 K_0 K_4 K_2,$$

and

$$Q' = K_1 K_2 K_1 K_5 K_6 K_5 K_1 K_2 K_1 K_4 K_1 K_2 K_1 K_5 K_6 K_5 K_2 K_1 K_4 K_2.$$

On substituting in (18.51), we obtain an abstract definition for the group in terms of the K's, and so ultimately in terms of the *two* operations (1234567) and K_0. It is possible that this new abstract definition could be simplified by the exercise of some ingenuity.

19. *The hundred and twenty tritangent planes**.

19.1. If $m < 9$, a set of m points of general position in a plane determines a finite number of rational curves which have the property of being completely specified by their multiplicities at these points. It is shown by Du Val, in a paper which will shortly appear, that these rational curves are in correspondence with the vertices of $(PA)_m$. These curves have one variable intersection with any cubic passing simply through the m points. When $m < 8$, such cubics represent the prime sections of the Del Pezzo surface† F_2^{9-m} (of order $9-m$, in $9-m$ dimensions), and the rational curves represent the lines on these surfaces. Thus *there is a perfect correspondence between the lines on the Del Pezzo surface* F_2^n *and the vertices of* $(PA)_{9-n}$.

Since F_2^3 is the cubic surface in ordinary space, while F_2^2 is the double plane branching along a quartic curve of genus three, these are the cases considered in Chapter 18.

Since plane cubics through eight points all pass through a ninth, there is no corresponding surface when $m = 8$. If, however, we consider sextic curves passing doubly through the eight points, we obtain a surface having on it 240 conics which correspond to the vertices of $(PA)_8$. This surface

* *Chapter* 19 *was added* 11 *June*, 1932.

† *Rend. di Palermo*, 1 (1887), 241.

is, in fact, a double quadric cone in ordinary space, having for branch curve the sextic of genus four in which it is cut by a general cubic surface. The section of the double cone by one of the 120 tritangent planes of this curve breaks up into a pair of conics which coincide in space but lie on different sheets of the double cone. Hence *the* 120 *tritangent planes of this special quadri-cubic curve correspond to the* 120 *pairs of opposite vertices of* $(PA)_8$. This theorem is due to Todd.

Let i, j, k denote any three numbers from among 1, 2, ..., 8. The rational curves determined by eight base points are actually* as follows:

8 points c_i, which can be regarded as curves of order zero;

28 lines a_{ij}, of which a_{78} joins c_7, c_8;

56 conics b_{ijk}, of which b_{678} is determined by the five points $c_1, c_2, \ldots, c_5$;

56 cubics c_{ij}, of which c_{18} goes once through $c_2, c_3, \ldots, c_7$, and twice through c_8;

56 quartics a_{ijk}, of which a_{678} goes once through $c_1, c_2, \ldots, c_5$, and twice through c_6, c_7, c_8;

28 quintics b_{ij}, of which b_{78} goes twice through $c_1, c_2, \ldots, c_6$, and once through c_7, c_8;

8 sextics c_i', of which c_8' goes twice through $c_1, c_2, \ldots, c_7$, and thrice through c_8.

(The order of the suffixes of a or b is immaterial, but c_{ij} and c_{ji} are distinct.)

The corresponding vertices of $(PA)_8$ are respectively

$$c_{i9},\ a_{ij9},\ b_{ijk},\ c_{ij},\ a_{ijk},\ b_{ij9},\ c_{9i},$$

where, using the coordinates (10.21) for $(PA)_8$ $3\sqrt{2}$,

$$a_{789} \text{ is } (1, 1, 1, 1, 1, 1; -2, -2, -2),$$

$$b_{123} \text{ is } (2, 2, 2; -1, -1, -1, -1, -1, -1),$$

$$c_{19} \text{ is } (3; 0, 0, 0, 0, 0, 0, 0; -3),$$

and so on. The relation is such that two curves having $r-1$ free intersections correspond to two vertices whose mutual distance is $\sqrt{r}$ times the edge $(3\sqrt{2})$.

* Noether, *Math. Annalen*, 33 (1888), 534. Coble, *Algebraic geometry and theta-functions* (New York, 1929), 209.

19.2. The group of automorphisms of the rational curves, being the group of symmetries of $(PA)_8$, is

$$\begin{bmatrix} 3, 3, 3, 3 \\ 3, 3 \\ 3 \end{bmatrix}. \tag{19.21}$$

By (16.74), this group is generated, and in fact abstractly defined, by eight involutory operations

$$O,\ N,\ N_1,\ N_2,\ N_3,\ P,\ P_1,\ Q$$

which satisfy the relations

$$\begin{aligned} (ON)^3 = (NN_1)^3 = (N_1N_2)^3 = (N_2N_3)^3 &= 1, \\ (OP)^3 \quad = (PP_1)^3 \quad &= 1, \\ (OQ)^3 \quad &= 1, \end{aligned}$$

all other pairs being permutable.

The α_6 $\qquad c_{39}c_{49}c_{59}c_{69}c_{79}c_{89}a_{129}$

belongs to both the β_7's

$$c_{29}c_{39}c_{49}c_{59}c_{69}c_{79}c_{89}a_{129}a_{139}a_{149}a_{159}a_{169}a_{179}a_{189}$$

and $\qquad c_{19}c_{39}c_{49}c_{59}c_{69}c_{79}c_{89}a_{129}a_{239}a_{249}a_{259}a_{269}a_{279}a_{289},$

and is therefore of type 4_{01}. On the other hand, the α_6

$$c_{29}c_{39}c_{49}c_{59}c_{69}c_{79}c_{89}$$

belongs to the former of these β_7's and also to the $\alpha_7(=4_{20})$

$$c_{19}c_{29}c_{39}c_{49}c_{59}c_{69}c_{79}c_{89},$$

so that it is of type 4_{10}.

It follows that the requirements of § 17.2 are satisfied if we take

X_0	to be the vertex					c_{89},
X_1	,,	,,	centre of the edge			$c_{79}c_{89}$,
X_2	,,	,,	,,	,,	α_2	$c_{69}c_{79}c_{89}$,
X_3	,,	,,	,,	,,	α_3	$c_{59}c_{69}c_{79}c_{89}$,
X_4	,,	,,	,,	,,	α_4	$c_{49}c_{59}c_{69}c_{79}c_{89}$,
X_{01}	,,	,,	,,	,,	α_6	$c_{39}c_{49}c_{59}c_{69}c_{79}c_{89}a_{129}$,
X_{11}	,,	,,	,,	,,	β_7	$c_{29}c_{39}c_{49}c_{59}c_{69}c_{79}c_{89}a_{129}a_{139}\dots a_{189}$,
X_{20}	,,	,,	,,	,,	α_7	$c_{19}c_{29}c_{39}c_{49}c_{59}c_{69}c_{79}c_{89}$.

These eight points determine the eight primes

$$3_h,\ 2_h,\ 1_h,\ 0_h,\ 0,\ 0_i,\ 1_i,\ 0_j$$

(by omitting them in turn, and joining the remaining seven to the centre X_{21}). By § 17.4, the generating operations

$$N_3,\ N_2,\ N_1,\ N,\ O,\ P,\ P_1,\ Q$$

are the reflections in these primes respectively.

In § 11.7 we saw that the group is generated by the two special symmetries

(19.22) $\qquad (12345678) \quad \text{and} \quad V_{123}.$

(The former is simply a cyclic permutation of the first eight coordinates, while the latter, defined in § 10.6, interchanges a_{123} and b_{123}, c_{14} and a_{234}, c_{41} and b_{234}, a_{456} and b_{789}, and so on, but leaves c_{12}, c_{45}, a_{345} and b_{345} unaltered.) It is therefore natural to consider the involutory operations

$$(12),\ (23),\ (34),\ (45),\ (56),\ (67),\ (78),\ \text{and}\ V_{123}.$$

(78) alters X_0 but leaves the other X's invariant,

(67) ,, X_1 ,, ,, ,, ,

(56) ,, X_2 ,, ,, ,, ,

(45) ,, X_3 ,, ,, ,, ,

(34) ,, X_4 ,, ,, ,, ,

(23) ,, X_{01} ,, ,, ,, ,

(12) ,, X_{11} ,, ,, ,, ,

V_{123} ,, X_{20} ,, ,, ,, .

Therefore

$$(19.23)\quad \begin{cases} N_3=(78),\quad N_2=(67),\quad N_1=(56),\quad N=(45),\quad O=(34), \\ P=(23),\quad P_1=(12),\quad \text{and}\quad Q=V_{123}. \end{cases}$$

By § 11.7 all the transpositions can be deduced from (19.22); and by substituting the resultant expressions for O, N, N_1, etc., we could obtain an abstract definition (albeit extremely cumbrous) in terms of these two operations.

19.3. The 28 bitangents of the plane quartic can be denoted by unordered pairs of the numbers 1, 2, ..., 8. Pascal* has devised an analogous notation for the 120 tritangent planes of the quadri-cubic curve, by unordered triads of the numbers 0, 1, ..., 9. In the special case when the quadric through the curve is a cone, the relation with $(PA)_8$ is as follows.

If i, j, k are any three of the numbers 1, 2, ..., 9, the opposite vertices a_{ijk} and b_{ijk} correspond to the plane ijk, and the opposite vertices c_{ij} and c_{ji} correspond to the plane $ij0$.

Pascal's notation does not immediately exhibit the full symmetry of the configuration of tritangent planes. Bath† has discovered a *bifid substitution* analogous to that used for permuting the bitangents of the plane quartic. The substitution

$$0123/456789,$$

for example, interchanges 014 and 234, 456 and 789, but leaves 012 and 345 unaltered. Thus V_{123}, *when regarded as permuting the joins of opposite vertices* ("diameters", say) *instead of the vertices themselves, is identical with* 0123/456789; and similarly for any V_{ijk}.

But the interchanges involved when the digit 0 occurs *after* the stroke in Bath's symbol show that a substitution such as 1234/567890 is *not* a symmetry of $(PA)_8$. In fact, the diameters 123 and 345 are perpendicular, whereas their transforms 123 and 125 are inclined at $\frac{1}{3}\pi$. Hence, *when the canonical curve lies on a cone, the only bifid substitutions that remain valid are those in which the digit* 0 *occurs before the stroke.*

19.4. It is well known that the canonical curve of genus p is of order $2p-2$, in $p-1$ dimensions, and has $2^{p-1}(2^p-1)$ $(p-1)$-tangent primes. These correspond to the odd theta-characteristics of genus p, and so their group of automorphisms is the special Abelian linear group‡ $A(2p, 2)$, of order

$$\prod_{r=1}^{p} 2^{2r-1}(2^{2r}-1). \tag{19.41}$$

When $p = 3$, this order is

$$2.3.8.15.32.63 = 1\,451\,520.$$

In fact, since the bitangents of the plane quartic correspond to the pairs of opposite vertices of 3_{21}, and since in seven dimensions reflection in a

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* *Annali di Mat.* (2), 20 (1892), 198. Actually, Pascal uses "10" instead of "0".

† *Journal London Math. Soc.*, 3 (1928), 84.

‡ Dickson, *Linear groups*, 89, 100.

point is a *negative* transformation, $A(6, 2)$ *is simply isomorphic with*

$$\begin{bmatrix} 3, & 3, & 3 \\ & 3, & 3, \\ & & 3 \end{bmatrix}'.$$

When $p = 4$, (19.41) becomes

$$2.3.8.15.32.63.128.255 = 47\ 377\ 612\ 800.$$

If the curve lies on a cone, one of the 136 even theta-characteristics is special*; so the group of automorphisms is a sub-group of index 136 in $A(8, 2)$, namely the corresponding *first hypoabelian group*†, of order

$$348\ 364\ 800 = 96.10!$$

Since the tritangent planes of the special quadri-cubic curve correspond to the pairs of opposite vertices of 4_{21}, their group of automorphisms is a self-conjugate sub-group of index 2 in (19.21). But this sub-group is not the same as

$$\begin{bmatrix} 3, & 3, & 3, & 3 \\ & 3, & 3 & \\ & & 3 & \end{bmatrix}',$$

since in eight dimensions reflection in a point is a *positive* transformation. These two sub-groups of the whole group of symmetries of 4_{21} have a common self-conjugate sub-group of index 2, namely the group of *positive* symmetries of the *diameters* of 4_{21}, which is the simple group $FH(8, 2)$‡, of order

$$174\ 182\ 400.$$

Since (19.21) is generated by (19.22), we can use the same symbols to represent the corresponding generators of the first hypoabelian group, provided we identify opposite vertices of 4_{21} by writing

$$T = 1$$

or

$$(19.42) \qquad V_{124} V_{235} V_{346} V_{457} V_{561} V_{672} V_{713} = (89)$$

(see §10.7). In fact, the abstract definition of the latter group is derived from that given in §19.2 by inserting one extra relation, namely (19.42) expressed in terms of O, N, N_1, etc.

* The corresponding theta-function vanishes for zero values of the arguments. See Schottky, *Journal für Math.*, 103 (1887), 185.

† Dickson, *Linear groups*, 201.

‡ *Ibid.*, 216.

19.5. We proceed to prove that the whole group $A(8, 2)$ is generated by these same operations along with the transposition

(90).

Since the operations of the first hypoabelian group include all the permutations of 1, 2, ..., 9, it will be sufficient if we prove that $A(8, 2)$ is generated by all the permutations of 0, 1, 2, ..., 9 together with all Bath's bifid substitutions. Let G denote the group so generated. If G were not the same as $A(8, 2)$, it would have to be a sub-group; therefore we merely have to prove that G is of order 47 377 612 800.

Of the 120 tritangent planes of the general quadri-cubic curve, a pair such as

079, 089

leads by permutations and bifid substitutions to a set of 7140 pairs, namely

1260 of type *abc*, *abd*;

3780 ,, *abc*, *ade*;

2100 ,, *abc*, *def*.

Since $7140 = \binom{120}{2}$, this shows that *all pairs of tritangent planes are equivalent.*

Again, a triad such as

069, 079, 089

leads to a set of 152320 triads, namely:

2520 of type *abc*, *abd*, *abe*;

840 ,, *abc*, *abd*, *acd*;

7560 ,, *abc*, *abd*, *cde*;

37800 ,, *abc*, *abd*, *aef*;

25200 ,, *abc*, *abd*, *efg*;

75600 ,, *abc*, *ade*, *bfg*;

2800 ,, *abc*, *def*, *ghi*.

Defining the *sum* of the symbols of three planes as the set of digits obtained by juxtaposing the symbols and cancelling repeated digits*, we see that these particular triads are such that the sum of their symbols has one or

* Pascal, *Annali di Mat.* (2), 20 (1892), 199.

five or nine digits. (For example, $abc+abd+abe=abcde$.) Following Pascal, we call these *even* triads. The sum of the symbols of any of the remaining 128520 possible triads ("odd" triads) has either three or seven digits. The geometrical significance of this distinction is that the nine points of contact of an even triad of tritangent planes do not lie on a quadric (other than that through the whole curve).

If two planes of an even triad are fixed, there are 64 possibilities for the third. For instance, a plane making an even triad with 079 and 089 may involve both or neither of the digits 7, 8. It is easily seen that all such planes can be derived from any one of them by permutations and bifid substitutions not affecting the two fixed planes.

If an even triad is fixed, there are 36 possibilities for a fourth plane which makes an even triad with every pair of the fixed triad. For instance, a plane making an even triad with every pair of 069, 079, 089 may either be 678 or involve none of the digits 6, 7, 8.

A plane making an even triad with every pair of

$$059,\ 069,\ 079,\ 089$$

must not involve any of the digits 5, 6, 7, 8; so there are $\binom{6}{3}=20$ possibilities, and these can be put in correspondence with the vertices of $t_2\alpha_5$.

A plane making an even triad with every pair of

$$049,\ 059,\ 069,\ 079,\ 089$$

must not involve any of 4, 5, 6, 7, 8; so there are 10 possibilities*. But since the vertex figure of $t_2\alpha_5$ is $[\alpha_2, \alpha_2]$, which has only nine vertices, we should expect one of these ten to be special. Such is easily seen to be the case, the special plane being, of course, 123.

There is no plane that will make an even triad with every pair of

$$123,\ 049,\ 059,\ 069,\ 079,\ 089.$$

But there are four possibilities if we replace 123 by any other symbol formed with three of the digits 0, 1, 2, 3, 9. *E.g.*, with 039 we can have any of

$$012,\ 019,\ 029,\ 129.$$

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* Noether, *Math. Annalen*, 14 (1879), 270. His

$$(\alpha_\rho),\ (\varrho\varrho'\alpha_\rho\alpha_\sigma\alpha_\tau),\ (\varrho\,\alpha_\rho\,\alpha_\sigma),\ (\varrho'\alpha_\rho\alpha_\sigma)$$

are Pascal's

$$0\rho 9,\ \rho\sigma\tau,\ 0\rho\sigma,\ \rho\sigma 9,$$

respectively.

Finally, 019 is the only plane that will make an even triad with every pair of

$$029,\ 039,\ 049,\ 059,\ 069,\ 079,\ 089^*.$$

Let us now translate these results into terms of groups. Those operations of G that keep one plane fixed form a sub-group G_1 of index 120. Those operations that keep two planes fixed form a sub-group G_2 of index 119 in G_1. Those that keep an even triad fixed form a sub-group G_3 of index 64 in G_2. And so on, through the sequence of numbers 36, 20, 9, 4. Finally, the only operations that will keep fixed an octad like

$$019,\ 029,\ 039,\ 049,\ 059,\ 069,\ 079,\ 089$$

are (90) and identity; so G_7 is of order 2.

We can now deduce the orders of all these groups. In particular, G_3 is of order

$$2.4.9.20.36 = 51\ 840,$$

and G itself is of order

$$2.4.9.20.36.64.119.120 = 47\ 377\ 612\ 800.$$

Therefore G is $A(8, 2)$, as we desired to prove.

19.6. The above procedure is closely analogous to that of Chapter 9, where we obtained successive vertex figures of certain polytopes. Since G_6, G_5, G_4 are the groups of symmetries of $(IA)_3$, $(IA)_4$, $(IA)_5$ respectively†, while G_3 has the same order as $\begin{bmatrix} 3, 3 \\ 3, 3 \\ 3 \end{bmatrix}$, it is natural to expect some relation between G_3 and $(IA)_6$.

Let j, j' be two of the numbers 1, 2, 3; k, k' two of 4, 5, 6; l, l' two of 7, 8, 9. The 36 planes which form even triads with every pair of

$$123,\ 456,\ 789$$

are $$jj'0,\ kk'0,\ ll'0,\ jkl.$$

In the case when the quadri-cubic curve lies on a cone, these 36 planes correspond to the pairs of opposite vertices

$$c_{jj'} \text{ and } c_{j'j},\quad c_{kk'} \text{ and } c_{k'k},\quad c_{ll'} \text{ and } c_{l'l},\quad a_{jkl} \text{ and } b_{jkl}$$

of the $(IA)_6 3\sqrt{2}$ (10.32). Since these planes acquire no extra auto-

* Pascal, *Atti R. Acc. Lincei* (*Rend.*) (5), 2 (1893), 122.

† (8.16).

morphisms when the quadri-cubic curve is taken to be general, G_3 *is simply isomorphic with* $\begin{bmatrix} 3, 3 \\ 3, 3 \\ 3 \end{bmatrix}$; or, in geometrical language, *those automorphisms of the tritangent planes which keep fixed an even triad* (*i.e.*, three planes whose points of contact do not lie on an extra quadric) *constitute a group simply isomorphic with the group of automorphisms of the lines on the cubic surface.*

This theorem is due to Pascal*, who regards the set of tritangent planes with one omitted as corresponding to a configuration of 119 planes in four dimensions. Pairs of these planes meet in a point or in a line according as the corresponding tritangent planes make an even or odd triad with the one that was omitted. Let two planes meeting only in a point be called *skew* ("gobbo"). There are 64 planes skew to any particular one; and, skew to any one among these, there are 36. Pascal shows that these 36 correspond to the double-sixes of lines on the cubic surface, which we recognize as corresponding to the diameters of $(IA)_6$. He gives a detailed account† of this configuration of 36 planes in four dimensions, from which we see that the relation with $(IA)_6$ is as follows.

The 36 planes, 360 skew pairs, 1080 skew triads (of the "second kind"), 1080 skew tetrads, and 216 skew pentads correspond to the pairs of opposite vertices, edges, α_2's, α_3's, and α_4's; while the 120 skew triads (of the "first kind") and the 135 tetrahedra correspond to the diagonal hexagons (or pairs of diagonal $\alpha_2\sqrt{3}$'s) and diagonal $\beta_4\sqrt{2}$'s.

19.7. We shall now prove that $A(8, 2)$ *is generated by the two operations*

(19.71) (012345678) *and* 0123/456789

[*cf.* (11.71)].

It is convenient to abbreviate the latter symbol to 0123/. (10.63) and (10.64) obviously generalize to give

$$fghi/\,.\,(ij) = (ij)\,.\,fghj/ = fghj/\,.\,fghi/,$$

$$bcde/\,.\,bfgh/ = bfgh/\,.\,bijk/,$$

where *bcdefghijk* is a permutation of 0123456789. Therefore $A(8, 2)$ is generated by

(19.72) $hijk/$ $(h, i, j, k < 9)$.

* *Atti R. Acc. Lincei* (*Rend.*) (5), 2 (1893), 68.

† *Ibid.*, 71.

The operations (19.71) together lead to

$$1234/, \quad 2345/, \quad 5678/, \quad 6780/,$$

which give the transpositions

$$(15), \quad (50)$$

and

$$(01),$$

whence we can deduce all the permutations of 0, 1, ..., 8, and so also the rest of the operations (19.72).

19.8. Since $A(8, 2)$ is generated by the symmetries of the diameters of 4_{21} along with the transposition (90), it is natural to extend (19.23) by writing

$$Q_1 = (90). \tag{19.81}$$

(We must suppose the operations O, N, N_1, etc., modified in the manner described at the end of § 19.4, so that

$$Q = 0123/456789.)$$

It is easily seen that Q_1 is permutable with

$$N_3, \ N_2, \ N_1, \ N, \ O, \ P, \ P_1,$$

whereas

$$(QQ_1)^3 = 1.$$

There are, of course, other relations; but these show that $A(8, 2)$ *may be regarded as a sub-group of the (infinite) group of symmetries of the Minkowskian polytope*

$$4_{22} \quad \text{or} \quad (IA)_9.$$

This result is not surprising when we recall that G_r is precisely the group of symmetries of $(IA)_{9-r}$ when $r > 3$, and is a sub-group (of index 2) when $r = 3$.

Instead of (19.81), we might have written

$$P_2 = (01),$$

thus exhibiting $A(8, 2)$ as a sub-group of the group of symmetries of the Minkowskian polytope

$$4_{31} \quad \text{or} \quad (SA)_9.$$

19.9. At the end of a paper on $(PA)_6$ and $(PA)_7$*, I have given geometrical interpretations for the elements, diagonals, etc., of $(PA)_7$ as

* *Proc. Camb. Phil. Soc.*, 24 (1928), 1–9. (The symbol Ta_4, used there, should have been t_1a_4.)

special sets of bitangents of the plane quartic. We shall now attempt to do the same for $(PA)_8$ and the tritangent planes. Naturally the elements of $(PA)_8$ correspond only to sets of tritangent planes of the "special" quadri-cubic curve (lying on a cone); but there are usually analogous sets in the general case. Thus, corresponding to the 3360 pairs of opposite edges of $(PA)_8$, we have 3360 special pairs of tritangent planes of the special curve, and $119 \cdot 120/2 = 7140$ pairs of tritangent planes of the general curve. Again, corresponding to the 30240 pairs of opposite triangular faces and the 1120 pairs of opposite diagonal triangles, we have $30240+1120$ special triads of tritangent planes of the special curve, and $64 \cdot 7140/3 = 152320$ *even* triads of tritangent planes of the general curve.

Most of the numbers in the second column of the table at the end of this chapter are taken from § 12.8 (the last column of page 414). The number of diagonal $\alpha_2\sqrt{3}$'s of $(PA)_8$ is $28 \cdot 240/3 = 2240$, since those at any vertex correspond to the diameters of $(PA)_7$. The number of diagonal $\{3, 4, 3\}$'s is $315 \cdot 240/24 = 3150$, since those at any vertex correspond to the diagonal cubes of $(PA)_7$ (or to the pairs of opposite diagonal tetrahedra). In each $\{3, 4, 3\}$ (*e.g.*, that lying in the 4-space

$$x_1+x_2 = x_3+x_4 = x_5+x_6, \quad x_7 = x_8 = x_9, \quad \Sigma x = 0)$$

we can inscribe three γ_4's or three $\beta_4\sqrt{2}$'s, making 9450 of each altogether.

The $\beta_4\sqrt{2}$'s inscribed in the diagonal $\{3, 4, 3\}$'s are particularly interesting since they correspond to tetrads of tritangent planes whose twelve points of contact all lie on an extra quadric. The "sum" of the symbols of such a tetrad has either no digits or all the ten.

In the case of the general quadri-cubic curve, every odd triad determines a fourth plane making with it a tetrad of this kind. *E.g.*, with the triads

235, 145, 136 and 012, 034, 056

we must associate 246 and 789

respectively. Thus the number of such tetrads is

$$128520/4 = 32\,130^*.$$

Let us take one of these tetrads, and associate with it as many more planes as possible, in such a way that every triad is odd. We find that just four more planes can be added, that this can be done in five ways, and that each new set of four is a tetrad of the same kind. We are thus led to consider octads of planes such as those which correspond to the

* Pascal, *Atti R. Acc. Lincei* (*Rend.*) (5), 2 (1893), 204.

diagonal $\beta_8\sqrt{2}$'s of $(PA)_8$. *Each octad can be divided into two of the* 32130 *tetrads in seven ways. E.g.*, in the octad

(19.91) 124, 235, 346, 457, 561, 672, 713, 890,

the three planes whose symbols involve a particular one of the digits 1, 2, 3, 4, 5, 6, 7 form with 890 a tetrad of the proper kind. Thus the number of such octads is

$$5 \cdot 32130/14 = 11\,475.$$

For the *special* quadri-cubic curve, we consider only those octads that correspond to diagonal $\beta_8\sqrt{2}$'s; then each tetrad belongs to three (instead of five) octads, and the number is

$$3 \cdot 9450/14 = 2\,025.$$

TABLE OF THE PRINCIPAL SETS OF TRITANGENT PLANES.

Number of planes in a set.	Number of sets for the *special* quadri-cubic curve.	Corresponding configuration in 4_{21}.	Typical set	Number of sets for the *general* quadri-cubic curve.
1	120	α_0	089	120
2	3360	α_1	079, 089	7140
3	30240	α_2	069, 079, 089	152320
	1120	$\alpha_2\sqrt{3}$ or $\{6\}$	123, 456, 789	
4	120960	α_3	059, 069, 079, 089	1370880
5	241920	α_4	049, 059, 069, 079, 089	5483520
4	9450*	$\beta_4\sqrt{2}$	012, 034, 056, 789	32130
8	9450	γ_4	235, 145, 136, 246 146, 236, 245, 135	
12	3150	$\{3, 4, 3\}$	Combination of the two above	
6	241920	$\alpha_5 = 4_{00}$	039, 049, 059, ..., 089	8225280
	—		123, 049, 059, ..., 089	913920
7	69120	$\alpha_6 = 4_{10}$	029, 039, 049, ..., 089	4700160
	34560	$\alpha_6 = 4_{01}$	129, 039, 049, ..., 089	
8	8640	$\alpha_7 = 4_{20}$	019, 029, 039, ..., 089	587520
14	1080	$\beta_7 = 4_{11}$	029, 039, 049, ..., 089 129, 139, 149, ..., 189	
8	2025	$\beta_8\sqrt{2}$	124, 235, ..., 713, 890†	11475

* The diagonal $\beta_8\sqrt{2}$'s lead to many more $\beta_4\sqrt{2}$'s, but these 9450 are special (being inscribed in $\{3, 4, 3\}$'s).

† (19.91).

NOTES.

20.1. *Corrections to Part* 1

[*Phil. Trans. Royal Soc.*, A, 229 (1930), 329–425].

329, line 14. *For* Π^{r+u} *read* Π_r^{+u}.

335, line 27. *For* ; *read* , .

337, line 6. *For* generally *read* usually.

346, line $16\frac{1}{2}$. *Ignore the comma after* 3.

347, line 2. *Interchange* τ *and* $\sqrt{5}$.

347, line 10. *Ignore the upright stroke.*

352, line 17. *For* e *read* E.

353, line 19. *Ignore the stop after* 4.81. *For* ,... *read* , ...

360, line 21. *Interchange* 2 *and* τ (*or any other pair*).

360, line 23. *Interchange* 1 *and* τ^{-1} (*or any other pair*).

361, line 15. *For* ± 3 *read* 3.

369, line 9. *For* an *n*th *read* a t_n.

388, line 21. *Insert* — *after* -1, . *Ignore the comma before the second semicolon.*

406, line 23. *For* $[1357.2468]\,\mathrm{T} = \mathrm{T}_{1357}\,\mathrm{ST}_{2468}$ *read*

$$\mathrm{ST}\,[1357.2468] = \mathrm{ST}_{1357}\,\mathrm{ST}_{2468}.$$

408, line 15. *For* automorphism *read* automorphisms.

20.2. *Miller's proof that every finite uniform polytope has a circumcentre**.

A set of points are said to be *equivalent* if, for every pair A, B of the points, there exists a congruent transformation which changes A into B, leaving the set unchanged as a whole. In §1.8 we made the assumption that *a finite set of equivalent points necessarily lie on a sphere-analogue.* J. C. P. Miller, assisted by J. A. Todd and L. C. Young, has

* *Cf.* E. Catalan, "Mémoire sur la théorie des polyèdres", *Journal de l'Ecole Polytechnique*, 41 (1865), 33. It is interesting to note that Catalan's definition (p. 25) of "polyèdre semi-régulier du premier genre" should admit Miller's non-uniform solid (§2.1); so, too, the reciprocal of Miller's solid is really one "du second genre".

constructed a proof to justify this assumption. For simplicity he employs the terminology of three dimensions; to obtain the general statement, we have merely to read "m-dimensional sphere-analogue" or "$(m-1)$-sphere" for "sphere", and "$(m-1)$-dimensional sphere-analogue" or "$(m-2)$-sphere" for "circle".

LEMMA. *There is a unique smallest sphere which encloses all the points.*

Proof of lemma. Since the set of points, say Π, is supposed finite, we may take an enclosing sphere of finite radius. If the smallest of such spheres is not unique, consider two distinct smallest spheres. Since these are equal and both enclose Π, they must intersect. Describe a new sphere concentric with, and passing through, their circle of intersection. This sphere will be smaller than the others, and, containing their common part, must enclose Π; which is absurd. Thus the lemma is established.

Proof of theorem. S, the smallest sphere enclosing Π, must evidently have at least one point A of Π on it. Any other point B of Π must lie on or within S. If possible, let B be definitely within. Since the points are equivalent, there exists a congruent transformation which changes A into B but leaves Π unchanged as a whole. This transformation changes S, which passes through A, into an equal sphere S' passing through B. Since B does not lie on S, S' must be distinct from S. Since S encloses Π, and Π is transformed into itself, S' must enclose Π. But this contradicts the lemma. Hence B, which was arbitrarily chosen from the points of Π, lies on S; and so all the points lie on S.

20.3. *Uniform (degenerate) polytopes not uniquely determined by their vertex figures.*

In §2.1 we made the assumption "that, given any uniform polytope, there is no other uniform polytope of different shape having the same vertex neighbourhood." There is no reason to doubt the validity of this assumption when the polytope is *finite*; but J. C. P. Miller has refuted it in the degenerate case, by describing two distinct uniform polytopes, say M_4 and M_4', which have the same vertex neighbourhood. Their common vertex figure is obtained if we cut a cuboctahedron of unit edge in halves along an equatorial hexagon, and replace one half by a hexagonal pyramid of unit altitude. Thus each polytope consists of a net of tetrahedra, octahedra, and triangular prisms, filling three-dimensional space. In order to avoid repetition, we shall at once describe the analogous polytopes filling m-dimensional space.

In §6.8 we saw that the vertex figure of $\alpha_m h$ is $e\alpha_m$, and that the bounding figures of the latter polytope correspond to all the elements of either of two reciprocal α_m's. It follows that with every bounding figure of $\alpha_m h$ can be associated a type symbol $0_{nn'}$ (distinct from $0_{n'n}$), where $n+n'=m-1$, in such a way that every $0_{nn'}$ at one vertex corresponds to an α_n of a definite one of the two reciprocal α_m's of which the actual vertex figure at that vertex is an "expansion" (in the sense of §6.9). *E.g.*, the triangles of $\alpha_2 h$ can be labelled alternately 0_{10} and 0_{01}. Further, there is a definite type symbol $0_{n(n'-1)}$ for the $(m-1)$-dimensional element in which a $0_{nn'}$ meets a $0_{(n+1)(n'-1)}$*. *E.g.*, every edge of $\alpha_2 h$ is of type 0_{00}.

Now, by considering definite integer values of a particular coordinate, say x_{m+1}, in (6.81), we see that it is possible (in $m+1$ ways) to select a series of parallel $(m-1)$-spaces, together containing all the vertices of $\alpha_m h$, and each filled with elements of $\alpha_m h$ forming an $\alpha_{m-1} h$. Every bounding figure of each $\alpha_{m-1} h$ is already marked with a type symbol of the form $0_{n(n'-1)}$. The new polytope M_{m+1}† is constructed by cutting $\alpha_m h$ along each one of these $(m-1)$-spaces, shifting the resultant layers apart, and inserting a layer of prisms $[0_{n(n'-1)}, \alpha_1]$. The modification M'_{m+1} is derived by sliding every *alternate* layer of elements of $\alpha_m h$, between its two bounding $(m-1)$-spaces, in such a way that each inserted prism, instead of joining two $0_{n(n'-1)}$'s, joins a $0_{n(n'-1)}$ to a $0_{(n'-1)n}$.

Clearly the vertex figure of either of these new polytopes is obtained if we cut an $e\alpha_m$ in halves along an equatorial $e\alpha_{m-1}$ [such as that obtained by fixing $x_{m+1}=0$ in (6.82)] and replace one half by the pyramid-analogue $(e\alpha_{m-1} \overset{\sqrt{2}}{\text{———}} \alpha_0)$.

Exceptionally, M_3' is the same as M_3, viz. alternate strips of triangles and squares filling a plane. (Vertex figure : a cyclic pentagon, of sides $\sqrt{2}$, 1, 1, 1, $\sqrt{2}$.) But consider the degenerate prism $[M_3, \delta_2]$, consisting of alternate layers of triangular prisms and of cubes, filling three-dimensional space. It is clear that every *alternate* layer of triangular prisms can be turned bodily through a right angle, so as to give a new uniform polytope, say $[M_3, \delta_2]'$, having the same vertex figure as $[M_3, \delta_2]$.

(This vertex figure is, of course, a bipyramid of slant edge $\sqrt{2}$ on the above-mentioned vertex figure of M_3.)

20.4. *Coordinates for pentagonal polytopes.*

The $\{5, 3\}4\tau^{-1}$ of §3.6 corresponds in position to the second $t_1\{3, 5\}2\tau^{-1}$ of §5.7; *i.e.*, the latter, apart from size, is an *actual*

* There are, in fact, type symbols for *all* the elements of $\alpha_m h$, except vertices.

† Not to be confused with Schoute's M_n, which is our γ_n.

truncation of the former. So, too, the $\{5, 3\}2\tau^{-1}$ of § 3.6 corresponds in position to the *first* $t_1\{3, 5\}2\tau^{-1}$. But, in order to bring the $\{3, 5\}2$ of § 3.6 into the position corresponding to the last-mentioned pair of polyhedra, it is necessary to perform a transposition among the coordinates τ, 1, 0.

Similarly, the $\{5, 3, 3\}2\tau^{-2}$ of § 3.6 and the $t_1\{3, 3, 5\}2\tau^{-1}$ of § 5.7 (both corrected as in § 20.1) correspond in position to the $t_1\{5, 3, 3\}2\tau^{-2}$ of § 5.7. But, in order to bring the $\{3, 3, 5\}2\tau^{-1}$ of § 3.6 into the corresponding position, it is necessary to perform a transposition among τ, 1, τ^{-1}, 0. After this alteration has been made, if $(x_1;\ x_2;\ x_3;\ x_4)$ is any vertex of the $\{3, 3, 5\}2\tau^{-1}$, then $(x_1+x_2;\ x_1-x_2;\ x_3+x_4;\ x_3-x_4)$* is a vertex of the $\{3, 3, 5\}2\sqrt{2}\tau^{-1}$ at the bottom of the page (346).

20.5. *Du Val's coordinates for* 5_{21}.

By applying the transformation

$$x'_{2r-1} = \tfrac{1}{2}(x_{2r-1}+x_{2r}),\quad x'_{2r} = \tfrac{1}{2}(x_{2r-1}-x_{2r})\quad (r = 1, 2, 3, 4)$$

to the coordinates (§ 9.1) for $5_{21}2\sqrt{2}$, we obtain, as the vertices of

$$5_{21}2,$$

the totality of points whose eight Cartesian coordinates are either all even or all odd or four even and four odd, with a restriction in the third case. If x_i, x_j, x_k, x_l are the four even coordinates (or the four odd ones), the suffixes i, j, k, l always form one half of one of the following bifid symbols:

$$[1234.5678],\quad [1256.3478],\quad [1278.3456],$$

$$[1357.2468],$$

$$[1368.2457],\quad [1458.2367],\quad [1467.2358].$$

A suitable permutation of the coordinates transforms these bifid symbols into those occurring in (9.36), which have the simple property that all are derivable from any one by cyclic permutation of the digits 1, 2, 3, 4, 5, 6, 7.

Of the points so defined, those distant 2 from the origin must be the vertices of

$$4_{21}2.$$

---

* *Cf.* Robinson, "On the orthogonal groups in four dimensions", *Proc. Camb. Phil. Soc.*, 27 (1930), 37-48. His δ_4 (= {3, 4, 3}) must not be confused with our δ_4 (= {4, 3, 4}).

These are easily seen to consist of the vertices

$$\pm(2, 0, 0, 0, 0, 0, 0, 0)$$

of a $\beta_8 2\sqrt{2}$, together with the vertices of fourteen $\gamma_4 2$'s of the form

$$x_e = x_f = x_g = x_h = 0, \quad \pm x_i = \pm x_j = \pm x_k = \pm x_l = 1,$$

where $[efgh.ijkl]$ is one of our special bifid symbols. In contrast with (9.21), this set of coordinates has the advantage that every axial prime is a prime of symmetry of the polytope.

By selecting the points of this set distant 2 from (0, 0, 0, 0, 0, 0, 0; 2), we obtain the vertices of

$$3_{21} 2$$

in the perfectly symmetrical form

$$x_e = x_f = x_g = x_h = 0, \quad \pm x_i = \pm x_j = \pm x_k = 1,$$

where $(i\ j\ k)$ is one of the seven triads

$$(1\ 2\ 4), \quad (2\ 3\ 5), \quad (3\ 4\ 6), \quad (4\ 5\ 7), \quad (5\ 6\ 1), \quad (6\ 7\ 2), \quad (7\ 1\ 3),$$

and e, f, g, h are the rest of the numbers 1, 2, 3, 4, 5, 6, 7. The essential properties of these triads are that every two digits determine a unique third forming a triad with them, that every two triads have a single digit in common, and that every digit belongs to just three triads. The digits of the first triad are the residues, mod 7, of the powers of 2 (or quadratic residues), and the rest of the triads are derived by cyclic permutation of the seven digits.

The vertices of 3_{21} have not hitherto been expressed by *rational* coordinates in *seven* dimensions. We observe, incidentally, that these fifty-six points are the vertices of seven cubes.

For the vertices of 5_{21}, we have now two expressions in eight dimensions and one in nine. By considering fundamental systems of theta-characteristics, Du Val has discovered an expression in *ten* dimensions. His statement is as follows.

The vertices of

$$5_{21} 5\sqrt{2}$$

are all the points in ten dimensions whose coordinates satisfy the equations

$$x_1+x_2+x_3+x_4+x_5 = x_6+x_7+x_8+x_9+x_{10} = 0$$

and the congruences

$$x_1 \equiv x_2 \equiv x_3 \equiv x_4 \equiv x_5 \equiv 2x_6 \equiv 2x_7 \equiv 2x_8 \equiv 2x_9 \equiv 2x_{10} \pmod 5.$$

The consequent coordinates for the vertices of

$$4_{21}\, 5\, \sqrt{2}$$

are

$$\begin{array}{llllllllll}
(0, & 0, & 0, & 0, & 0; & 5, & 0, & 0, & 0, & -5), \\
(1, & 1, & 1, & 1, & -4; & 3, & 3, & -2, & -2, & -2), \\
(2, & 2, & 2, & -3, & -3; & 1, & 1, & 1, & 1, & -4), \\
(3, & 3, & -2, & -2, & -2; & 4, & -1, & -1, & -1, & -1), \\
(4, & -1, & -1, & -1, & -1; & 2, & 2, & 2, & -3, & -3), \\
(5, & 0, & 0, & 0, & -5; & 0, & 0, & 0, & 0, & 0).
\end{array}$$

20.6. *Degenerate prisms.*

In § 17.7 we defined the fundamental simplex of the finite generalized prism. It may seem unsatisfactory that nothing has been said about the degenerate prism (§ 4.8). However, the extension is easily made.

Let $$\Sigma^{(1)}_{i-2},\ \Sigma^{(2)}_{i-2},\ \ldots,\ \Sigma^{(i)}_{i-2}$$

be the bounding simplexes of the (Euclidean) fundamental simplex Σ_{i-1} of a degenerate polytope Π_i, and let $(f\ g)$ be the angle between the spaces of $\Sigma^{(f)}_{i-2}$ and $\Sigma^{(g)}_{i-2}$. Similarly, let

$$\Sigma^{(i+1)}_{m-i-2},\ \Sigma^{(i+2)}_{m-i-2},\ \ldots,\ \Sigma^{(m)}_{m-i-2}$$

be the bounding simplexes of the fundamental simplex Σ_{m-i-1} of another degenerate polytope Π_{m-i}, and let $(k\ l)$ be the angle between $\Sigma^{(k)}_{m-i-2}$ and $\Sigma^{(l)}_{m-i-2}$. We should expect the fundamental simplex Σ_{m-1} of the degenerate prism $[\Pi_i, \Pi_{m-i}]$ to have bounding spaces corresponding to $\Sigma^{(1)}_{i-2}, \ldots, \Sigma^{(i)}_{i-2}, \Sigma^{(i+1)}_{m-i-2}, \ldots, \Sigma^{(m)}_{m-i-2}$, such that the angle between the spaces corresponding to $\Sigma^{(f)}_{i-2}$ and $\Sigma^{(g)}_{i-2}$ is $(f\ g)$, while the spaces corresponding to $\Sigma^{(g)}_{i-2}$ and $\Sigma^{(k)}_{m-i-2}$ are perpendicular. Now this is precisely the state of affairs in the (finite) prism $[\Sigma_{i-1}, \Sigma_{m-i-1}]$; the angle between the spaces of $[\Sigma^{(f)}_{i-2}, \Sigma_{m-i-1}]$ and $[\Sigma^{(g)}_{i-2}, \Sigma_{m-i-1}]$ is $(f\ g)$, while the spaces of $[\Sigma^{(g)}_{i-2}, \Sigma_{m-i-1}]$ and $[\Sigma_{i-1}, \Sigma^{(k)}_{m-i-2}]$ are perpendicular. In fact, the vertices of Σ_{m-1} all coincide with the point at infinity in the direction normal to the $(m-2)$-space of $[\Sigma_{i-1}, \Sigma_{m-i-1}]$, and its bounding $(m-2)$-spaces join this point at infinity to the bounding $(m-3)$-spaces of $[\Sigma_{i-1}, \Sigma_{m-i-1}]$. Since $[\Pi_{m-i}, \Pi_i]$ actually lies in (or rather fills) this $(m-2)$-space, we can neglect the

$(m-1)$-th dimension, and acknowledge a *fundamental prism* in $m-2$ dimensions in place of the fundamental simplex in $m-1$.

More generally, the degenerate prism

$$[\Pi_{m_1+1},\ \Pi_{m_2+1},\ \ldots]$$

has the fundamental prism

$$[\Sigma_{m_1},\ \Sigma_{m_2},\ \ldots],$$

where Σ_{m_1} is the fundamental simplex of Π_{m_1+1}, and so on. The reflections in the bounding spaces of this fundamental prism generate the direct product of the groups generated by the reflections in the bounding spaces of $\Sigma_{m_1}, \Sigma_{m_2}, \ldots$. This direct product is either the whole group of symmetries of the degenerate prism or a sub-group thereof.

The fundamental simplex of δ_2 (see §§ 3.5, 15.8) being

$$(\infty) = a_1 \tfrac{1}{2},$$

the fundamental prism of

$$[\delta_2,\ \delta_2,\ \ldots] = \delta_{m+1}$$

is

$$[a_1 \tfrac{1}{2},\ a_1 \tfrac{1}{2},\ \ldots] = \gamma_m \tfrac{1}{2}.$$

Since $\gamma_m \frac{1}{2}$ can be divided (by primes of symmetry through one of its vertices) into $m!$ repetitions of the simplex $(4, 3^{m-2}, 4)$, the corresponding group (viz., the direct product of m groups of the form $S^2 = T^2 = 1$) is a sub-group of index $m!$ in the group of symmetries of δ_{m+1}.

On the other hand, if $a, b, \ldots$ are all different, the (extended) group with $[a_1 \frac{1}{2} a, a_1 \frac{1}{2} b, \ldots]$ for fundamental region is precisely the group of symmetries of $[\delta_2 a, \delta_2 b, \ldots]$. The unextended group corresponding to the rectangle $[a_1 \frac{1}{2} a, a_1 \frac{1}{2} b]$—that is, the group of rotations (positive symmetries) of $[\delta_2 a, \delta_2 b]$—is discussed by Burnside (§ 298).

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ON THE REPRESENTATIONS OF A NUMBER AS THE SUM OF THREE OR MORE PRODUCTS

By T. ESTERMANN.

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1. *Introduction.*

In my paper* "On the representations of a number as the sum of three products", I proved the following theorem:

THEOREM 1. *Let $\nu_3(n)$ denote the number of representations of n in the form $x_1y_1+x_2y_2+x_3y_3$, where x_1, y_1, etc., are positive integers; let $\sigma_s^{(0)}(n)$ be the sum of the s-th powers of the positive divisors of n, and $\sigma_s^{(v)}(n)$ the v-th derivative of $\sigma_s^{(0)}(n)$ with respect to s. Then*

$$\nu_3(n) = n^2 \sum_{t=0}^{3} \log^t n \sum_{v=0}^{3-t} c_{3,t,v}\, \sigma_{-2}^{(v)}(n) + O(n^{\frac{3}{2}} \log^3 n),$$

where the c's are constants, and in particular $c_{3,3,0} = [2\zeta(3)]^{-1}$.

I shall now give a much simpler proof of the following result, of which Theorem 1 is a particular case:

THEOREM 2. *Let r be any integer greater than or equal to 3, and let $\nu_r(n)$ denote the number of representations of n in the form $x_1y_1+x_2y_2+\ldots+x_ry_r$. Then*

$$\nu_r(n) = n^{r-1} \sum_{t=0}^{r} \log^t n \sum_{v=0}^{r-t} c_{r,t,v}\, \sigma_{1-r}^{(v)}(n) + O(n^{r_1-2} \log^{r_2} n),$$

where the c's are constants, r_1 is the greater of the numbers r and $3\frac{1}{2}$, and $r_2 = r$ except when $r = 4$, in which case $r_2 = 5$.

The proof will be independent of the paper mentioned. Its method has already been applied to a more difficult problem in my paper† "On the representations of a number as the sum of two products", from which I shall quote some of the simpler results.

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* *Proc. London Math. Soc.* (2), 29 (1929), 453–478.

† *Proc. London Math. Soc.* (2), 31 (1930), 123–133.

2. *Application of the Farey-dissection.*

Let

$$F(\tau) = \sum_{n=1}^{\infty} d(n)\, e^{2\pi i n\tau} \quad (\Im\tau > 0), \tag{1}$$

where $d(n)$ is the number of positive divisors of n. Then it is obvious that

$$F^r(\tau) = \sum_{n=r}^{\infty} \nu_r(n)\, e^{2\pi i n\tau}, \tag{2}$$

and hence

$$\nu_r(n) = \int_{\tau_0}^{\tau_0+1} F^r(\tau)\, e^{-2\pi i n\tau}\, d\tau, \tag{3}$$

where τ_0 is any number whose imaginary part is positive. Taking $\tau_0 = i/n + (N+1)^{-1}$, where

$$N = [\sqrt{n}], \tag{4}$$

and applying the Farey-dissection of order N to the path of integration in (3), we obtain

$$\nu_r(n) = \sum_{k=1}^{N} \sum_{h}{}' \int_{i/n+h/k-\vartheta_1}^{i/n+h/k+\vartheta_2} F^r(\tau)\, e^{-2\pi i n\tau}\, d\tau, \tag{5}$$

where the accent indicates that h runs through the positive integers less than or equal to k and prime to k, and ϑ_1 and ϑ_2 are so determined (for every k and h) that $h/k-\vartheta_1$ and $h/k+\vartheta_2$ are the medians between h/k and its neighbours in the Farey-series. It is known that

$$(2kN)^{-1} \leqslant \vartheta_m \leqslant k^{-1}(N+1)^{-1} \quad (m = 1, 2). \tag{6}$$

Substituting $h/k+y$ for τ in (5), and using the abbreviation

$$\xi_k = e^{2\pi i/k}, \tag{7}$$

we get

$$\nu_r(n) = \sum_{k=1}^{N} \sum_{h}{}' \xi_k^{-nh}\, a_{k,h}, \tag{8}$$

where

$$a_{k,h} = \int_{i/n-\vartheta_1}^{i/n+\vartheta_2} F^r(h/k+y)\, e^{-2\pi i n y}\, dy. \tag{9}$$

3. *Investigation of* $F(h/k+y)$.

Throughout this and the next section, let

$$k \leqslant \sqrt{n}, \quad y = i/n+\vartheta, \quad -k^{-1}n^{-\frac{1}{2}} < \vartheta < k^{-1}n^{-\frac{1}{2}}. \tag{10}$$

The numbers in square brackets refer to formulae proved in the second paper mentioned in the introduction.

By [22],

$$F(h/k+y) = Z+U, \tag{11}$$

where, by [24] and [9],

$$Z = (-2\pi iky)^{-1}[\gamma - 2\log k - \log(-2\pi iy)], \tag{12}$$

and, by [23] and [37],

$$U = D(0;\ \xi_k^{\,h}) + (2\pi i)^{-1} I. \tag{13}$$

$R_k(m)$ being the least non-negative residue mod k of m, we have $|R_k(m) - \frac{1}{2}k| \leqslant \frac{1}{2}k$ for any m. Hence, by [25], [30], and [29],

$$|D(0;\ \xi_k^{\,h})| \leqslant \tfrac{1}{4} + \frac{k}{2\pi} \sum_{0<b<\frac{1}{2}k} \frac{1}{b} < k\log(2k). \tag{14}$$

Henceforth let $A_1, A_2, \ldots$ denote suitable positive constants (either absolute or depending only on r). Then, by [38], [45], and [9],

$$|I| < A_1 k|y|n. \tag{15}$$

I shall take $n \geqslant 2$ throughout. Then, by (13), (14), (15), and (10),

$$|U| < k\log n + A_1 k|y|n < A_2\sqrt{n}\log n, \tag{16}$$

and, by (12) and (10),

$$|Z| < A_3 k^{-1}|y|^{-1}\log n. \tag{17}$$

4. *An inequality for* $a_{k,h} - \int_{i/n-\vartheta_1}^{i/n+\vartheta_2} Z^r e^{-2\pi i n y}\,dy$.

It is obvious that

$$|(Z+U)^r - Z^r| < A_4(|Z|^{r-1}|U| + |U|^r). \tag{18}$$

Hence, by (11), (16), and (17),

$$|F^r(h/k+y) - Z^r| < A_5(k^{-1}|y|^{-1}\log n)^{r-1}(k\log n + k|y|n) + A_5 n^{\frac{1}{2}r}\log^r n. \tag{19}$$

Also

$$\int_{i/n-\vartheta_1}^{i/n+\vartheta_2} |y|^{-1}\,dy < 4\int_0^{\frac{1}{2}} (n^{-1}+\vartheta)^{-1}\,d\vartheta \leqslant 4\log n, \tag{20}$$

$$\int_{i/n-\vartheta_1}^{i/n+\vartheta_2} |y|^{-m}\,dy < \int_{-\infty}^{\infty} (n^{-2}+\vartheta^2)^{-\frac{1}{2}m}\,d\vartheta < A_6 n^{m-1} \quad (m \geqslant 2), \tag{21}$$

and, by (4) and (6),

$$\vartheta_1+\vartheta_2<2k^{-1}n^{-\frac{1}{2}}. \tag{22}$$

By (19), (20), (21), (22), and (10),

$$\int_{i/n-\vartheta_2}^{i/n+\vartheta_2}|F^r(h/k+y)-Z^r|\,dy<A_7k^{-r+2}n^{r-2}\log^r n+2A_5k^{-1}n^{\frac{1}{2}r-\frac{1}{2}}\log^r n \tag{23}$$

$$<A_8k^{-r+2}n^{r-2}\log^r n.$$

Hence, by (9),

$$\left|a_{k,h}-\int_{i/n-\vartheta_1}^{i/n+\vartheta_2}Z^r e^{-2\pi iny}\,dy\right|<A_9k^{-r+2}n^{r-2}\log^r n. \tag{24}$$

5. *The inequality for the error term in Theorem 2.*

Let

$$b_k=\int_{i/n-\infty}^{i/n+\infty}Z^r e^{-2\pi iny}\,dy. \tag{25}$$

Then, by (12), (6), and (4),

$$\left|b_k-\int_{i/n-\vartheta_1}^{i/n+\vartheta_2}Z^r e^{-2\pi iny}\,dy\right| \tag{26}$$

$$<A_{10}k^{-r}\left\{\int_{(2k\sqrt{n})^{-1}}^{n}\vartheta^{-r}\log^r n\,d\vartheta+\int_n^\infty\vartheta^{-r}\log^r\vartheta\,d\vartheta\right\}$$

$$<A_{11}k^{-1}n^{\frac{1}{2}r-\frac{1}{2}}\log^r n\quad(k\leqslant\sqrt{n}).$$

Hence, by (24),

$$|a_{k,h}-b_k|<A_{12}k^{-r+2}n^{r-2}\log^r n\quad(k\leqslant\sqrt{n}). \tag{27}$$

Putting (with Ramanujan)

$$c_k(n)=\sum_h{}'\,\xi_k^{-nh}, \tag{28}$$

we have, by (8), (27), and (4),

$$\left|\nu_r(n)-\sum_{k=1}^{N}c_k(n)\,b_k\right|<A_{12}n^{r-2}\log^r n\sum_{k=1}^{N}k^{-r+3}<A_{13}n^{r_1-2}\log^{r_2}n, \tag{29}$$

where r_1 and r_2 have the meanings stated in the introduction.

Putting $-2\pi iny=w$, we have, by (25) and (12),

$$b_k=k^{-r}n^{r-1}(2\pi i)^{-1}\int_{2\pi-i\infty}^{2\pi+i\infty}e^w w^{-r}(\gamma-2\log k-\log w+\log n)^r\,dw. \tag{30}$$

Now it is known that

$$(31) \qquad (2\pi i)^{-1} \int_{2\pi-i\infty}^{2\pi+i\infty} e^w w^{-s}\, dw = \frac{1}{\Gamma(s)} \quad (s > 1).$$

Let $a^{(0)}(s) = 1/\Gamma(s)$, and let $a^{(g)}(s)$ denote the g-th derivative of $a^{(0)}(s)$. Then, by (31),

$$(32) \qquad (2\pi i)^{-1} \int_{2\pi-i\infty}^{2\pi+i\infty} e^w w^{-s}(-\log w)^g\, dw = a^{(g)}(s).$$

Hence, by (30) and the multinomial theorem,

$$(33) \qquad b_k = k^{-r} n^{r-1} \sum_{l,\,m,\,g,\,t} \frac{r!}{l!\ m!\ g!\ t!}\, \gamma^l (-2\log k)^m a^{(g)}(r) \log^t n,$$

where l, m, g, and t run through integers greater than or equal to 0 such that $l+m+g+t = r$. By (33),

$$|b_k| < A_{14}\, n^{r-1} k^{-r} \log^r k \quad (k > \sqrt{n});$$

hence, by (4) and (28),

$$\left| \sum_{k>N} c_k(n)\, b_k \right| < A_{14}\, n^{r-1} \sum_{k>N} k^{-r+1} \log^r k < A_{15}\, n^{\frac{1}{2}r} \log^r n,$$

and hence, by (29),

$$(34) \qquad \left| v_r(n) - \sum_{k=1}^{\infty} c_k(n)\, b_k \right| < A_{16}\, n^{r_1-2} \log^{r_2} n.$$

6. *The dominant term.*

We have

$$(35) \qquad \sum_{k=1}^{\infty} c_k(n)\, k^{-s} = [\zeta(s)]^{-1} \sigma_{1-s}(n) \quad (s > 1),$$

where $\sigma_{1-s}(n) = \sigma^{(0)}_{1-s}(n)$ (see introduction), and $\zeta(s)$ is the Riemann ζ-function. This was first proved by Ramanujan*, then, in a simpler way, by Hardy†. It seems, however, to have been overlooked that (35) follows almost immediately from Ramanujan's formula

$$(36) \qquad c_k(n) = \sum_{\substack{l,\,m \\ lm=k \\ m \mid n}} \mu(l)\, m,$$

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* *Trans. Camb. Phil. Soc.*, 22 (1918), 264.

† *Proc. Camb. Phil. Soc.*, 20 (1920), 266.

for which Landau* has given a very simple proof. We have, by (36),

$$c_k(n)\,k^{-s} = \sum_{\substack{l,\,m \\ lm=k \\ m\mid n}} \mu(l)\,l^{-s}\,m^{1-s},$$

and hence

$$\sum_{k=1}^{\infty} c_k(n)\,k^{-s} = \sum_{\substack{l,\,m \\ m\ \mid\ n}} \mu(l)\,l^{-s}\,m^{1-s} = \sum_{l=1}^{\infty} \mu(l)\,l^{-s} \sum_{m\,\mid\,n} m^{1-s},$$

which proves (35).

Let $b^{(0)}(s) = [\zeta(s)]^{-1}$, and let $b^{(m)}(s)$ denote the m-th derivative of $b^{(0)}(s)$. Then, by (35) and Leibniz's theorem,

$$(37) \qquad \sum_{k=1}^{\infty} c_k(n)\,k^{-s}(-\log k)^m = \sum_{u+v=m} \frac{m!}{u!\,v!}\,b^{(u)}(s)(-1)^v\,\sigma_{1-s}^{(v)}(n).$$

Hence, by (33),

$$(38) \qquad \sum_{k=1}^{\infty} c_k(n)\,b_k$$

$$= n^{r-1} \sum_{l,\,u,\,v,\,g,\,t} \frac{r!}{l!\,u!\,v!\,g!\,t!}\,\gamma^l\,2^u\,b^{(u)}(r)(-2)^v\,\sigma_{1-r}^{(v)}(n)\,a^{(g)}(r)\log^t n$$

$$= n^{r-1} \sum_{t=0}^{r} \log^t n \sum_{v=0}^{r-t} c_{r,\,t,\,v}\,\sigma_{1-r}^{(v)}(n),$$

where

$$(39) \qquad c_{r,\,t,\,v} = \frac{r!}{t!\,v!}\,(-2)^v \sum_{\substack{l,\,u,\,g \\ l+u+g=r-t-v}} \frac{1}{l!\,u!\,g!}\,\gamma^l\,2^u\,b^{(u)}(r)\,a^{(g)}(r).$$

This, together with (34), proves Theorem 2.

* *Vorlesungen über Zahlentheorie*, 1 (1927), 188, Theorem 220.

ON QUANTITATIVE SUBSTITUTIONAL ANALYSIS

(*Sixth Paper*).

By ALFRED YOUNG.

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This paper is a continuation of the fourth paper in the series*. In the first paragraph a second proof of Theorem II in that paper is given. Section 2 gives a proof of Frobenius' generating function for the characters of the symmetric group by the methods of this analysis—a proof which I believe throws light on its nature.

In § 3 the quadratic invariant form, which forms the basis of what follows, is introduced. In § 4 this form is calculated in an elementary way for the representations of unit rank $T_{n-k,1^k}$ (where 1^k is an abbreviation for 1 repeated k times). The results elaborated in § 5 give an indication of the nature of the more general results which follow.

In § 6 is introduced the seminormal matrix in terms of which all substitutional functions may be expressed, and Theorem II is proved; this theorem enables substitutional expressions which do not involve the last letter to be written down at once. Theorem III of § 7 gives a means of calculating the seminormal transforming matrix.

Sections 8, 9, 10, 11 lead up to Theorem IV, which gives a means of writing down the seminormal matrix for any transposition involving consecutive letters; this matrix is of a very simple form, involving only units positive or negative on the leading diagonal and quadratic matrices which need no calculation along that diagonal, the other elements being zero. Given these transpositions, all other substitutions can at once be calculated. In § 12 Theorem V the same thing is done for the orthogonal

* *Proc. London Math. Soc.* (2), 31 (1931), 253–288.

matrix. In § 15 a numerical function, the tableau function, is introduced, which enables the principal constants in the quadratic invariant to be calculated. In § 17 the full constants for the case $T_{n-k,\,k}$, where there are only two rows, are obtained. The same method could have been used for the case of unit rank, but it was felt that it was better to do this at the outset by elementary methods. I have as yet not been able to obtain the full constants for the general case.

1. The following is an alternative proof of Theorem II, Q.S.A. IV.

Let Γ be a symbol which represents the operation of taking the sum of the $n!$ permutations of the letters in the substitutional expression which follows it. We will consider such an expression belonging to $T_{\alpha_1 \alpha_2 \ldots \alpha}$, and given by the matrix M.

Then $$\Gamma M = \Sigma\, s M s^{-1},$$

where s is in turn each permutation of the symmetric group. We may express s as a sum of matrices belonging to the various T's, and (since the product of expressions belonging to two different T's is zero) we need only consider the matrices

$$s T_{\alpha_1 \alpha_2 \ldots \alpha_h} = [\lambda_{rs}], \quad s^{-1} T_{\alpha_1 \alpha_2 \ldots \alpha_h} = [\mu_{rs}].$$

Then $$\Gamma M = \Sigma\, [\lambda_{rs}]\, M\, [\mu_{rs}] = \Sigma\, [\mu_{rs}]\, M\, [\lambda_{rs}].$$

Now $$\Gamma M = \Sigma\, \lambda_\rho t_\rho,$$

where t_ρ is the sum of all the members of a conjugate set and λ_ρ is numerical. Hence

$$\Gamma M = AE,$$

where A is a constant.

Let $M_{\alpha\beta}$ be that matrix for which all the elements are zero except that in the α-th row and β-th column, which is unity. In other words

$$M_{\alpha\beta} = P_\alpha \sigma_{\alpha\beta} N_\beta M_\beta.$$

Then $$[\lambda_{rs}]\, M_{\alpha\beta}\, [\mu_{rs}] = [\lambda_{r\alpha} \mu_{\beta s}].$$

Therefore $$\Gamma M_{\alpha\beta} = A_{\alpha\beta} E = \Sigma\, [\lambda_{r\alpha} \mu_{\beta s}];$$

hence $$\Sigma \lambda_{r\alpha} \mu_{\beta s} = 0,$$

unless $r = s$.

But under the sign of Σ the λ and μ are interchangeable as they stand; for s and s^{-1} severally become each permutation in turn. Hence

$$\Sigma \lambda_{r\alpha} \mu_{\beta s} = 0,$$

unless $\alpha = \beta$. Also

$$\Sigma \lambda_{r\alpha} \mu_{\alpha r} = \Sigma \lambda_{s\alpha} \mu_{\alpha s} = \Sigma \lambda_{r\beta} \mu_{\beta r}.$$

Thus $$\Gamma P_\alpha N_\alpha M_\alpha = \Gamma M_{\alpha\alpha} = \Gamma M_{\beta\beta} = \frac{1}{f} \Gamma E = \frac{n!}{f}.$$

And $$\Gamma P_\alpha \sigma_{\alpha\beta} N_\beta M_\beta = \Gamma M_{\alpha\beta} = 0,$$

unless $\alpha = \beta$.

2. We may obtain the generating function, given by Frobenius* for the characteristics of the symmetric group, in the following manner.

Let $$P_{\alpha\beta} = \{a_1 a_2 \dots a_\alpha\} \{a_{\alpha+1} \dots a_{\alpha+\beta}\},$$

where $\alpha \geqslant \beta$ and the sequence of the letters is that of the suffixes.

Consider $$P_{\alpha\beta} T_{\alpha+r, \beta-r} = P_{\alpha\beta} \sum_{s=1}^{f} P_s N_s M_s.$$

We express every term as the sum of standard forms

$$\Sigma B_{us} P_u \sigma_{us} N_s M_s.$$

For any term $P_s N_s M_s$, whose tableau F_s has one of the first α letters in the lower row, the sum of standard forms obtained from the product contains only terms for which u and s are different. For the others the product is always

$$\alpha!\, r!\, (\beta - r)!\, P_s N_s M_s + \sum_{s \neq u} B_{us} P_u \sigma_{us} N_s M_s;$$

moreover the number of these terms is $\binom{\beta}{r}$.

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* *Berliner Sitzungsberichte*, 1900, 516–534.

Applying the results and method of the last paragraph, we have

$$\Gamma P_{\alpha\beta}\, T_{\alpha+r,\,\beta-r} = \alpha!\,\beta!\,\frac{(\alpha+\beta)!}{f_{\alpha+r,\,\beta-r}}\, T_{\alpha+r,\,\beta-r}.$$

Hence

$$\frac{1}{(\alpha+\beta)!}\binom{\alpha+\beta}{\alpha}\Gamma P_{\alpha\beta} = \sum_{r=0}^{\beta}\frac{(\alpha+\beta)!}{f_{\alpha+r,\,\beta-r}}\, T_{\alpha+r,\,\beta-r}$$

$$= \frac{1}{1-S_{12}}\,\frac{(\alpha+\beta)!}{f_{\alpha,\,\beta}}\, T_{\alpha,\,\beta},$$

where S_{12} is an operator which removes a letter from the lower row to the upper one.

It is easy to see that this equation may be inverted, giving

$$\frac{(\alpha+\beta)!}{f_{\alpha,\,\beta}}\, T_{\alpha,\,\beta} = (1-S_{12})\,\frac{1}{(\alpha+\beta)!}\binom{\alpha+\beta}{\alpha}\Gamma P_{\alpha,\,\beta}.$$

More generally, we find in the same way that

$$\frac{1}{n!}\binom{n}{\alpha_1\,\alpha_2\,\ldots\,\alpha_h}\Gamma P_{\alpha_1\alpha_2\ldots\alpha_h} = \Pi S_{rs}^{\lambda_{rs}}\,\frac{n!}{f_{\alpha_1\alpha_2\ldots\alpha_h}}\, T_{\alpha_1\alpha_2\ldots\alpha_h};$$

where $P_{\alpha_1\alpha_2\ldots\alpha_h}$ is the product of positive symmetric groups of different letters of degrees $\alpha_1, \alpha_2, \ldots, \alpha_h$ respectively; also $r<s$, and S_{rs} represents the operation of moving one letter from the s-th row up to the r-th row; and the resulting term is regarded as zero, when any row becomes less than a row below it, or when letters from the same row overlap—as, for instance, happens when $\alpha_1 = \alpha_2$ in the case $S_{13}\,S_{23}$.

This equation also may be inverted, and gives us

$$\frac{n!}{f_{\alpha_1\alpha_2\ldots\alpha_h}}\, T_{\alpha_1\alpha_2\ldots} = \Pi(1-S_{rs})\,\frac{1}{n!}\binom{n}{\alpha_1\,\alpha_2\,\ldots\,\alpha_h}\Gamma P_{\alpha_1\alpha_2\ldots\alpha_h}. \qquad \text{(I)}$$

Let $t_{\beta_1\beta_2\ldots\beta_k} \equiv t_\beta$ be the sum of the different members of the conjugate set of permutations, belonging to the symmetric group of n letters, which has k cycles of $\beta_1, \beta_2, \ldots, \beta_k$ letters respectively. Then it was proved (Corollary to Theorem I, Q.S.A. IV, p. 256) that

$$\frac{n!}{f_\alpha}\, T_\alpha = \Sigma\,\chi_\beta^\alpha\, t_\beta. \qquad \text{(II)}$$

Let $$W_\beta = \prod_{r=1}^{k} (x_1^{\beta_r} + x_2^{\beta_r} + \ldots + x_n^{\beta_r}) = \Sigma\, B_\alpha^\beta x_1^{\alpha_1} x_2^{\alpha_2} \ldots x_h^{\alpha_h}.$$

Then the number of ways in which a permutation of the set t_β can be formed out of the *different* groups of $P_{\alpha_1 \alpha_2 \ldots \alpha_h}$ is the coefficient B_α^β of $x_1^{\alpha_1} x_2^{\alpha_2} \ldots x_h^{\alpha_h}$ in the expansion of W_β; in this enumeration the letters belonging to a given group of P are regarded as all identical, but when $\beta_r = \beta_{r+1}$ the two cycles are regarded as different. We now change the notation and denote by $\tau_{\gamma_1 \gamma_2 \ldots}$ the sum of the permutations of the symmetric group of n letters which belong to that conjugate set which has γ_1 cycles of one letter, γ_2 cycles of two letters, and so on. The number of these permutations is well known to be n_γ, where

$$n_\gamma \Pi \gamma_r!\ r^{\gamma_r} = n!.$$

Consider the number $n_\gamma^{(\alpha)}$ of the permutations of this set contained in $P_{\alpha_1 \alpha_2 \ldots \alpha_h}$.

To find this we use a function W_γ, which is merely what W_β becomes on change of notation, and observe that, since the equal cycles are really equivalent, we must divide B_α^γ by $\Pi \gamma_r!$; and, since the letters of the different groups of P are really distinct, we must multiply by $\Pi \alpha_r!$; and finally, since the letters of each cycle may be permuted cyclically without changing the cycle, we must divide by Πr^{γ_r}. Hence

$$n_\gamma^{(\alpha)} \Pi \gamma_r!\ r^{\gamma_r} = B_\alpha^\gamma \Pi \alpha_r!;$$

and therefore $$n_\gamma B_\alpha^\gamma = \binom{n}{\alpha_1 \alpha_2 \ldots \alpha_h} n_\gamma^{(\alpha)}.$$

Hence $$\binom{n}{\alpha_1 \alpha_2 \ldots \alpha_h} \Gamma P_{\alpha_1 \alpha_2 \ldots \alpha_h} = \Sigma\, n!\ B_\alpha^\gamma t_\gamma.$$

Then, using (I) and (II), we obtain

$$\Sigma \chi_\gamma^\alpha t_\gamma = \Pi (1 - S_{rs}) \Sigma B_\alpha^\gamma t_\gamma.$$

To deduce the generating function, we consider first the case $h = 2$. Then

$$\Sigma \chi_\gamma^{\alpha_1 \alpha_2} t_\gamma = (1 - S_{12}) \Sigma B_{\alpha_1 \alpha_2}^\gamma t_\gamma$$

$$= \Sigma B_{\alpha_1 \alpha_2}^\gamma t_\gamma - \Sigma B_{\alpha_1 + 1,\, \alpha_2 - 1}^\gamma t_\gamma.$$

Hence $\chi_\gamma^{\alpha_1 \alpha_2}$ is the coefficient of $x_1^{\alpha_1 + 1} x_2^{\alpha_2}$ in the expansion of $(x_1 - x_2) W_\gamma$.

In the same way we see that, whatever h may be, χ_γ^α is the coefficient of

$$x_1^{n-1+\alpha_1} x_2^{n-2+\alpha_2} x_h^{n-h+\alpha_h} x_{h+1}^{n-h-1} \dots$$

in the expansion of $\Delta \,.\, W_\gamma$, where Δ is the alternating function

$$\prod_{r<s}^{n} (x_r - x_s).$$

3. Corresponding to the set of forms $T_{\alpha_1 \alpha_2 \dots \alpha_h}$ there is a representation of the symmetric group in f variables $x_1, x_2, \dots, x_f$.

Each variable is related to a standard tableau F, and thus we may conveniently use a notation $[F]$ for the variable, exhibiting the tableau to which it is related.

A permutation s corresponds to a transformation of the variables; this transformation may be obtained by operating on the tableau with s. Let F thus become F_1, then F_1 is not necessarily standard, but

$$sPN = P_1 sN = \Sigma \lambda_r P_r s_r N,$$

where every P_r is standard. The linear transformation corresponding to s transforms $[F]$ into $\Sigma \lambda_r [F_r]$.

We may conveniently extend the notation and use the non-standard symbol $[F_1]$ as the equivalent of the sum $\Sigma \lambda_r [F_r]$. For every real linear group there is a quadratic function of the variables, which is invariant for all the transformations of the group, and is unique when the group is irreducible*. In our case the quadratic invariant is evidently given by $\Sigma [F]^2$, where the Σ includes all tableaux whether standard or not.

4. Let us consider an example, $T_{n-k, 1^k}$. Then the tableaux, and so the variables, are completely defined by the letters in the first column $a_1 a_{\lambda_1} \dots a_{\lambda_k}$. We write the general variable as $x_{\lambda_1 \lambda_2 \dots \lambda_k}$; we may allow the suffixes to be arranged in any sequence provided that we remember that the interchange of a pair of suffixes changes the sign of the variable.

In consequence of this, in a product term of the quadratic invariant the suffixes of the two factors are the same except for one pair. It is

* Burnside, *The theory of groups* (1911), 367.

easy to calculate directly that

$$\Sigma\,(k+1)\,x^2_{\lambda_1\lambda_2\ldots\lambda_k}+2\,\Sigma\,x_{\lambda_1\lambda_2\ldots\lambda_{k-1}\mu_1}x_{\lambda_1\lambda_2\ldots\lambda_{k-1}\mu_2}$$

may be taken as the quadratic invariant. Here $1<\lambda_1<\lambda_2\ldots<\lambda_{k-1}$, but the values of μ_1, μ_2 bear no direct relation to the sequence.

THEOREM I. *The quadratic invariant for* $T_{n-k,\,1^k}$ *may be written*

$$H=\Sigma\,\frac{1}{\Pi\,(\lambda_r)_2}\,X^2_{\lambda_1\lambda_2\ldots\lambda_k},$$

where $1<\lambda_1<\lambda_2\ldots<\lambda_k$, *and*

$$X_{\lambda_1\lambda_2\ldots\lambda_k}=\Sigma\,B^{\lambda}_{\mu_1}B^{\lambda}_{\mu_2}\ldots B^{\lambda}_{\mu_k}x_{\mu_1\mu_2\ldots\mu_k},$$

the numerical coefficients $B^{\lambda}_{\mu_r}$ *being thus defined*:

$$B^{\lambda}_{\mu_r}=0 \quad \textit{unless} \quad \lambda_{r+1}\geqslant\mu_r\geqslant\lambda_r,$$

$$B^{\lambda}_{\mu_r}=1 \quad \textit{if} \quad \lambda_{r+1}>\mu_r>\lambda_r,$$

$$B^{\lambda}_{\mu_r}=\lambda_r \quad \textit{if} \quad \lambda_r=\mu_r,$$

$$B^{\lambda}_{\mu_r}=1-\lambda_{r+1} \quad \textit{if} \quad \lambda_{r+1}=\mu_r.$$

Since the representation of the group is irreducible, there is only one invariant quadratic. It is therefore sufficient to show that this quadratic is invariant for the group. The symmetric group of degree n is generated by the $n-1$ transpositions of consecutive letters $(a_r a_{r+1})$; therefore we need consider only these transpositions. Consider the result of operating with $(a_r a_{r+1})$ on the variable $x_{\lambda_1\lambda_2\ldots\lambda_k}$. There are three cases:

(i) Neither r nor $r+1$ appear among the suffixes of x; the variable is unchanged.

(ii) Both r and $r+1$ appear among the suffixes of x; the variable is changed in sign.

(iii) One, say r, but not $r+1$ appears amongst the suffixes; then x is changed into another variable y in which the suffix r is replaced by $r+1$.

Let us now consider the effect of this transposition on $X_{\lambda_1\lambda_2\ldots\lambda_k}$. We distinguish the same three cases:

(i) Neither r nor $r+1$ appears among the suffixes. Then, these suffixes being arranged in ascending order of magnitude, r and $r+1$ must lie between a consecutive pair, say λ_s and λ_{s+1}.

Then the coefficient of $x_{\mu_1\ldots\mu_k}$ is zero when r or $r+1$ appears amongst its suffixes, unless only one appears and that as μ_s. Moreover, when $\mu_s = r$ or $r+1$, $B^{\lambda}_{\mu_s} = 1$. Thus the linear function X is unaltered by the transformation.

(ii) When both r and $r+1$ appear among the suffixes of X, we may take $\lambda_s = r$, $\lambda_{s+1} = r+1$. Let Y be the transformed form of X.

Then the coefficient of $x_{\mu_1\ldots}$ is zero unless $\mu_s = r$ or $r+1$; and thus, when only one of r, $r+1$ appears among the suffixes of x, it is μ_s. Now for $\mu_s = r$, $B^{\lambda}_{\mu_s} = r$; and for $\mu_s = r+1$, $B^{\lambda}_{\mu_s} = -r$. Moreover, when both r and $r+1$ appear among the suffixes of x, the sign of x is changed by the transformation. Thus

$$Y = -X.$$

(iii) $$\lambda_s = r, \quad \lambda_{s+1} > r+1, \quad \lambda_{s-1} < r.$$

We shall call this for brevity X_r, and associate with it the function in which all the λ's are the same except $\lambda_s = r+1$, and call that X_{r+1}. We want to find the transformed functions Y_r and Y_{r+1}.

The coefficient of x_μ is zero in both cases when r or $r+1$ appears as any suffix except μ_s or μ_{s-1}.

When neither r nor $r+1$ appears in the suffixes of x we have a sum of terms $\Sigma A_\mu x_\mu$, the same for both X_r and X_{r+1}, and also for Y_r and Y_{r+1}. In X_r we have the following coefficients:

$$\mu_s = r, \ B_{\mu_s} = r; \quad \mu_s = r+1, \ B_{\mu_s} = 1; \quad \mu_{s-1} = r, \ B_{\mu_{s-1}} = -(r-1).$$

In X_{r+1} we have the following coefficients:

$$\mu_s = r+1, \ B_{\mu_s} = r+1; \quad \mu_{s-1} = r, \ B_{\mu_{s-1}} = 1;$$

$$\mu_{s-1} = r+1, \ B_{\mu_{s-1}} = -r.$$

Then, if y is a variable for which $\mu_{s-1} = r$, $\mu_s = r+1$, y will appear

in X_r with coefficient $-(r-1)C$, and in X_{r+1} with coefficient $(r+1)C$. Let u_r be a variable x in which $\mu_s = r$, but $r+1$ is not among the suffixes and u_{r+1} the same when μ_s is changed to $r+1$; let v_r, v_{r+1} be similar variables when r or $r+1$ is in the position μ_{s-1}.

Then

$$X_r = \Sigma A_\mu x_\mu - (r-1)\Sigma Cy + \Sigma D(ru_r + u_{r+1}) - (r-1)\Sigma Ev_r;$$

$$X_{r+1} = \Sigma A_\mu x_\mu + (r+1)\Sigma Cy + (r+1)\Sigma Du_{r+1} + \Sigma E(v_r - rv_{r+1});$$

and hence

$$X_r + Y_r = 2\Sigma A_\mu x_\mu + (r+1)\Sigma D(u_r + u_{r+1}) - (r-1)\Sigma E(v_r + v_{r+1})$$

$$= X_{r+1} + Y_{r+1},$$

and

$$\frac{Y_r - X_r}{r-1} = 2\Sigma Cy - \Sigma D(u_r - u_{r+1}) + \Sigma E(v_r - v_{r+1})$$

$$= -\frac{Y_{r+1} - X_{r+1}}{r+1}.$$

Now in the sum H, X_r and X_{r+1} appear in the form

$$g\left[\frac{1}{r(r-1)}X_r^2 + \frac{1}{r(r+1)}X_{r+1}^2\right] = g\left[\frac{1}{r(r-1)}Y_r^2 + \frac{1}{r(r+1)}Y_{r+1}^2\right],$$

where g is a constant. Hence H is invariant for the transformation $(a_r a_{r+1})$.

We have yet to consider the transformation $(a_1 a_2)$. The only variable affected is $x_{2,\mu_2\ldots\mu_k}$, and this variable occurs only in $X_{2,\lambda_2\ldots\lambda_k}$. Thus every X_λ which has not 2 for its first suffix is unchanged by the transformation. Now $x_{2,\mu_2\ldots}$ becomes on transformation $-x_{2,\mu_2\ldots} - x_{3,\mu_2\ldots}$.

Let $X_{2,\lambda_2\ldots}$ become Y. In Y there may at first appear variables $x_{\nu_1\nu_2\ldots\nu_k}$ which have zero coefficients in X. Let us consider the possibilities, as follows:

(i) Two suffixes lie between λ_s and λ_{s+1}; this variable arises from two terms of X in which one or other of the suffixes is replaced by 2. They will appear with the same coefficient but a different sign, and their sum is zero.

(ii) Next we may have three ν's between λ_{s-1} and λ_{s+1}, one of them being λ_s; say ν_{s-1}, λ_s, ν_{s+1}.

This term arises from three terms of X, each with $\mu_1 = 2$; they are

$$\mu_{s-1} = \nu_{s-1},\ \mu_s = \lambda_s,\ \text{coefficient } \pm 2\lambda_s C,$$

$$\mu_{s-1} = \nu_{s-1},\ \mu_s = \nu_s,\ \text{coefficient } \mp 2C,$$

$$\mu_{s-1} = \lambda_s,\quad \mu_s = \nu_s,\ \text{coefficient } \mp 2(\lambda_s - 1)\, C,$$

and again the sum is zero.

We may treat the case of four ν's between λ_{s-2} and λ_{s+1}, viz. ν_{s-2}, λ_{s-1}, λ_s, ν_{s+1}, in the same way; the sum again is zero.

Proceeding thus we find that the coefficients of all variables which do not appear in X are zero in Y.

As regards those terms $x_{\mu_1 \mu_2 \dots \mu_k}$ whose coefficients are not zero in X, we have:

(i) $\mu_1 = 2$, the term is merely changed in sign.

(ii) $\mu_1 < \lambda_2$. $B_{\mu_1} = 1$. The coefficient is C in X.

In Y it also arises from $x_{2, \mu_2 \dots \mu_k}$ and from no other term, and this gives a coefficient $-2C$; and thus finally the coefficient is $-C$.

(iii) $\mu_1 = \lambda_2$, $\mu_2 < \lambda_3$.

In Y this arises by replacing either μ_1 or μ_2 by 2, giving coefficients $-C$, $\lambda_2 C$; in X the coefficient was $-(\lambda_2 - 1)\, C$, in Y the final coefficient is $(\lambda_2 - 1)\, C$, a mere change of sign. Proceeding thus we find that

$$Y = -X.$$

Thus H is also invariant for $(a_1 a_2)$ and therefore for the whole group. It is therefore the quadratic invariant.

5. We may now take $X_{\lambda_1 \dots \lambda_k}$ as new variables; in this case the quadratic invariant is a sum of squares with numerical coefficients, the seminormal form. The transformations of the symmetric group for these variables follow a very simple law. They are derived from the transpositions of consecutive letters. We express the corresponding transformations as matrices. That for $(a_1 a_2)$ consists merely of ± 1 along the leading diagonal. That for $(a_r a_{r+1})$ is made up as follows: some of the elements in the leading diagonal are ± 1, and in these cases the row and

column on which they lie are filled elsewhere with zeros; for the rest there is a series of quadratic matrices about the remaining elements of the leading diagonal, each being

$$\begin{bmatrix} -1/r & (r-1)/r \\ (r+1)/r & 1/r \end{bmatrix}$$

for the transposition $(a_r a_{r+1})$. The rest of the elements of the matrix are zero.

It is to be observed here that the sequence of variables follows the law that $X_{\lambda_1 \dots \lambda_k}$ precedes $X_{\mu_1 \dots \mu_k}$, when the first of the differences

$$\lambda_k - \mu_k, \quad \lambda_{k-1} - \mu_{k-1}, \quad \dots$$

which does not vanish is positive.

Should it be desired to obtain an orthogonal representation of the group, we may take as variables*

$$\xi_{\lambda_1 \dots \lambda_k} = \frac{1}{\sqrt{\{(\lambda_1)_2 \, (\lambda_2)_2 \dots (\lambda_k)_2\}}} X_{\lambda_1 \lambda_2 \dots \lambda_k}.$$

It is easily seen that the matrix corresponding to $(a_r a_{r+1})$ is of the same form as in the case already given, the only difference being that the quadratic submatrices about the leading diagonal are

$$\begin{bmatrix} -1/r & \sqrt{(r^2-1)}/r \\ \sqrt{(r^2-1)}/r & 1/r \end{bmatrix}.$$

6. Let us now consider the general case $T_{a_1 a_2 \dots a_k}$. There are f standard tableaux and corresponding to them f variables which will for the general argument be written $x_1, x_2, \dots, x_f$ according to the sequence of tableaux. Then, as described, we may obtain a quadratic invariant

$$H = \Sigma A_{rr} x_r^2 + 2 \Sigma A_{rs} x_r x_s.$$

We may express H as a sum of squares. Thus we begin with x_f and the first square is

$$\frac{1}{A_{ff}} \{A_{ff} x_f + A_{ff-1} x_{f-1} + \dots + A_{f1} x_1)^2,$$

* We use $(\lambda)_k$ as an abbreviation for $\lambda \, (\lambda-1) \dots (\lambda-k+1)$.

the rest of H does not contain x_f; we then absorb all the terms containing x_{f-1} in the next square; the final result may be written

$$H = \Sigma\, 1/B_r X_r^2;$$

where $$X_r = \lambda_{rr} x_r + \lambda_{rr-1} x_{r-1} + \ldots + \lambda_{r1} x_1.$$

Here all the coefficients are positive or negative integers. Let us take as new variables

$$\xi_r = \frac{1}{\lambda_{rr}} X_r = x_r + \frac{\lambda_{rr-1}}{\lambda_{rr}} x_{r-1} + \ldots. \qquad \text{(III)}$$

We now turn our attention to substitutional expressions. All such have been expressed in the form

$$\Sigma\, C_{rs} P_r \sigma_{rs} N_s M_s$$

or as the matrix $[C_{rs}] = C$.

Let us now use new substitutional units derived from the old ones by the transformation (III), *i.e.*

$$\xi_{rs} = P_r \sigma_{rs} N_s M_s + \frac{\lambda_{rr-1}}{\lambda_{rr}} P_{r-1} \sigma_{r-1s} N_s M_s + \ldots.$$

Then, if the new matrix is C',

$$C = MC',$$

where

$$M = \begin{bmatrix} 1 & \frac{\lambda_{21}}{\lambda_{22}} & \frac{\lambda_{31}}{\lambda_{33}} & \ldots & \frac{\lambda_{f1}}{\lambda_{ff}} \\ 0 & 1 & \frac{\lambda_{32}}{\lambda_{33}} & \ldots & \frac{\lambda_{f2}}{\lambda_{ff}} \\ 0 & 0 & 1 & \ldots & \\ \ldots & & \ldots & \ldots & \ldots \\ 0 & 0 & 0 & \ldots & 1 \end{bmatrix}.$$

Thus $$C' = M^{-1} C.$$

At present the right-hand side $N_s M_s$ of the units has not been transformed. In order that the new matrix expression for unity may still be E, we must perform the inverse transformation at the same time on the right. The new matrix will then be

$$D = M^{-1} CM.$$

When C is the matrix form of the transformation which corresponds to the permutation s, it gives the transformation for the variables x; then that for the variables ξ is given by D.

Now this transformation with units in the leading diagonal appropriate to the seminormal quadratic invariant arranged according to the tableau sequence is definite and unique. We may take this sequence to be according to the position of the last letter, then according to the last but one, and so on. We have already obtained a transformation of the same form for passing from substitutional expressions of $n-1$ letters to those of n letters. In Q.S.A. IV, Theorem IV (p. 270), it was shown that if S is a substitutional expression of the letters $a_1 a_2 \dots a_{n-1}$, and $T_{a_1 \dots a_h} = T$ involves a last letter a_n, then

$$ST = IKI^{-1};$$

where K is a T matrix which consists of the matrices $[T_\lambda]$, $\lambda = h, \dots, 1$, down the leading diagonal, and $[T'_\lambda]$ is the matrix for $S\Delta_\lambda T$ ($\Delta_\lambda T$ meaning T with its λ-th suffix reduced by one); and the matrix $I = [I_{\lambda\mu}]$, where $I_{\lambda\mu}$ are partial matrices, $I_{\lambda\lambda}$ occupying the same place in I as $[T_\lambda]$ does in K, and

$$I_{\lambda\lambda} = E, \quad I_{\lambda\mu} = 0 \quad (\mu < \lambda).$$

Now the product and also the inverses of matrices which have all the elements zero below the leading diagonal and all the elements of the leading diagonal unity have the same property. Let $[T_\lambda']$ be the matrix $[T_\lambda]$ transformed to its seminormal form; and let K' be the matrix obtained from K by replacing each matrix $[T_\lambda]$ by $[T_\lambda']$. Then K is transformed to K' by a matrix with the property just described. The expression S which involves only the first $n-1$ letters when expressed as a matrix in the seminormal form becomes

$$JK'J^{-1},$$

where J has the same property.

Now this transformation is unique so far as matrices of this nature are concerned. Moreover, the property that the quadratic invariant for the group contains no product terms is a property of the substitutions of the group. But the substitutions represented by K' already have this property, hence $J = 1$.

We shall refer to this representation of $T_{a_1 a_2 \dots a_h}$ as the seminormal representation, and to the substitutional units which correspond to

the elements of the seminormal matrix as the seminormal units; it will be convenient to use the notation ϖ_{rs} for these units.

The important fact for the seminormal representation, which has just been deduced, may be enunciated thus:

THEOREM II. *The seminormal matrix for $S.T$, where S is any substitutional expression involving $a_1, a_2, \ldots, a_{n-1}$, and $T = T_{a_1 a_2 \ldots}$ involves a last letter a_n, is that T matrix which consists of the matrices $[T_\lambda]$, $\lambda = h, h-1, \ldots, 1$, down the leading diagonal; where $[T_\lambda]$ is the seminormal matrix for $S.\Delta_\lambda T$.*

It should be noticed that the values of λ are given in descending order, as according to the usual sequence, tableaux with a_n in a given row precede those with a_n in a row higher up. In this particular the enunciation of Theorem IV in Q.S.A. IV is at fault.

7. For the sake of clearness the matrices hitherto used for $T_{a_1 \ldots a_h}$ corresponding to the units $P_r \sigma_{rs} N_s M_s$ will be called the natural matrices, and these units the natural units. It has been shown that there is a matrix M of special character, by means of which we can pass from the natural matrix C to the seminormal matrix D of a given expression, that in fact

$$D = M^{-1} C M.$$

We call M the seminormal transforming matrix for $T_{a_1 \ldots a_h}$. Consider now an expression S involving only the first $n-1$ letters. We have seen that, if $[T_\lambda]$ is the natural matrix and $[T_\lambda']$ the seminormal matrix for $S.\Delta_\lambda T_{a_1 \ldots a_h}$, then the natural matrix for $S.T_{a_1 \ldots a_h}$ is

$$I \begin{bmatrix} [T_h] & 0 & 0 & \ldots \\ 0 & [T_{h-1}] & 0 & \ldots \\ 0 & 0 & [T_{h-2}] & \ldots \\ \ldots & \ldots & \ldots & \ldots \end{bmatrix} I^{-1} = IKI^{-1};$$

and the seminormal matrix is

$$\begin{bmatrix} [T_h'] & 0 & \ldots \\ 0 & [T_{h-1}'] & \ldots \\ \ldots & \ldots & \ldots \\ \ldots & \ldots & \ldots \end{bmatrix},$$

but this is equal to $J^{-1}KJ$,

where J is the matrix

$$\begin{bmatrix} M_h & 0 & 0 & \cdots \\ 0 & M_{h-1} & 0 & \cdots \\ 0 & 0 & M_{h-2} & \cdots \\ \cdots & \cdots & \cdots & \cdots \end{bmatrix},$$

M_λ being the seminormal transforming matrix for

$$\Delta_\lambda T_{\alpha_1 \ldots \alpha_h}.$$

We notice that S may be the perfectly general substitutional expression of the first $n-1$ letters, in which case $[T_\lambda]$ is a group matrix, and, as Schur has proved, is permutable with no other matrix except E. Then, since the matrices $[T_\lambda]$ are all different, the only matrix permutable with K is L, where L is the matrix obtained from J by replacing each M_r by $p_r E_r$, p_r being an arbitrary constant, and E_r the unit matrix of appropriate degree. Then, since

$$M^{-1}IKI^{-1}M = J^{-1}KJ,$$

K is permutable with $I^{-1}MJ^{-1}$, and

$$I^{-1}MJ^{-1} = L.$$

But both I and MJ^{-1} are matrices with units in the leading diagonal and zeros below it, hence in L every p_r is unity, so that

$$L = E,$$

and therefore $$M = IJ.$$

THEOREM III. *The seminormal transforming matrix for $T_{\alpha_1 \alpha_2 \ldots \alpha_h} = T$ being M, and that for $\Delta_\lambda T$ being M_λ, then*

$$M = IJ$$

where I is the transforming matrix corresponding to the addition of the last letter in T, and J is that matrix formed by the matrices M_λ in the leading diagonal, with zeros elsewhere.

8. Consider the matrix for $(a_{n-1}a_n)$ in the seminormal form. We first subdivide the seminormal matrix according to the positions of the last two letters. When we take the position of the last letter only we have a system of subdivision according to the sequence of tableaux for

$$\Delta_h T, \quad \Delta_{h-1} T, \quad \ldots, \quad \Delta_1 T.$$

Taking a further subdivision we use the tableaux for

$$\Delta_h^2 T, \quad \Delta_{h-1}\Delta_h T, \quad \ldots, \quad \Delta_1\Delta_h T; \qquad \Delta_h\Delta_{h-1} T, \quad \Delta_{h-1}^2 T, \quad \ldots$$

in this sequence.

By Theorem II the most general substitutional expression involving the first $n-2$ letters only is represented by the appropriate group matrices $[T_{\lambda\mu}]$ down the leading diagonal and zeros elsewhere; call this X. The transposition $(a_{n-1}a_n)$ is permutable with all such expressions. Now Schur* has shown that the only matrices permutable with X are those which are made up of quadratic submatrices of the form

$$\begin{bmatrix} p_{11}E_{\lambda\mu} & p_{12}E_{\lambda\mu} \\ p_{21}E_{\lambda\mu} & p_{22}E_{\lambda\mu} \end{bmatrix},$$

occupying the rows and columns defined by the tableaux for $\Delta_\lambda\Delta_\mu T$ and $\Delta_\mu\Delta_\lambda T$: submatrices $qE_{\lambda\lambda}$ occupying the position on the leading diagonal of $[T_{\lambda\lambda}]$ in X, and zeros everywhere else. The seminormal matrix for $(a_{n-1}a_n)$ is then of this character.

It may be noticed that there are obviously more independent matrices than those appropriate to $A+B(a_{n-1}a_n)$ of this nature. Let Γ_{n-2} be an operation which represents taking the sum of all the $(n-2)!$ permutations of the symmetric group of the first $n-2$ letters in the following expression. Then $\Gamma_{n-2}(a_1a_{n-1})$, or $\Gamma_{n-2}(a_1a_n)$, or in fact $\Gamma_{n-2}S$, where S is any substitutional expression, is of this nature.

9. Consider now the transposition $(a_{n-1}a_n)$. We first take the case where the rows of the tableaux for T are all of unequal length. Then the natural matrix for $(a_{n-1}a_n)$ consists of quadratic matrices of the form $\begin{bmatrix} O & E \\ E & O \end{bmatrix}$, and of matrices E along the leading diagonal and zero elsewhere.

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* *Berliner Sitzungsberichte*, 1905, 406–432 (§3).

To make this clear we give an example, the natural matrix for $(a_5 a_6)$ in T_{42}.

The matrix has nine rows and columns, but since we are concerned only with the last two letters we can group together the first four letters. Arranging the tableaux according to their proper sequence, they will be:

$$T_4 \text{ (1 row)}, \quad T_{3,1} \text{ (3 rows)}, \quad T_{3,1} \text{ (3 rows)}, \quad T_{2,2} \text{ (2 rows)} ;$$

and the required matrix is

$$\begin{bmatrix} E_4 & 0 & 0 & 0 \\ 0 & 0 & E_{31} & 0 \\ 0 & E_{31} & 0 & 0 \\ 0 & 0 & 0 & E_{22} \end{bmatrix}.$$

As a second example, $(a_5 a_6)$ in T_{321} is

$$\begin{bmatrix} 0 & 0 & E_{31} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & E_{22} & 0 \\ E_{31} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & E_{211} \\ 0 & E_{22} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & E_{211} & 0 & 0 \end{bmatrix}.$$

Let C be the natural and D the seminormal matrix for $(a_{n-1} a_n)$, then

$$CM = MD. \tag{IV}$$

The matrices will be subdivided as in § 8, so that

$$M = [m_{rs}].$$

Let r refer to the row defined by $\Delta_\lambda{}^2 T$; then in the matrix on the left of (IV) the only elements in the r-th row are $E_{\lambda\lambda} m_{rs}$, the only elements on the r-th column in the right-hand matrix are $m_{qr} D_{rr}$, and hence, equating

the elements in the r-th row and column,

$$m_{rr} D_{rr} = E_{\lambda\lambda} m_{rr};$$

or $D_{rr} = E_{\lambda\lambda}$.

Let r refer to the tableau $\Delta_\lambda \Delta_\mu T$ and s to the tableau $\Delta_\mu \Delta_\lambda T$, where $\lambda < \mu$ and therefore r precedes s. Then, so far as the r-th and s-th rows are concerned, C is

$$\begin{bmatrix} 0 & E_{\lambda\mu} \\ E_{\lambda\mu} & 0 \end{bmatrix},$$

the columns being the r-th and s-th, the other elements of both these rows and columns being zeros. The same is true for these rows and columns of D except that the quadratic matrix is

$$\begin{bmatrix} p_{11} E_{\lambda\mu} & p_{12} E_{\lambda\mu} \\ p_{21} E_{\lambda\mu} & p_{22} E_{\lambda\mu} \end{bmatrix},$$

the p's being constants.

Consider the elements which occupy the position of this quadratic matrix on both sides of (IV). We have

$$\begin{bmatrix} m_{sr} m_{ss} \\ m_{rr} m_{rs} \end{bmatrix} = \begin{bmatrix} p_{11} m_{rr} + p_{21} m_{rs} & p_{12} m_{rr} + p_{22} m_{rs} \\ p_{11} m_{sr} + p_{21} m_{ss} & p_{12} m_{sr} + p_{22} m_{ss} \end{bmatrix},$$

and here $m_{sr} = 0$, for $s > r$.

Hence $$p_{11} m_{rr} = -p_{21} m_{rs}, \quad m_{rr} = p_{21} m_{ss},$$

$$m_{rs} = p_{22} m_{ss};$$

thus $$\frac{m_{rr}}{p_{21}} = \frac{m_{rs}}{-p_{11}} = m_{ss} = p_{12} m_{rr} + p_{22} m_{rs};$$

i.e. $$p_{11} = -p_{22}, \quad 1 = p_{12} p_{21} - p_{11} p_{22},$$

the matrices m_{rr}, m_{rs}, m_{ss} being all similar. To find the matrices m_{rr}, etc., we use the equation

$$M = IJ.$$

On the leading diagonal we have on the first subdivision the matrices M_λ;

and above this diagonal matrices $I_{\mu\lambda}M_\lambda$ $(\lambda<\mu)$. Along the leading diagonal of M_λ there are the matrices $M_{\mu\lambda}=M_{\lambda\mu}$; thus we see that

$$m_{rr}=m_{ss}=M_{\lambda\mu},$$

and $p_{21}=1$.

To find m_{rs}: this comes in the matrix

$$I_{\mu\lambda}M_\lambda;$$

it lies on the rows for which, in the tableaux, a_n is in the μ-th row and a_{n-1} in the λ-th row. To find its value reference must be made to Theorem III Q.S.A. IV, from which Theorem IV was derived. We are concerned in $I_{\mu\lambda}$ with the derivation of those terms of T which have a_n in the μ-th row from the terms of $\Delta_\lambda T$. This was effected by a substitutional product applied to the left-hand side of the form $K_{\lambda,h}K_{\lambda,h-1}\dots K_{\lambda,\lambda+1}$, where

$$K_{\lambda,\nu}=1+\sum_{t=1}^{a_\nu}\frac{1}{a_\lambda-a_\nu+\nu-\lambda}\quad(a_{\nu t}a_n);$$

for this purpose the length of the λ-th row is to be reckoned as $a_\lambda-1$.

We are interested in that part of the product which, when expanded out and expressed as standard forms, gives terms with a_n in the μ-th row. Thus all those factors $K_{\lambda\nu}$ in which $\nu>\mu$ may be replaced by unity. Now the matrix $I_{\mu\lambda}$ has to be subdivided into matrices for which the rows are given by the tableaux for

$$T_{h,\mu},\quad T_{h-1,\mu},\quad\dots,\quad T_{1,\mu}$$

and the columns by the tableaux for

$$T_{h,\lambda},\quad T_{h-1,\lambda},\quad\dots,\quad T_{1,\lambda}.$$

We are concerned only with the row defined by $T_{\lambda\mu}$, *i.e.* which represents a_{n-1} as being finally in the λ-th row. Let $I_{\mu\lambda}=[i_{\nu\rho}]$ according to this subdivision and let $M_\lambda=[n_{\nu\rho}]$.

Then the element of $I_{\mu\lambda}M_\lambda$ which we want, *i.e.* m_{rs}, is

$$\sum_\nu i_{\lambda\nu}n_{\nu\mu}.$$

We know that $n_{\nu\mu}$ is zero unless $\nu\geqslant\mu$ and $n_{\mu\mu}=M_{\lambda\mu}$. Also $i_{\lambda\nu}$ is that part of $I_{\mu\lambda}$ which represents terms in which a_{n-1} was originally in the

ν-th row and is finally in the λ-th row. Consider how this can happen when $\nu > \mu$. The last letter a_n is lowered from the λ-th row to the μ-th row. It is useless to consider terms which have a_n in a row lower than the μ-th, for the process of expressing non-standard terms in terms of standard can never raise the last letter. Moreover, since a_{n-1} is originally in the ν-th, a row lower than the μ-th, a_{n-1} cannot be raised by this process without reducing a_n to the ν-th row; and thus, when $\nu > \mu$,

$$i_{\lambda\nu} = 0.$$

We have only then to consider

$$i_{\lambda\mu} n_{\mu\mu} = i_{\lambda\mu} M_{\lambda\mu}.$$

This can arise only from the term of the operating product K,

$$\frac{1}{a_\lambda - a_\mu + \mu - \lambda} (a_{n-1} a_n);$$

and this term is standard.

Hence $$i_{\lambda\mu} = \frac{1}{a_\lambda - a_\mu + \mu - \lambda} E;$$

and $$m_{rs} = \frac{1}{a_\lambda - a_\mu + \mu - \lambda} M_{\lambda\mu} = \rho M_{\lambda\mu}.$$

The quadratic matrices then which appear in the seminormal matrix for $(a_{n-1} a_n)$ are of the form

$$\begin{bmatrix} -\rho & 1-\rho^2 \\ 1 & \rho \end{bmatrix}, \qquad \text{(V)}$$

where ρ^{-1} has the value $a_\lambda - a_\mu + \mu - \lambda$.

10. It is necessary now to consider the general case in which some of the tableau rows may be of the same length. Let $a_\tau = a_{\tau+1} > a_{\tau+2}$. Then a_n cannot appear in the τ-th row of a standard tableau; the particular question that must be considered is the effect of the transposition $(a_{n-1} a_n)$ on those natural units whose P tableaux have a_{n-1} at the end of the τ-th row and a_n at the end of the $(\tau+1)$-th row We subdivide the natural matrix $[C_{rs}] = C$ for $(a_{n-1} a_n)$ as before according to the positions of the

last two letters. And we specify the row or column defined by the positions of a_{n-1}, a_n now considered in the tableau as the t-th.

Then $$C_{tt} = -E_{\tau\tau+1},$$

$$C_{st} = 0, \quad \text{when} \quad s > t.$$

But there will be elements C_{st} which it is impossible to give offhand when $s < t$; they arise from the equations for expressing non-standard forms in terms of standard forms.

Consider now the element in the t-th row and t-th column on both sides of equation (IV). We have to remember that there may be another pair of equal rows. For the sake of argument suppose that this is the case, and that this second pair is higher up in the tableau, and that when a_{n-1} and a_n both lie in them the tableau is specified as the u-th, where in consequence $t < u$. In this case C_{tu} will not in general be zero, but since m_{ut} is zero it will not enter our equation, which becomes

$$-E_{\tau,\tau+1} m_{tt} = m_{tt} D_{tt},$$

or $$D_{tt} = -E_{\tau,\tau+1}.$$

It is sufficient now to notice that these unknown elements in the C matrix which arise when there are equal rows do not affect the argument of the last paragraph, because in the terms which are considered they always appear multiplied by a zero matrix of M.

Thus the seminormal matrix for $(a_{n-1}a_n)$ consists entirely of the quadratic matrices (V), unit matrices in those places on the leading diagonal defined by tableaux with a_{n-1}, a_n in the same row, and negative unit matrices in those places in the leading diagonal defined by tableaux with a_{n-1}, a_n in the same column.

11. Let us apply Theorem II to the fact just obtained. We see that the seminormal matrix for $(a_r a_{r+1})$ is known at once when we know the seminormal matrices for this transposition for all T's which refer to the first $r+1$ letters.

Now the rows and columns of the matrix for $T_{a_1 a_2 \ldots a_h}$ are defined by the sequence of tableaux F_1, F_2, ..., F_f. Let us specify them more particularly with respect to the positions of a_r and a_{r+1}. Let F_ϵ be a tableau in which a_r, a_{r+1} are in the same row, F_ζ one in which they are in the same column; and let F_ξ, F_η be a pair of tableaux which are identical

except for the positions of a_r and a_{r+1}, F having a_r in the λ-th row and the γ-th column—or briefly in the position (λ, γ)—and a_{r+1} in the position (μ, δ), and F_η having these positions interchanged. We suppose that $\lambda < \mu$, then $\xi < \eta$. The standard condition gives $\gamma > \delta$. Then the seminormal matrix for $(a_r a_{r+1})$ has unity on the leading diagonal and zeros elsewhere on the ϵ-th row and column; -1 on the leading diagonal and zeros elsewhere on the ζ-th row and column; and the ξ-th and η-th rows and columns have the quadratic matrix

$$\begin{bmatrix} -\rho & 1-\rho^2 \\ 1 & \rho \end{bmatrix}$$

where they intersect and zeros elsewhere, where

$$\rho^{-1} = \gamma - \delta + \mu - \lambda.$$

This is the sum of the projections of the straight line joining a_r and a_{r+1} on the first row and the first column of the tableau; we call it for brevity the axial projection of $a_r a_{r+1}$. We may then enunciate the result as follows:

THEOREM IV. *The seminormal matrix for $(a_r a_{r+1})$ has zero everywhere, except (i) unity on the leading diagonal where the tableau has a_r, a_{r+1} in the same row; (ii) -1 on the leading diagonal where the tableau has a_r, a_{r+1} in the same column; (iii) a quadratic matrix*

$$\begin{bmatrix} -\rho & 1-\rho^2 \\ 1 & \rho \end{bmatrix},$$

the elements being in the positions of the intersections of a pair of rows and columns whose tableaux differ only by the exchange of a_r and a_{r+1}—the value of ρ being the reciprocal of the axial projection of $a_r a_{r+1}$ in the tableaux.

12. The seminormal matrix corresponds to the variables defined by (III), and the quadratic invariant

$$\Sigma \frac{\lambda_{rr}^2}{B_r} \xi_r^2.$$

To find an orthogonal representation of the group, new variables η_r must

be chosen, for which the form $\Sigma \eta_r^2$ is invariant. This may be done by choosing

$$\eta_r = \frac{\lambda_{rr}}{\sqrt{B_r}} \xi_r = p_r \xi_r.$$

This is equivalent to transformation by a matrix which has zeros everywhere except in the leading diagonal; in other words, the r-th column of the matrices is multiplied by the same number p_r, and the r-th row is divided by p_r. In fact, we may write $\bar{D}$ as the orthogonal matrix where

$$\bar{D} = \bar{M}^{-1} D \bar{M},$$

and $\bar{M}$ has zeros everywhere except on the leading diagonal, which has the elements

$$p_1, p_2, \ldots, p_f.$$

The transformation $(a_r a_{r+1})$ for the orthogonal matrix is the same as for the seminormal matrix except that the quadratic matrices which there appear have now the form

$$\begin{bmatrix} -\rho & (1-\rho^2)\varpi^{-1} \\ \varpi & \rho \end{bmatrix}.$$

Since this matrix must be orthogonal,

$$\varpi = \sqrt{(1-\rho^2)}.$$

THEOREM V. *The orthogonal matrix for $(a_r a_{r+1})$ is identical with the seminormal matrix except that the quadratic matrices which appear in it are replaced by the matrices*

$$\begin{bmatrix} -\rho & \sqrt{(1-\rho^2)} \\ \sqrt{(1-\rho^2)} & \rho \end{bmatrix},$$

ρ having the same meaning as before.

13. We call the rows and columns which appear in the quadratic matrix in the argument of the last paragraph, the s-th and t-th, $s < t$. Then

$$\sqrt{(1-\rho^2)} = \varpi = \frac{p_s}{p_t} = \frac{\lambda_{ss}}{\lambda_{tt}} \sqrt{\left(\frac{B_t}{B_s}\right)}, \qquad \text{(VI)}$$

a relation which connects the coefficients of the original quadratic invariant § 6,

$$H = \Sigma \frac{1}{B_r} X_r^2.$$

This relation will be required later.

14. Consider the product

$$(a_r a_{r+1}) \varpi_{su},$$

where ϖ_{su} is a seminormal unit, and s, t give the rows and columns of the last paragraph.

$$(a_r a_{r+1}) \varpi_{su} = -\rho \varpi_{su} + \varpi_{tu},$$

$$(a_r a_{r+1}) \varpi_{tu} = (1-\rho^2) \varpi_{su} + \rho \varpi_{tu}.$$

Now

$$\varpi_{su} = \sum_\tau A_{u\tau} \left\{ P_s \sigma_{s\tau} N_\tau M_\tau + \frac{\lambda_{s\,s-1}}{\lambda_{ss}} P_{s-1} \sigma_{s-1\,\tau} N_\tau M_\tau + \ldots \right\};$$

and

$$\varpi_{tu} = \sum_\tau A_{u\tau} \left\{ P_t \sigma_{t\tau} N_\tau M_\tau + \frac{\lambda_{t\,t-1}}{\lambda_{tt}} P_{t-1} \sigma_{t-1\,\tau} N_\tau M_\tau + \ldots \right.$$
$$\left. + \frac{\lambda_{ts}}{\lambda_{tt}} P_s \sigma_{s\tau} N_\tau M_\tau + \ldots \right\}.$$

We are not concerned with the values of the constants $A_{u\tau}$.

Consider the effect of multiplying the term beginning with P_v by $(a_r a_{r+1})$. Since $v < t$, the position of a_n in the tableau F_v is either in the same row or lower down than in the tableau F_t; if $(a_r a_{r+1}) F_v$ is non-standard, the equations expressing the resulting term in standard units cannot raise the position of a_n. Thus the only terms which can yield P_s or P_t are those terms P_v which have a_n in the same row as in F_s or F_t. We apply the same argument to each letter in turn, and conclude that the only terms commencing with P_v which, on multiplication by $(a_r a_{r+1})$ and then expressing in standard units, contain the terms P_s or P_t are those for which F_v is the same as F_s and F_t so far as the letters $a_{r+2}, a_{r+3}, \ldots, a_n$ are concerned. As regards the position of a_r, a_{r+1} in F_v we may remove these last $n-r-1$ letters, and consider only tableaux F' of fixed form of $r+1$ letters. Regarding the possible tableaux F_v we see that unless a_r, a_{r+1} terminate two consecutive equal rows $(a_r a_{r+1}) F_v$ is standard, and so the

term only yields P_s or P_t when $v = s$ or t. In the case when a_r, a_{r+1} are at the ends of equal rows in F_v, the pair of equal rows must lie below the a_r row in F_s otherwise $v > t$; and thus both letters a_r, a_{r+1} lie below this row. After the transposition the standardising equations cannot raise them up to the level of that row without altering the position of later letters. Thus

$$(a_r a_{r+1})\,\varpi_{su} = \sum_\tau A_{u\tau} P_t \sigma_{t\tau} N_\tau M_\tau + \ldots$$

$$= -\rho \sum_\tau A_{u\tau} P_s \sigma_{s\tau} N_\tau M_\tau + \ldots$$

$$+ \sum_\tau A_{u\tau} \{P_t \sigma_{t\tau} N_\tau M_\tau + \lambda_{ts}/\lambda_{tt} P_s \sigma_{s\tau} N_\tau M_\tau + \ldots\},$$

giving $$\lambda_{ts}/\lambda_{tt} = \rho.$$

And

$$(a_r a_{r+1})\,\varpi_{tu} = \sum_\tau A_{u\tau} \{P_s \sigma_{s\tau} N_\tau M_\tau + \lambda_{ts}/\lambda_{tt} P_t \sigma_{t\tau} N_\tau M_\tau + \ldots\}$$

$$= (1-\rho^2) \sum_\tau A_{u\tau} \{P_s \sigma_{s\tau} N_\tau M_\tau + \ldots\}$$

$$+ \rho \sum_\tau A_{u\tau} \{P_t \sigma_{t\tau} N_\tau M_\tau + \lambda_{ts}/\lambda_{tt} P_s \sigma_{s\tau} N_\tau M_\tau + \ldots\},$$

with the same result.

Let us now consider the variables; the transformation $(a_r a_{r+1})$ becomes

$$\xi_s' = -\rho \xi_s + \xi_t,$$

$$\xi_t' = (1-\rho^2)\,\xi_s + \rho \xi_t;$$

changing to the variables X_r of § 6 and using Y_r as the transformed variable, we have

$$Y_s = -\rho X_s + \frac{\lambda_{ss}}{\lambda_{tt}} X_t,$$

$$Y_t = (1-\rho^2) \frac{\lambda_{tt}}{\lambda_{ss}} X_s + \rho X_t,$$

$$Y_s + X_s = (1-\rho) X_s + \frac{\lambda_{ss}}{\lambda_{tt}} X_t,$$

$$Y_t + X_t = (1-\rho^2) \frac{\lambda_{tt}}{\lambda_{ss}} X_s + (1+\rho) X_t.$$

If $\lambda_{ss}/\lambda_{tt} = 1+\rho$, then

$$Y_s+X_s = Y_t+X_t,$$

and

$$\frac{Y_s-X_s}{1+\rho} = -\frac{Y_t-X_t}{1-\rho},$$

exactly as was found for the case $T_{k,1^k}$.

The full justification for this ratio of λ_{ss} to λ_{tt} will be found later. Meanwhile it should be noticed that, the absolute values of λ_{rr} being still unfixed, the coefficients can all be made integral with this ratio.

15. We will now define a numerical function, the tableau function, by means of the standard tableau.

Let F be a standard tableau; it may be defined particularly as a matrix

$$F \equiv [\gamma_{r,s}];$$

$\gamma_{r.s}$ is the element in the r-th row and the s-th column; in particular, γ_{rs} is the suffix of the letter which appears in this position in the standard tableau F. The standard condition is

$$\gamma_{rs} > \gamma_{ts} \quad \text{when} \quad r > t,$$

$$\gamma_{rs} > \gamma_{rt} \quad \text{when} \quad s > t.$$

Then $\alpha_1, \alpha_2, \ldots, \alpha_h$ being, as usual, the numbers of letters in the first, second, etc., rows of F, the tableau function $[F]$ is the product of $\alpha_2+2\alpha_3+\ldots+(h-1)\alpha_h$ factors, which are defined thus. The element γ_{rs} gives rise to $r-1$ factors, one in relation to each row above it.

Consider the t-th row $(r > t)$; we have a factor γ_{rst}. We know that $\gamma_{ts} < \gamma_{rs}$, but γ_{tu} may be either less or greater than γ_{rs} when $u > s$; it must be greater than γ_{ts}.

Let us suppose that $\gamma_{tu} < \gamma_{rs}$ and $\gamma_{tu+1} > \gamma_{rs}$ (when $u = \alpha_t$ we shall always regard the non-existent γ_{tu+1} as greater than γ_{rs}). Then we define

$$\gamma_{rst} = u+r-s-t+1.$$

(If it is necessary to specify u more particularly we may call it u_{rst}.)

Then we define our function as

$$[F] = \Pi\, \gamma_{rst}. \qquad \text{(VII)}$$

We may take the first coefficient in the linear function X_r, *i.e.* the coefficient of the variable x_r, which corresponds to the standard tableau F_r, as the number which we have defined as $[F_r]$, the tableau function of F_r.

For consider the transposition $(a_r a_{r+1})$ which interchanges the tableaux F_s and F_t, defined as above; in F_s the position of a_r will be taken as (λ, σ), of a_{r+1} as (μ, τ); so that $\gamma_{\lambda\sigma} = r$, and $\gamma_{\mu\tau} = r+1$.

Then $$\gamma_{\mu\tau\lambda} = \sigma+\mu-\tau-\lambda+1.$$

In $[F_t]$ all the factors are the same as in $[F_s]$ with the single exception of $\gamma_{\mu\tau\lambda}$, which becomes $\sigma+\mu-\tau-\lambda$.

Now $\sigma+\mu-\tau-\lambda$ is what we have called the axial projection of $a_r a_{r+1}$ for the tableau, and have written ρ^{-1}; thus

$$[F_s]/(1+\rho) = [F_t].$$

This is the same as the ratio of the coefficients λ_{ss}, λ_{tt} given in § 14.

16. We now introduce a function which we call the second tableau function. For a given tableau it is defined by the same elements as the first tableau function, and we write it

$$[F]_2 = \Pi(\gamma_{rst}-1). \qquad \text{(VIII)}$$

Then, with the notation of § 15,

$$\frac{[F_s]_2}{[F_t]_2} = \frac{1}{1-\rho}.$$

Equation (VI) of § 13 gives

$$\frac{B_s}{B_t} = \frac{\lambda_{ss}^2}{(1-\rho^2)\lambda_{tt}^2} = \frac{1+\rho}{1-\rho} = \frac{[F_s]\,[F_s]_2}{[F_t]\,[F_t]_2}.$$

In the expression of § 6 for the quadratic invariant we may then put

$$B_r = [F_r]\,[F_r]_2;$$

and $\lambda_{rr} = [F_r]$.

It will be noticed that these values agree with the results already found for $T_{n-k,\,1^k}$.

Consider the coefficient of x_f^2 in H; the only term in which it appears is X_f^2; thus its coefficient is

$$[F_f]/[F_f]_2.$$

Let $\beta_1 = h$ be the number of letters in the first column of the tableaux, β_2 the number in the second, and so on; then the tableau F_f is

$$\begin{array}{ccccccc}
a_1 & a_{\beta_1+1} & \cdots & \cdots & \cdots & \cdots & a_n \\
a_2 & a_{\beta_1+2} & \cdots & \cdots & \cdots & & \\
\cdots & \cdots & \cdots & \cdots & \cdots & & \\
\cdots & \cdots & \cdots & \cdots & \cdots & &
\end{array}$$

$$\begin{array}{l}
a_{\beta_2}\, a_{\beta_1+\beta_2} \\
\vdots \\
a_{\beta_1}
\end{array}$$

Then
$$\prod_{s=1}^{\alpha_r} \gamma_{rst} = (r-t+1)^{\alpha_r};$$

and
$$\prod_{s=1}^{\alpha_r} (\gamma_{rst}-1) = (r-t)^{\alpha_r}.$$

Hence

$$[F_f]/[F_f]_2 = 2^{\alpha_2} 3^{\alpha_3} \ldots h^{\alpha_h} = \beta_1!\ \beta_2! \ldots \beta_k!.$$

Let us define H exactly as the sum of the squares of all the terms derived from standard or non-standard tableaux which have different rows, a permutation in the rows only being regarded as not producing a difference; for such a permutation does not alter the corresponding function of the variables x. The number of the terms in H is then

$$\binom{n}{\alpha_1\, \alpha_2 \ldots \alpha_h}.$$

Consider the coefficient of x_f^2 in H. It is the number of forms $P_r N_r$ which when expressed in terms of forms with a standard P contain the term P_f. Let $P_r N_r$ be such a form; then, when $\alpha_1 > \alpha_2$, a_n must lie in the

same place in the first row of F_r; it cannot be raised by the standardizing equations. Similarly a_{n-1}, a_{n-2},..., $a_{n-\alpha_1,+\alpha_2+1}$ must all lie in the first row of F_r. In the same way when $\alpha_2 > \alpha_3$ the pair of letters $a_{n-\alpha_1+\alpha_2}$, $a_{n-\alpha_1+\alpha_2-1}$ must occupy places one in the first and other in the second row, for unless one is in the first row our equations will not allow either to be raised to it; and unless the other is in the second row it cannot be raised there. If both lie in the first row of F_r, then in the final result P_f does not appear at all, or appears twice, once with a + and once with a − sign. Proceeding thus we see that F_r, when the letters in each row are arranged according to the fixed sequence, is obtained from F_f by a permutation which only interchanges letters in the same column, and each such permutation will result in a linear function in which P_f appears with the coefficient ± 1; and each gives a different term of the sum H. The number of such is

$$\beta_1!\ \beta_2!\ \dots\ \beta_k! = [F_f]/[F_f]_2.$$

THEOREM VI. *The quadratic invariant obtained from the sum of the squares obtained from all tableaux with different rows is equal to*

$$\sum_{r=1}^{f} \frac{1}{[F_r][F_r]_2} X_r^2,$$

where

$$X_r = [F_r]x_r + \lambda_{rr-1}x_{r-1} + \dots + \lambda_{r1}x_1.$$

17. THEOREM VII. *The exact quadratic invariant for $T_{n-k,k}$ is*

$$\sum_{r=1}^{f} \frac{1}{[F_\rho][F_\rho]_2} X_\rho^2,$$

where

$$X_\rho = [F_\rho]x_\rho + \lambda_{\rho,\rho-1}x_{\rho-1} + \dots.$$

When the suffixes of the letters of the lower rows of the tableaux F_ρ, F_σ are $\lambda_1, \lambda_2, \dots, \lambda_k$; $\mu_1, \mu_2, \dots, \mu_k$, respectively, the values of the coefficients are

$$[F_\rho] = \lambda_1[\lambda_2 - 2](\lambda_3 - 4)\dots(\lambda_k - 2k + 2),$$

$$[F_\rho]_2 = (\lambda_1 - 1)(\lambda_2 - 3)\dots(\lambda_k - 2k + 1),$$

$$\lambda_{\rho\sigma} = B_1^{\rho} B_2^{\rho} \dots B_k^{\rho};$$

where

$$B_t^{\rho} = 0, \quad \mu_t < \lambda_t,$$

$$B_t^{\rho} = \lambda_{t+u} - 2t + 2 - u, \quad \mu_t = \lambda_{t+u},$$

$$B_t^{\rho} = u, \quad \lambda_{t+u} > \mu_t > \lambda_{t+u-1},$$

λ_{k+1} *being reckoned as greater than* n.

In § 6 we saw that the seminormal transforming matrix M is

$$M = [m_{rs}] = [\lambda_{sr}/\lambda_{ss}],$$

where of course $\lambda_{sr} = 0$ for $r > s$.

In Theorem III it was proved that

$$M = IJ,$$

where J is a matrix which may be subdivided according to the position of a_n; when thus expressed it contains zeros everywhere except on the leading diagonal. In our case, using suffixes as for T, and thus writing $M_{n-k,k}$ for M, we have

$$J = \begin{bmatrix} M_{n-k,k-1} & 0 \\ 0 & M_{n-k-1,k} \end{bmatrix}.$$

In Q.S.A. IV, § 12, it was shown that

$$I = I_{n-k,k} = \begin{bmatrix} E_{n-k,k-1} & \dfrac{1}{n-2k+1} K_{n-k,k} \\ 0 & E_{n-k-1,k} \end{bmatrix},$$

with a necessary alteration in nomenclature. An unfortunate error appeared in the formula quoted, $1/(\alpha-\beta+2)$ being given as the coefficient of $M_{\alpha\beta}$ which should be $1/(\alpha-\beta+1)$. Further it was shown that

$$K_{n-k,k} = \begin{bmatrix} K_{n-k,k-1} & 0 \\ E_{n-k-1,k-1} & K_{n-k-1,k} \end{bmatrix}.$$

We shall use these results to establish our theorem by induction, assuming its truth for all values of n less than the one under consideration.

Let $$K_{\alpha,\beta-\lambda} K_{\alpha-1,\beta-\lambda+1} \dots K_{\alpha-\lambda,\beta} = K_{\alpha,\beta}^{\lambda+1}$$

$$= K_{\alpha,\beta-1}^{\lambda} K_{\alpha-\lambda,\beta}.$$

Then it easily follows by induction that

$$K^{\lambda}_{\alpha,\beta} = K^{\lambda-1}_{\alpha,\beta-1} K_{\alpha-\lambda+1,\beta}$$

$$= \begin{bmatrix} K^{\lambda-1}_{\alpha,\beta-2} & 0 \\ (\lambda-1) K^{\lambda-2}_{\alpha-1,\beta-2} & K^{\lambda-1}_{\alpha-1,\beta-1} \end{bmatrix} \begin{bmatrix} K_{\alpha-\lambda+1,\beta-1} & 0 \\ E_{\alpha-\lambda,\beta-1} & K_{\alpha-\lambda,\beta} \end{bmatrix}$$

$$= \begin{bmatrix} K^{\lambda}_{\alpha,\beta-1} & 0 \\ \lambda K^{\lambda-1}_{\alpha-1,\beta-1} & K^{\lambda}_{\alpha-1,\beta} \end{bmatrix}.$$

Let $$M^{p}_{\alpha,\beta} = \frac{1}{(\alpha-\beta+1)_p} K^{p}_{\alpha,\beta} M_{\alpha-p,\beta},$$

then multiplication of these two quadratic matrices yields

$$M^{p}_{\alpha,\beta} = \begin{bmatrix} \dfrac{\alpha-\beta+2}{\alpha-\beta+2-p} M^{p}_{\alpha,\beta-1} & M^{p+1}_{\alpha,\beta} \\ \dfrac{p}{\alpha-\beta+2-p} M^{p-1}_{\alpha-1,\beta-1} & M^{p}_{\alpha-1,\beta} \end{bmatrix}. \qquad \text{(VIII)}$$

When $p = 0$, $$M^{0}_{\alpha,\beta} = M_{\alpha,\beta}.$$

Consider now the matrix $M_{n-k,k}$.

$$M_{n-k,k} = IJ = \begin{bmatrix} E_{n-k,k-1} & \dfrac{1}{n-2k+1} K_{n-k,k} \\ 0 & E_{n-k-1,k} \end{bmatrix} \begin{bmatrix} M_{n-k,k-1} & 0 \\ 0 & M_{n-k-1,k} \end{bmatrix}$$

$$= \begin{bmatrix} M_{n-k,k-1} & M^{1}_{n-k,k} \\ 0 & M_{n-k-1,k} \end{bmatrix}.$$

The positions of the elements of this matrix are defined by the positions of a_n in the tableaux; in fact, in the first row they may be defined as

$$\begin{pmatrix} a_n, & a_n \end{pmatrix}, \begin{pmatrix} a_n, & \\ & a_n \end{pmatrix} \text{ and in the second row as } \begin{pmatrix} a_n, & \\ & a_n \end{pmatrix}, \begin{pmatrix} a_n, & a_n \end{pmatrix}.$$

The assumption that the theorem is known to be true for values of n less than the given value establishes the theorem for the whole of this matrix except the part $\begin{pmatrix} a_n, & \\ & a_n \end{pmatrix}$.

We consider then this last part of the matrix, and subdivide it according to the position of a_{n-1}. By VIII,

$$M^1_{n-k,\,k} = \begin{bmatrix} \dfrac{n-2k+2}{n-2k+1} M^1_{n-k,\,k-1} & M^2_{n-k,\,k} \\ \dfrac{1}{n-2k+1} M_{n-k-1,\,k-1} & M^1_{n-k-1,\,k} \end{bmatrix}.$$

Here again the whole matrix is known except the element $M^2_{n-k,\,k}$ in the position $\begin{pmatrix} a_{n-1}, \\ a_{n-1}, \end{pmatrix}$, or, with reference to the original matrix, the position $\begin{pmatrix} a_{n-1}a_n, \\ a_{n-1}a_n, \end{pmatrix}$. We keep on repeating this process, and therefore consider in detail the matrix $M^p_{n-k,\,k}$ which may be defined as the element

$$\begin{pmatrix} a_{n-p+1}a_{n-p+2}\dots a_n, \\ a_{n-p+1}a_{n-p+2}\dots a_n, \end{pmatrix}$$

of $M_{n-k,\,k}$.

We consider this matrix subdivided as in VIII according to the position of a_{n-p}, and write it

$$\begin{bmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{bmatrix}.$$

The matrix $M^p_{n-k,\,k-1}$ concerned with $n-1$ letters is the element

$$\begin{pmatrix} a_{n-p}a_{n-p+1}\dots a_{n-1} \\ a_{n-p}a_{n-p+1}\dots a_{n-1}, \end{pmatrix}$$

of the matrix $\mathrm{M}_{n-k,\,k-1}$.

The matrices m_{11} and $M^p_{n-k,\,k-1}$ are defined by the same tableaux so far as the first $n-p-1$ letters are concerned. Now the matrix

$$M = [\lambda_{sr}/\lambda_{ss}],$$

where the lower rows of the tableaux F_s, F_r are

$$\lambda_1, \lambda_2, \dots, \lambda_k;\ \mu_1, \mu_2, \dots, \mu_k,$$

respectively, the former corresponding to the column tableaux of M. Then in these two matrices $\lambda_1, \lambda_2, \dots, \lambda_{k-1};\ \mu_1, \mu_2, \dots, \mu_{k-p-1}$ are the same.

Let $$\lambda_{ss} = [F_s] = c_1 c_2 \dots c_k,$$

and $$\lambda_{sr} = b_1 b_2 \dots b_k,$$

the values of the factors b, c being given in the enunciation for values of n lower than the given one. The value of λ_{ss} is, in fact, that given for all values of n by Theorem VI. Then the values of $c_1, c_2, \dots, c_{k-1}$, and $b_1, b_2, \dots, b_{k-p-1}$, are the same according to the enunciation in corresponding elements of m_{11} and $M_{n-k,\,k-1}$. In the former case we should have

$$c_k = n-2k+2-p,$$

$$b_{k-p} = n-2k+2, \quad b_{k-p+1} = p, \quad b_{k-p+2} = p-1, \quad \dots, \quad b_{k-1} = 1;$$

and in the latter case we have

$$b_{k-p} = p, \quad b_{k-p-1} = p-1, \quad \dots, \quad b_{k-1} = 1;$$

and hence, by definition, the ratio of the corresponding elements according to the enunciation is the same as that given by (VIII), viz.:

$$m_{11} = \frac{n-2k+2}{n-2k+2-p} M^p_{n-k,\,k-1}.$$

Thus the induction holds for this part of the matrix. The matrix $M^{p-1}_{n-k-1,\,k-1}$ concerned with $n-2$ letters is the element

$$\begin{pmatrix} & a_{n-p}\, a_{n-p+1} \dots a_{n-2} \\ a_{n-p}\, a_{n-p+1} \dots a_{n-2}, & \end{pmatrix}$$

of $M_{n-k-1,\,k-1}$; and m_{21} is the element

$$\begin{pmatrix} a_{n-p}, & \\ & a_{n-p} \end{pmatrix}$$

of $M^p_{n-k,\,k}$. The two matrices are defined by the same tableaux so far as the first $n-p-1$ letters are concerned.

Then the values of c and b given in the enunciation which differ are, for m_{21}:

$$c_k = n-2k+2-p; \quad b_{k-p+1} = p, \quad b_{k-p+2} = p-1, \quad \dots, \quad b_k = 1;$$

and, for $M^{p-1}_{n-k-1,\,k-1}$:

$$b_{k-p+1} = p-1, \quad b_{k-p+2} = p-2, \quad \dots, \quad b_{k-1} = 1.$$

Thus the ratio of the coefficients should give

$$m_{21} = \frac{p}{n-2k+2-p} M^{p-1}_{n-k-1,\,k-1},$$

which we have found to be the case.

The matrix $M^{p}_{n-k-1,\,k}$ concerned with $n-1$ letters is the element

$$\begin{pmatrix} & a_{n-p}\, a_{n-p+1} \cdots a_{n-1} \\ a_{n-p}\, a_{n-p+1} \cdots a_{n-1}, & \end{pmatrix}$$

of $M_{n-k-1,\,k}$; and m_{22} is the element

$$\begin{pmatrix} a_{n-p}, & a_{n-p} \end{pmatrix}$$

of $M^{p}_{n-k,\,k}$. It is easy to see that the coefficients are the same according to the enunciation as in formula VIII. Thus we have proved the induction except for the element

$$m_{12} = M^{p+1}_{n-k,\,k}.$$

This matrix may be treated in the same way. When $p = n-2k$, $M_{n-k,\,k}$ is really defined by the tableaux of $T_{k,\,k}$ of the first $2k$ letters. But here λ_k, μ_k are both fixed, for the last letter of the lower row in a standard tableau (the two rows being equal) must be a_{2k}; and hence

$$M^{n-2k}_{n-k,\,k} = M^{n-2k}_{n-k,\,k-1}.$$

Thus the theorem is true for any value of n, provided that it is true for all lower values. It is easy to establish its truth directly for the first few values of n, and hence it is always true.

18. For some purposes it is more convenient to use the matrices appropriate to the variables X_r, instead of the seminormal matrices which have been used, namely the matrices corresponding to the variable ξ, where

$$\xi_r [F_r] = X_r.$$

We shall call these new matrices the exact seminormal matrices; they are obtained by multiplying each column F_r by $[F_r]$ and dividing the F_r row by $[F_r]$. Thus the elements on the leading diagonal are unaltered, but the seminormal element a_{rs} becomes $a_{rs}[F_s]/[F_r]$.

Consider now the quadratic matrices of Theorem IV, corresponding to the transposition $(a_r a_{r+1})$. We have seen that such a quadratic matrix occupies those elements which lie at the intersection of the s-th and t-th rows and columns, where the tableaux F_s and F_t merely differ by the interchange of a_r and a_{r+1}.

Then in § 15 it was shown that

$$[F_s] = (1+\rho)\,[F_t],$$

where ρ is the reciprocal of the axial projection of $a_r a_{r+1}$ in either tableau. Then the quadratic matrices of the Theorem IV become

$$\begin{bmatrix} -\rho & 1-\rho \\ 1+\rho & \rho \end{bmatrix}$$

in the exact seminormal matrix.

Theorem II holds good as it stands for this matrix.

SOME THEOREMS ON SELF-RECIPROCAL FUNCTIONS.

By B. M. MEHROTRA.

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1. In their joint paper entitled "Formulae connecting different classes of self-reciprocal functions"*, Hardy and Titchmarsh have given a number of examples to illustrate their various theorems. Some of them, as given there, do not seem to originate from a single principle. Following their method, it has been shown in this note that all these examples can be covered by one theorem.

I wish to express my thanks to Prof. E. C. Titchmarsh for his having suggested the problem to me and for his constant help and advice in my work.

2. Following Hardy and Titchmarsh†, we shall say that $f(x)$ belongs to $A(\omega, a)$, where $0 < \omega \leqslant \pi$, $a < \frac{1}{2}$, if (i) it is an analytic function of $x = re^{i\theta}$ regular in the angle A defined by $r > 0$, $|\theta| < \omega$, and (ii) it is $O(|x|^{-a-\delta})$ for small x, and $O(|x|^{a-1+\delta})$ for large x, for every positive δ and uniformly in any angle $|\theta| \leqslant \omega - \eta < \omega$.

3. We shall make use of the following result of Hardy and Titchmarsh‡:

A necessary and sufficient condition that a function $f(x)$ of $A(\omega, a)$ should be its own Hankel transform of order ν is that it should be of the form

$$f(x) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} 2^{\frac{1}{2}s}\,\Gamma(\tfrac{1}{4}+\tfrac{1}{2}\nu+\tfrac{1}{2}s)\,\psi(s)\,x^{-s}\,ds, \qquad (1)$$

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* *Proc. London Math. Soc.* (2), 33 (1932), 225–232.

† G. H. Hardy and E. C. Titchmarsh, "Self-reciprocal functions", *Quart. J. of Math.* (Oxford series), 1 (1930), 196–231, § 2.4.

‡ *Ibid.*, Theorem 4.

where $\psi(s)$ is regular and satisfies the condition

$$\psi(s) = \psi(1-s)$$

in the strip $$a < \sigma < 1-a; \tag{2}$$

and $$\psi(s) = O\{e^{(\frac{1}{4}\pi-\omega+\eta)|t|}\} \tag{3}$$

for every positive η and uniformly in any strip interior to (2); and where c is any value of σ in (2).

4. Our main theorem is:

If $f(x)$ is its own J_μ transform and belongs to $A(\omega, a)$, then the function

$$g(x) = \frac{1}{x}\int_0^\infty P\left(\frac{y}{x}\right) f(y)\, dy$$

is its own J_ν transform, provided that

$$P(x) = \frac{1}{2\pi i}\int_{k-i\infty}^{k+i\infty} \Gamma(\tfrac{1}{4}+\tfrac{1}{2}\mu+\tfrac{1}{2}s)\,\Gamma(\tfrac{3}{4}+\tfrac{1}{2}\nu-\tfrac{1}{2}s)\,\chi(s)\,x^{-s}\,ds, \tag{4}$$

where $\chi(s)$ is regular and satisfies the condition

$$\chi(s) = \chi(1-s) \tag{5}$$

in the strip (2); *and*

$$\chi(s) = O\{e^{(\frac{1}{2}\pi-\omega+\eta)|t|}\} \tag{5'}$$

for every positive η and uniformly in any strip interior to (2), *and k is any value of σ in* (2).

By (1) we have

$$g(x) = \frac{1}{2\pi i x}\int_0^\infty P\left(\frac{y}{x}\right) dy \int_{c-i\infty}^{c+i\infty} 2^{\frac{1}{2}s}\,\Gamma(\tfrac{1}{4}+\tfrac{1}{2}\mu+\tfrac{1}{2}s)\,\psi(s)\,y^{-s}\,ds, \tag{6}$$

where $\psi(s) = \psi(1-s)$. Hence

$$g(x) = \frac{1}{2\pi i x}\int_{c-i\infty}^{c+i\infty} 2^{\frac{1}{2}s}\,\Gamma(\tfrac{1}{4}+\tfrac{1}{2}\mu+\tfrac{1}{2}s)\,\psi(s)\,ds \int_0^\infty P\left(\frac{y}{x}\right) y^{-s}\,dy,$$

provided that the inversion is justified. Therefore

$$g(x) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} 2^{\frac{1}{2}s}\,\Gamma(\tfrac{1}{4}+\tfrac{1}{2}\mu+\tfrac{1}{2}s)\,\psi(s)\,x^{-s}\,ds \int_0^\infty u^{-s}\,P(u)\,du.$$

Now, using Mellin's inversion formula, by (4) we have

$$\int_0^\infty x^{s-1} P(x)\, dx = \Gamma(\tfrac{1}{4}+\tfrac{1}{2}\mu+\tfrac{1}{2}s)\,\Gamma(\tfrac{3}{4}+\tfrac{1}{2}\nu-\tfrac{1}{2}s)\,\chi(s),$$

where $\chi(s) = \chi(1-s)$. Changing s into $1-s$, we get

$$\int_0^\infty x^{-s} P(x)\, dx = \Gamma(\tfrac{3}{4}+\tfrac{1}{2}\mu-\tfrac{1}{2}s)\,\Gamma(\tfrac{1}{4}+\tfrac{1}{2}\nu+\tfrac{1}{2}s)\,\chi(s).$$

Hence

$$g(x) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} 2^{\frac{1}{2}s}\,\Gamma(\tfrac{1}{4}+\tfrac{1}{2}\mu+\tfrac{1}{2}s)\,\Gamma(\tfrac{3}{4}+\tfrac{1}{2}\mu-\tfrac{1}{2}s)\,\Gamma(\tfrac{1}{4}+\tfrac{1}{2}\nu+\tfrac{1}{2}s)\,\psi(s)\,\chi(s)\,x^{-s}\,ds$$

$$= \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} 2^{\frac{1}{2}s}\,\Gamma(\tfrac{1}{4}+\tfrac{1}{2}\nu+\tfrac{1}{2}s)\,\psi_1(s)\,x^{-s}\,ds,$$

where $$\psi_1(s) = \Gamma(\tfrac{1}{4}+\tfrac{1}{2}\mu+\tfrac{1}{2}s)\,\Gamma(\tfrac{3}{4}+\tfrac{1}{2}\mu-\tfrac{1}{2}s)\,\chi(s)\,\psi(s).$$

Now, as $|t|\to\infty$, $$|\Gamma(A+\tfrac{1}{2}it)| \sim Ce^{-\frac{1}{4}\pi|t|}\,|\tfrac{1}{2}t|^{A-\frac{1}{2}}. \tag{7}$$

Therefore, by (3), (5′), and (7), we have

$$\psi_1(s) = O\{e^{(\frac{1}{4}\pi-2\omega+2\eta)|t|}\}; \tag{8}$$

and, since $\psi_1(s)$ satisfies the condition

$$\psi_1(s) = \psi_1(1-s),$$

it follows that $g(x)$ is a function of the class $A(2\omega, a)$ and is its own J_ν transform.

It remains to justify the inversion of the integral on the right-hand side of (6). Consider the integral

$$I = \int_0^\infty \left| P\left(\frac{y}{x}\right)\right| dy \int_{-\infty}^\infty 2^{\frac{1}{2}c}\,|\Gamma(\kappa+\tfrac{1}{2}it)|\,|\psi(s)|\,y^{-c}\,dt, \tag{9}$$

where $\kappa = \frac{1}{2}c+\frac{1}{2}\mu+\frac{1}{4}$. Using (7), we find that the right-hand side of (9) is not greater than a constant multiple of

$$x^{-c}\int_0^\infty |P(u)|\,u^{-c}\,du \int_{-\infty}^\infty t^{\kappa-\frac{1}{2}}\,e^{-\frac{1}{4}\pi|t|}\,|\psi(s)|\,dt. \tag{10}$$

Hence the inversion is justified if the two integrals in (10) exist.

The t-integral will exist if $f(x)$ belongs* to $A(\omega, a)$. Also, from (2), it is clear that the u-integral will exist if $P(x)$ belongs to $A(\omega, a)$. Now the modulus of the integrand in (4) is, by (5′) and (7), less than a constant multiple of

$$r^{-\sigma} e^{-|t|(\omega-\eta-|\theta|)} \left|\tfrac{1}{2}t\right|^{\frac{1}{2}\mu+\frac{1}{2}\nu-\frac{1}{2}}. \tag{11}$$

The integral is therefore uniformly convergent throughout any domain of x interior to the angle A; and $P(x)$ is regular in A, and is $O(|x|^{-c})$ in any smaller angle. Since c may be taken as near to a or $1-a$ as we please, $P(x)$ belongs to $A(\omega, a)$.

5. If, in the above theorem, we put $\mu = \nu$, we get what we may call Corollary I. However, we can replace it in this case by the following simpler form:

If $f(x)$ is its own J_ν transform and belongs to $A(\omega, a)$, the function

$$g(x) = \frac{1}{x}\int_0^\infty P\left(\frac{y}{x}\right) f(y)\, dy$$

is its own J_ν transform, provided that $P(x)$ belongs to $A(\omega, a)$ and satisfies the condition

$$P\left(\frac{1}{x}\right) = xP(x). \tag{12}$$

Let
$$\phi(s) = \int_0^\infty x^{s-1} P(x)\, dx, \tag{13}$$

where $P(x)$ satisfies (12). Then

$$\phi(1-s) = \int_0^\infty x^{-s} P(x)\, dx = \int_0^\infty x^{s-2} P\left(\frac{1}{x}\right) dx = \int_0^\infty x^{s-1} P(x)\, dx,$$

by (12). Therefore
$$\phi(1-s) = \phi(s). \tag{14}$$

We may deform the integral (13), by Cauchy's theorem, into an integral along any radius-vector $\theta = \beta$, where $|\beta| < \omega$, and we may suppose that β has the same sign as t. Then

$$|\phi(s)| = \left| e^{i\beta\sigma-\beta t} \int_0^\infty r^{\sigma+it-1} P(re^{i\beta})\, dr \right| \leqslant e^{-\beta t} \int_0^\infty r^{\sigma-1} |P(re^{i\beta})|\, dr$$

$$= O(e^{-|\beta||t|}). \tag{15}$$

* For proof of this see Hardy and Titchmarsh, *ibid.*, Theorem 4.

Proceeding as in the above theorem, we have

$$g(x) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} 2^{\frac{1}{2}s}\,\Gamma(\tfrac{1}{4}+\tfrac{1}{2}\nu+\tfrac{1}{2}s)\,\psi(s)\,x^{-s}\,ds \int_0^\infty u^{-s}\,P(u)\,du$$

$$= \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} 2^{\frac{1}{2}s}\,\Gamma(\tfrac{1}{4}+\tfrac{1}{2}\nu+\tfrac{1}{2}s)\,\psi(s)\,\phi(1-s)\,x^{-s}\,ds.$$

Now the function $\psi_2(s) = \psi(s)\,\phi(1-s)$ satisfies the condition

$$\psi_2(s) = \psi_2(1-s);$$

and, by (3) and (15),

$$\psi_2(s) = O\{e^{(\frac{1}{4}\pi-\omega+\eta-|\beta|)|t|}\},$$

and, since $|\beta|$ may be taken as near ω as we please, we have

$$\psi_2(s) = O\{e^{(\frac{1}{4}\pi-2\omega+\eta)|t|}\}.$$

Therefore, from (3), it follows that $g(x)$ belongs to $A(2\omega, a)$ and is its own J_ν transform.

Examples of COROLLARY I.

(i) Let $$P(x) = \frac{1}{(1+x^\alpha)^{1/\alpha}},$$

where α is any positive number. Then

$$g(x) = \int_0^\infty \frac{f(y)\,dy}{(x^\alpha+y^\alpha)^{1/\alpha}}.$$

The cases when $\alpha = 1$ and $\alpha = 2$ were given in the paper of Hardy and Titchmarsh quoted at the beginning of this note.

(ii) Let $$P(x) = x^{\frac{1}{2}-\nu}(1+x^2)^{\nu-1},$$

where $0 \leqslant \nu-\frac{1}{2} < c < \frac{3}{2}-\nu$. Then

$$g(x) = \int_0^\infty (xy)^{\frac{1}{2}-\nu}(x^2+y^2)^{\nu-1} f(y)\,dy.$$

In particular, putting $f(y) = y^{\frac{1}{2}+\nu} e^{-\frac{1}{2}y^2}$, we get*

$$g(x) = x^{\frac{1}{2}-\nu} e^{\frac{1}{2}x^2} \int_x^\infty t^{2\nu-1} e^{-\frac{1}{2}t^2}\, dt.$$

(iii) Let $$P(x) = x^{-\frac{1}{2}} F(x),$$

where $F(x) = F(1/x)$, and $F(x) = O(x^a)$ for small x, where $a > \frac{1}{2} - c$. Then

$$g(x) = \int_0^\infty \frac{1}{\sqrt{(xy)}} F\left(\frac{y}{x}\right) f(y)\, dy.$$

In particular, putting $f(y) = y^{\frac{1}{2}+\nu} e^{-\frac{1}{2}y^2}$, we get

$$g(x) = x^{\frac{1}{2}+\nu} \int_0^\infty \omega^\nu F(\omega)\, e^{-\frac{1}{2}x^2\omega^2}\, d\omega,$$

where $F(\omega) = F(1/\omega)$†.

(iv) A still more general example is obtained by putting

$$P(x) = \frac{x^{\frac{1}{2}(al-1)}}{(1+x^a)^l},$$

where $0 \leqslant \frac{1}{2}(1-al) < c < \frac{1}{2}(1+al)$.

If in this we put $l = 1/a$, we obtain example (i) again. If we put $a = 2$, $l = 1-\nu$, we get example (ii).

6. If, in the above theorem, we put $\mu = -\frac{1}{2}$, $\nu = \frac{1}{2}$, we get

COROLLARY II. *If $f(x)$ is its own cosine-transform, and belongs to $A(\omega, a)$, the function*

$$g(x) = \frac{1}{x} \int_0^\infty P\left(\frac{y}{x}\right) f(y)\, dy$$

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* This function was proved to be its own J_ν transform by W. N. Bailey, "On a function which is its own reciprocal in the Fourier-Bessel integral transform", *Journal London Math. Soc.*, 5 (1930), 92–5.

† This function was proved to be its own J_ν transform by W. N. Bailey in "Some classes of functions which are their own reciprocals in the Fourier-Bessel integral transform" *Journal London Math. Soc.*, 5 (1930), 258–65.

is its own sine-transform, provided that

$$P(x) = \frac{1}{2i}\int_{k-i\infty}^{k+i\infty} \frac{1}{\sin \frac{1}{2}s\pi}\,\chi(s)\,x^{-s}\,ds,$$

where $\chi(s)$ *satisfies* (5) *and* (5′).

Example (i). Let $\chi(s) = 1$, and $0 < k < 1$. Then

$$P(x) = \frac{1}{1+x^2}, \quad g(x) = x\int_0^\infty \frac{f(y)\,dy}{x^2+y^2}.$$

(ii) Let $$\chi(s) = \frac{\sqrt{\pi}}{\Gamma(\frac{1}{2}+\frac{1}{2}s)\,\Gamma(1-\frac{1}{2}s)},$$

and $0 < k < 1$. Then

$$P(x) = \frac{1}{\sqrt{(1-x^2)}} \quad (x < 1),$$

$$= 0 \qquad\qquad (x > 1),$$

and $$g(x) = \int_0^x \frac{f(y)\,dy}{\sqrt{(x^2-y^2)}}.$$

7. If, in the above theorem, we put $\mu = \frac{1}{2}$, $\nu = -\frac{1}{2}$, we get

COROLLARY III. *If* $f(x)$ *is its own sine-transform and belongs to* $A(\omega, a)$, *the function*

$$g(x) = \frac{1}{x}\int_0^\infty P\left(\frac{y}{x}\right) f(y)\,dy$$

is its own cosine-transform, provided that

$$P(x) = \frac{1}{2i}\int_{k-i\infty}^{k+i\infty} \frac{1}{\cos \frac{1}{2}s\pi}\,\chi(s)\,x^{-s}\,ds,$$

where $\chi(s)$ *satisfies* (5) *and* (5′).

Example (i). Let $\chi(s) = \frac{1}{2}$, and $0 < k < 1$. Then

$$P(x) = \frac{x}{1+x^2}, \quad g(x) = \int_0^\infty \frac{yf(y)}{x^2+y^2}\,dy.$$

(ii) Let $$\chi(s) = \frac{\sqrt{\pi}}{\Gamma(\frac{1}{2}+\frac{1}{2}s)\,\Gamma(1-\frac{1}{2}s)},$$

and $0 < k < 1$. Then

$$P(x) = \frac{1}{\sqrt{(x^2-1)}} \quad (x > 1),$$

$$= 0 \qquad\qquad (x < 1),$$

$$g(x) = \int_x^\infty \frac{f(y)\,dy}{\sqrt{(y^2-x^2)}}.$$

8. In the paper cited above, Hardy and Titchmarsh proved that, if $f(x)$ is its own cosine-transform, another function $g(x)$ depending on $f(x)$ can be found which is its own sine-transform. Following their method, I proceed to show that a similar theorem holds when cosine and sine-transforms are replaced by J_μ and J_ν transforms respectively.

If $f(x)$ is its own J_μ transform and belongs to $A(\omega, a)$, the function

$$g(x) = \int_0^\infty P(xy) f(y)\,dy$$

is its own J_ν transform, provided that

$$P(x) = \frac{1}{2\pi i}\int_{k-i\infty}^{k+i\infty} 2^s\,\Gamma(\tfrac{1}{4}+\tfrac{1}{2}\mu+\tfrac{1}{2}s)\,\Gamma(\tfrac{1}{4}+\tfrac{1}{2}\nu+\tfrac{1}{2}s)\,\chi(s)\,x^{-s}\,ds, \qquad (16)$$

where $\chi(s)$ satisfies (5) *and* (5′).

By (1) we have

$$g(x) = \frac{1}{2\pi i}\int_0^\infty P(xy)\,dy \int_{c-i\infty}^{c+i\infty} 2^{\frac{1}{2}s}\,\Gamma(\tfrac{1}{4}+\tfrac{1}{2}\mu+\tfrac{1}{2}s)\,y^{-s}\,\psi(s)\,ds$$

$$= \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} 2^{\frac{1}{2}s}\,\Gamma(\tfrac{1}{4}+\tfrac{1}{2}\mu+\tfrac{1}{2}s)\,\psi(s)\,ds \int_0^\infty P(xy)\,y^{-s}\,dy.$$

As in the previous theorem, it can be easily shown that the inversion of the integral is legitimate. Hence

$$g(x) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} 2^{\frac{1}{2}s}\,\Gamma(\tfrac{1}{4}+\tfrac{1}{2}\mu+\tfrac{1}{2}s)\,\psi(s)\,x^{s-1}\,ds \int_0^\infty P(u)\,u^{-s}\,du.$$

Now, using Mellin's inversion formula, by (16), we have

$$\int_0^\infty P(x)\,x^{s-1}\,dx = 2^s\,\Gamma(\tfrac{1}{4}+\tfrac{1}{2}\mu+\tfrac{1}{2}s)\,\Gamma(\tfrac{1}{4}+\tfrac{1}{2}\nu+\tfrac{1}{2}s)\,\chi(s),$$

where $\chi(s)=\chi(1-s)$. Changing s into $1-s$, we have

$$\int_0^\infty P(x)\,x^{-s}\,dx = 2^{1-s}\,\Gamma(\tfrac{3}{4}+\tfrac{1}{2}\mu-\tfrac{1}{2}s)\,\Gamma(\tfrac{3}{4}+\tfrac{1}{2}\nu-\tfrac{1}{2}s)\,\chi(s).$$

Hence

$$\begin{aligned} g(x) &= \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} 2^{1-\frac{1}{2}s}\,\psi(s)\,\Gamma(\tfrac{1}{4}+\tfrac{1}{2}\mu+\tfrac{1}{2}s)\,\Gamma(\tfrac{3}{4}+\tfrac{1}{2}\mu-\tfrac{1}{2}s) \\ &\qquad\qquad \times\Gamma(\tfrac{3}{4}+\tfrac{1}{2}\nu-\tfrac{1}{2}s)\,x^{s-1}\,\chi(s)\,ds \\ &= \frac{1}{2\pi i}\int_{1-c-i\infty}^{1-c+i\infty} 2^{\frac{1}{2}+\frac{1}{2}s}\,\Gamma(\tfrac{3}{4}+\tfrac{1}{2}\mu-\tfrac{1}{2}s)\,\Gamma(\tfrac{1}{4}+\tfrac{1}{2}\mu+\tfrac{1}{2}s) \\ &\qquad\qquad \times\Gamma(\tfrac{1}{4}+\tfrac{1}{2}\nu+\tfrac{1}{2}s)\,x^{-s}\,\chi(s)\,\psi(s)\,ds \\ &= \frac{1}{2\pi i}\int_{1-c-i\infty}^{1-c+i\infty} 2^{\frac{1}{2}s}\,\Gamma(\tfrac{1}{4}+\tfrac{1}{2}\nu+\tfrac{1}{2}s)\,\psi_1(s)\,x^{-s}\,ds, \end{aligned}$$

where $$\psi_1(s)=\sqrt{2}\,\Gamma(\tfrac{1}{4}+\tfrac{1}{2}\mu+\tfrac{1}{2}s)\,\Gamma(\tfrac{3}{4}+\tfrac{1}{2}\mu-\tfrac{1}{2}s)\,\psi(s)\,\chi(s).$$

As before, it can be shown that $\psi_1(s)$ satisfies (8′) and, since $\psi_1(s)=\psi_1(1-s)$, it follows from (3) that $g(x)$ belongs to $A(2\omega,\ a)$ and is its own J_ν transform.

The symmetry of the result (16) shows that, if $f(x)$ is its own J_ν transform, then $g(x)$ is its own J_μ transform.

Example (i). Let $\chi(s)=2^{\frac{1}{2}\mu+\frac{1}{2}\nu-\frac{3}{2}}$, where $\Re(s+\frac{1}{2})>\Re(-\mu)$. Then, by a formula given by Watson*,

$$P(x)=x^{\frac{1}{2}\mu+\frac{1}{2}\nu+\frac{1}{2}}K_{\frac{1}{2}\nu-\frac{1}{2}\mu}(x);$$

$$g(x)=x^{\frac{1}{2}\mu+\frac{1}{2}\nu+\frac{1}{2}}\int_0^\infty y^{\frac{1}{2}\mu+\frac{1}{2}\nu+\frac{1}{2}}K_{\frac{1}{2}\nu-\frac{1}{2}\mu}(xy)f(y)\,dy.$$

(ii) Let $$\chi(s)=\frac{2^{\frac{1}{2}\nu-\frac{1}{2}\mu-\frac{1}{2}}}{\Gamma(\frac{1}{4}+\frac{1}{2}\mu+\frac{1}{2}s)\,\Gamma(\frac{3}{4}+\frac{1}{2}\mu-\frac{1}{2}s)},$$

where $0<\Re(\frac{1}{2}+\nu+s)<\Re(\frac{3}{2}+\frac{1}{2}\nu+\frac{1}{2}\mu)$. Then, using a formula given by Watson†, we have

$$P(x)=x^{\frac{1}{2}\nu-\frac{1}{2}\mu+\frac{1}{2}}J_{\frac{1}{2}\mu+\frac{1}{2}\nu}(x);$$

$$g(x)=\int_0^\infty (xy)^{\frac{1}{2}\nu-\frac{1}{2}\mu+\frac{1}{2}}J_{\frac{1}{2}\mu+\frac{1}{2}\nu}(xy)f(y)\,dy.$$

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* Watson, *Theory of Bessel functions*, § 13.21 (8).

† *Ibid.*, § 13.24 (1).

(iii) Let $$\chi(s) = \frac{2^{\frac{1}{2}\mu-\frac{1}{2}\nu-\frac{1}{2}}}{\Gamma(\frac{1}{4}+\frac{1}{2}\nu+\frac{1}{2}s)\,\Gamma(\frac{3}{4}+\frac{1}{2}\nu-\frac{1}{2}s)},$$

where $0 < \Re(\frac{1}{2}+\mu+s) < \Re(\frac{3}{2}+\frac{1}{2}\mu+\frac{1}{2}\nu)$. Then, using the same formula*, we have

$$P(x) = x^{\frac{3}{2}\mu-\frac{1}{2}\nu+\frac{1}{2}}\, J_{\frac{1}{2}\mu+\frac{1}{2}\nu}(x);$$

$$g(x) = \int_0^\infty (xy)^{\frac{1}{2}\mu-\frac{1}{2}\nu+\frac{1}{2}}\, J_{\frac{1}{2}\mu+\frac{1}{2}\nu}(xy) f(y)\, dy.$$

[*Added July*, 1932.—Since this paper was written, W. N. Bailey† has published a paper in which some of the above results are obtained by a completely different method. My Corollary I is equivalent to his Theorem I$_A$ (except that he does not state conditions of validity), and he gives Example iv to Corollary I, and my last two examples. His other examples are readily obtained from my main theorem.]

* See foot-note † on preceding page.

† *Journal London Math. Soc.*, 6 (1931), 242–247.

A REMARKABLE SERIES OF ORTHOGONAL FUNCTIONS (I)

By R. E. A. C. PALEY.

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1. We define a sequence of functions $\phi_0(t)$, $\phi_1(t)$, ..., $\phi_n(t)$, ... by the following conditions:

$$\phi_0(t) = 1 \quad (0 \leqslant t < \tfrac{1}{2}),$$

$$\phi_0(t) = -1 \quad (\tfrac{1}{2} \leqslant t < 1),$$

$$\phi_0(t+1) = \phi_0(t),$$

$$\phi_n(t) = \phi_0(2^n t) \quad (n = 1, 2, \ldots).$$

We call the functions $\phi_n(t)$ Rademacher's† functions.

By means of Rademacher's functions, we define a new system as follows. Let $n = 2^{n_1} + 2^{n_2} + \ldots + 2^{n_\nu}$; then we write $\psi_0(t) = 1$ and

$$\psi_n(t) = \phi_{n_1}(t)\,\phi_{n_2}(t) \ldots \phi_{n_\nu}(t) \quad (n > 0). \tag{1.1}$$

This system has already been obtained in a different way by Walsh‡. Walsh has proved that the equation (1.1) defines a normalized set of orthogonal functions for the interval (0, 1). Every function $f(t)$ absolutely integrable in the interval (0, 1) can be expanded (quite formally) by means of the functions $\psi_n(t)$ in the form of a series

$$f(t) \sim \sum_{n=0}^{\infty} c_n \psi_n(t), \tag{1.2}$$

where the numbers c_n are defined by means of the equation

$$c_n = \int_0^1 f(t)\,\psi_n(t)\,dt. \tag{1.21}$$

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† See Rademacher, **9**; Khintchine, **6**; Paley and Zygmund, **8**; Kaczmarz and Steinhaus, **5**.

‡ Walsh, **12**.

Walsh established a connection between the series (1.2) and the corresponding expansion in terms of another series of orthogonal functions. These are the Haar† system of orthogonal functions $\{\chi\}$, which are defined in the following way. We write

$$\chi_0(t) = 1,$$

$$\chi_1(t) = 1 \quad (0 \leqslant t < \tfrac{1}{2}),$$

$$\chi_1(t) = -1 \quad (\tfrac{1}{2} \leqslant t < 1),$$

$$\chi_2^{(1)}(t) = \sqrt{2}, \quad \chi_2^{(2)}(t) = 0 \quad (0 \leqslant t < \tfrac{1}{4}),$$

$$\chi_2^{(1)}(t) = -\sqrt{2}, \quad \chi_2^{(2)}(t) = 0 \quad (\tfrac{1}{4} \leqslant t < \tfrac{1}{2}),$$

$$\chi_2^{(1)}(t) = 0, \quad \chi_2^{(2)}(t) = \sqrt{2} \quad (\tfrac{1}{2} \leqslant t < \tfrac{3}{4}),$$

$$\chi_2^{(1)}(t) = 0, \quad \chi_2^{(2)}(t) = -\sqrt{2} \quad (\tfrac{3}{4} \leqslant t < 1).$$

We divide the interval into 2^n equal sub-intervals, and denote by $i_n^{(m)}$ the interval $(m-1)/2^n \leqslant t < m/2^n$. Then we define the n-th group of functions $\{\chi\}$ in the following way:

$$\chi_n^{(m)}(t) = \sqrt{2^{n-1}} \text{ in the interval } i_n^{(2m-1)},$$

$$= -\sqrt{2^{n-1}} \text{ in the interval } i_n^{(2m)},$$

$$= 0 \text{ elsewhere} \quad (m = 1, 2, \ldots, 2^{n-1}).$$

Walsh proves that, if the function $f(t)$ is expanded by means of the functions $\{\chi\}$ in a series

$$f(t) \sim c_0 + \sum_{n=1}^{\infty} \sum_{m=1}^{2^{n-1}} c_n^{(m)} \chi_n^{(m)}(t), \tag{1.3}$$

then the 2^n-th partial sums of the two series (1.2), (1.3) are equal. He is able to deduce a number of properties of the series (1.2) from the corresponding properties of the Haar orthogonal system. In a more recent paper‡ Kaczmarz uses the same method to obtain further properties of the system $\{\psi\}$. Most of the theorems which we obtain in this paper follow from a recent theorem of Hardy and Littlewood§ on averages.

We begin by giving a fresh proof that the system $\{\psi\}$ forms a normalized orthogonal system for the interval (0, 1); in other words that

$$\int_0^1 \psi_m(t)\,\psi_n(t) = 0 \quad (m \neq n), \tag{1.31}$$

$$= 1 \quad (m = n).$$

† Haar, 1.

‡ Kaczmarz, 4.

§ Hardy and Littlewood, 2.

The second result is immediate. For the first we have only to observe† that

$$\int_0^1 \phi_{n_1}^{\alpha_1} \phi_{n_2}^{\alpha_2} \dots \phi_{n_\nu}^{\alpha_\nu} dt \tag{1.4}$$

vanishes, unless the numbers $\alpha_1, \alpha_2, \dots, \alpha_\nu$ are all even, in which case the integral is equal to 1.

The system is also complete‡. In fact, the functions

$$\psi_m(t) \quad (0 \leqslant m \leqslant 2^n - 1)$$

are all constant in the intervals

$$i_n^{(\nu)} \equiv (\nu-1)\, 2^{-n} \leqslant t < \nu 2^{-n} \quad (\nu = 1, 2, \dots, 2^n).$$

Suppose that $f(t) \in L$ is a function for which

$$\int_0^1 f(t)\, \psi_m(t) = 0 \quad (0 \leqslant m \leqslant 2^n - 1). \tag{1.5}$$

We write

$$I_n^{(\nu)} = \int_{(\nu-1)\,2^{-n}}^{\nu 2^{-n}} f(t)\, dt \quad (\nu = 1, 2, \dots, 2^n).$$

Then (1.5) gives

$$\int_0^1 f(t)\, \psi_n(t) = \sum_{\nu=1}^{2^n} I_n^{(\nu)} \psi_m(i_n^{(\nu)}) = 0. \tag{1.6}$$

The equation (1.6) is satisfied for $0 \leqslant m \leqslant 2^n - 1$. Now, in virtue of (1.31), the determinant

$$|\psi_m(i_n^{(\nu)})| \quad (0 \leqslant m \leqslant 2^n - 1, \ 1 \leqslant \nu \leqslant 2^n),$$

does not vanish. Thus the numbers $\psi_m(i_n^{(\nu)})$ are linearly independent, and it follows from (1.6) that

$$I_n^{(\nu)} = \int_{(\nu-1)\,2^{-n}}^{\nu 2^{-n}} f(t)\, dt = 0 \quad (\nu = 1, 2, \dots, 2^n). \tag{1.7}$$

Now suppose that (1.5) is satisfied for all m, and let $F(t)$ denote the continuous function

$$\int_0^t f(\theta)\, d\theta.$$

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† See Paley and Zygmund, **8**, 340–341.

‡ See also Kaczmarz, **4**, 190-191.

Then we have, by (1.7),

$$F\{(\nu-1)\,2^{-n}\} = F(\nu 2^{-n}) \quad (\nu = 1, 2, \ldots, 2^n, \ n = 1, 2, \ldots),$$

and so $F(t)$ is a constant. It follows that $f(t)$ is equivalent to zero. Thus the functions $\psi_n(t)$ form a normal complete orthogonal system.

2. Walsh† observes that the behaviour of the functions (1.1) is in many respects similar to that of the trigonometrical functions. In particular, he discusses this similarity in the case of convergence at a point, and of uniqueness theorems. It will appear in this paper that in other directions this similarity is even more striking.

We begin with some more or less elementary theorems. These are the consequences of Hardy and Littlewood's maximal theorem. We first prove a generalization of Kaczmarz's result, that if $f(t) \in L$ and $s_n(t)$ denotes the n-th partial sum‡ of its expansion in terms of the system $\{\psi\}$, then $s_{2^n}(t) \to f(t)$ for almost all t. Kaczmarz's proof deduces this result from the corresponding theorem for Haar's functions§. If $f(t) \in L^k$, where $k > 1$, we are able to prove a rather stronger result for both systems of functions by an entirely different method.

We go on to consider some results of a more difficult nature. Let $f_n(t)$ denote the difference $s_{2^{n+1}} - s_{2^n}$. We show that

$$(2.1) \qquad B_k \int_0^1 \{\Sigma f_n^2(t)\}^{\frac{1}{2}k}\,dt \leqslant \int_0^1 |f(t)|^k\,dt \leqslant B_k \int_0^1 \{\Sigma f_n^2(t)\}^{\frac{1}{2}k}\,dt \quad (1 < k < \infty),$$

B_k denoting a constant which depends only on k (it may denote different constants in different contexts). The corresponding result for Fourier series is true‖, but the proof for Walsh-Kaczmarz functions is considerably easier. From the result (2.1) we now deduce the $\{\psi\}$ analogue of M. Riesz's well known result¶

$$\int_0^1 |s_n(t)|^k\,dt \leqslant B_k \int_0^1 |f(t)|^k\,dt \quad (1 < k < \infty).$$

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† Walsh, **12**.

‡ By the n-th partial sum we mean

$$\sum_{m=0}^{n-1} \psi_m(t) \int_0^1 f(\theta)\,\psi_m(\theta)\,d\theta.$$

§ Haar, **1**.

‖ Littlewood and Paley, **7**.

¶ M. Riesz, **10**.

We next consider a generalization of the result (2.1). Suppose that $\{\lambda_n\}$ denotes an increasing lacunary sequence, that is to say a sequence for which $\lambda_{n+1}/\lambda_n \geqslant q > 1$. We denote by $\delta_n(t)$ the difference

$$s_{\lambda_n}(t) - s_{\lambda_{n-1}}(t).$$

Then

$$(2.2) \qquad B_{k,\delta}\int_0^1 \{\Sigma\,\delta_n^2\}^{\frac{1}{2}k}\,dt \leqslant \int_0^1 |f(t)|^k\,dt \leqslant B_{k,\delta}\int_0^1 \{\Sigma\,\delta_n^2\}^{\frac{1}{2}k}\,dt \quad (1 < k < \infty).$$

The constants $B_{k,\delta}$ depend only on k and δ. From (2.2) we may obtain a generalization of the theorem

$$s_{2^n} \to f \quad \text{p.p. in } (0, 1).$$

In fact, if $\{\lambda_n\}$ denotes a lacunary sequence of the type already discussed, then

$$(2.3) \qquad s_{\lambda_n}(t) \to f(t)$$

for almost all t in (0, 1). The results (2.2), (2.3) have already† been established for Fourier series.

3. We begin with the following lemma, due to Hardy and Littlewood‡. We denote by B, here and in the sequel, a positive absolute constant, and by B_k, B_δ, ... constants which depend only on k, δ, ... (B, B_k, B_δ, ... may denote different constants in different contexts).

LEMMA 1. *Let $f(t)$ be a function absolutely integrable in* (0, 1). *Let $F(t)$ denote the upper bound*

$$F(t) = \sup \frac{1}{2x}\int_{t-x}^{t+x} |f(\theta)|\,d\theta.$$

Then

$$\int_0^1 F^k(t)\,dt \leqslant B_k \int_0^1 |f(t)|^k\,dt \quad (k > 1);$$

$$\int_0^1 F(t)\,dt \leqslant B\int_0^1 |f(t)| \overset{+}{\log} |f(t)|\,dt + B,$$

the right-hand side in each case being supposed to exist.

The function $F(t)$ is what Hardy and Littlewood call the maximum average of f at the point t. They show that numerous functions common in analysis, such, for example, as the Cesàro mean of positive order of the

† Littlewood and Paley, 7.

‡ Hardy and Littlewood, 2.

Fourier series of f at the point t, are maximized by $F(t)$. For the functions $\{\psi\}$ we have

LEMMA 2. *With the notation of the last lemma*

$$|s_{2^n}(t)| \leqslant 2F(t).$$

We have
$$s_{2^n}(t) = \sum_{m=0}^{2^n-1} \psi_m(t) \int_0^1 f(\theta)\psi_m(\theta)\,d\theta$$
$$= \int_0^1 f(\theta)\left[\sum_{m=0}^{2^n-1} \psi_m(t)\psi_m(\theta)\right] d\theta.$$

The kernel $\sum_{m=0}^{2^n-1} \psi_m(t)\psi_m(\theta)$ is identically equal to

$$\prod_{m=0}^{n-1} \{1+\phi_m(t)\phi_m(\theta)\}, \tag{3.1}$$

and the expression (3.1) vanishes except in an interval of length 2^{-n} enclosing the point t. In this interval it takes the value 2^n. Thus

$$|s_{2^n}(t)| \leqslant 2^n \int_{t-2^{-n}}^{t+2^{-n}} |f(\theta)|\,d\theta \leqslant 2F(t).$$

This proves the lemma.

From the last two lemmas we deduce the following theorem.

THEOREM I. *Let $n(t)$ denote an integer which varies arbitrarily with t. Then, with the previous notation,*

$$\int_0^1 |s_{2^{n(t)}}(t)|^k\,dt \leqslant B_k \int_0^1 |f(t)|^k\,dt \quad (k>1); \tag{3.2}$$

$$\int_0^1 |s_{2^{n(t)}}(t)|\,dt \leqslant B\int_0^1 |f(t)| \overset{+}{\log} |f(t)|\,dt + B;$$

$$|s_{2^{n(t)}}(t)| \leqslant \max |f(t)|,$$

the right-hand side in each case being supposed to exist.

Suppose that $2^{l-1} \leqslant n < 2^l - 1$, and that $n = 2^{l-1} + (m-1)$. We write $\chi_n(t) = \chi_l^{(m)}(t)$, and rewrite the series (1.3) as

$$\sum_{n=0}^{\infty} c_n \chi_n(t).$$

If t is fixed we observe that exactly one of the functions

$$\chi_n(t) \quad (2^{l-1} \leqslant n \leqslant 2^l - 1,\ l \text{ fixed})$$

differs from zero.

From Theorem I we may easily deduce the following result.

THEOREM II. *Let $n(t)$ denote an integer which varies arbitrarily with t. Let $s'_{n(t)}(t)$ denote the $n(t)$-th partial sum of the expansion of $f(t)$ by means of the Haar system of orthogonal functions. Then*

$$\int_0^1 |s'_{n(t)}(t)|^k dt \leqslant B_k \int_0^1 |f(t)|^k dt \quad (k > 1);$$

$$\int_0^1 |s'_{n(t)}(t)| dt \leqslant B \int_0^1 |f(t)| \overset{+}{\log} |f(t)| dt + B;$$

$$|s'_{n(t)}| \leqslant \max f(t),$$

the right-hand side in each case being supposed to exist.

Suppose that $2^{m(t)} < n(t) \leqslant 2^{m(t)+1}$. Let

$$f(t) \sim \sum_{n=0}^{\infty} c_n' \chi_n(t)$$

denote the Haar expansion of $f(t)$. Then, since at most one of the terms

$$c_n' \chi_n(t) \quad (2^{m(t)} \leqslant n < 2^{m(t)+1})$$

can differ from zero, it follows that $s'_{n(t)}(t)$ is equal to either $s'_{2^{m(t)}}(t)$ or $s'_{2^{m(t)+1}}(t)$. But since†

$$s'_{2^{m(t)}}(t) = s_{2^{m(t)}}(t),$$

the required result follows at once from Theorem I.

THEOREM III. *Let n denote a fixed positive integer. Then*

$$\int_0^1 |s_{2^n}(t)| dt \leqslant \int_0^1 |f(t)| dt,$$

the right-hand side being supposed to exist.

Let $g(t)$ denote an arbitrary function of unit modulus, and let $\gamma_{2^n}(t)$ denote the 2^n-th partial sum of the $\{\psi\}$ expansion of $g(t)$. Then, in virtue of Theorem I, we have $|\gamma_{2^n}(t)| \leqslant 1$. Let

$$f(t) \sim \sum_{m=0}^{\infty} a_m \psi_m(t), \quad g(t) \sim \sum_{m=0}^{\infty} b_m \psi_m(t).$$

† Kaczmarz, **4**, 191.

Then, since $s_{2^n}(t)$ is a polynomial, we may use (1.21) to obtain

$$\int_0^1 s_{2^n}(t)\,g(t)\,dt = \sum_{m=0}^{2^n-1} a_m b_m = \int_0^1 \gamma_{2^n}(t) f(t)\,dt \leqslant \int_0^1 |f(t)|\,dt,$$

from which the desired result follows.

The following result has been established by Kaczmarz†.

THEOREM IV. *If $f(t) \in L$, then, for almost all t in* (0, 1),

$$s_{2^n}(t) \to f(t).$$

4. For the theorem which follows we need the following two lemmas. These, with Hardy and Littlewood's Lemma 1, are the key lemmas to the theory. Lemma 3 is the averaging lemma, and expresses the fact that functions of the form (4.1) are of more or less uniform modulus in the interval (0, 1). Thus the functions $\phi_m(t)$ are averaging functions. Lemma 4 we use for interpolation. If we have established a result of the form (3.2) for two values of k, we may often apply the lemma to give a similar result for intermediate values of k. We use the lemma exclusively in this paper with $\gamma = \alpha$, and generally one of the two cases from which we start is that with $\gamma = \alpha = \frac{1}{2}$. Sometimes the other is that with $\gamma = \alpha = 0$, and in this case the expression (4.2) is to be interpreted as meaning

$$\text{Effective maximum } |T(f)| / \text{Effective maximum } |f|.$$

LEMMA 3. *Let $\Phi(t)$ denote the function*

$$\sum_{m=0}^{\infty} c_m \phi_m(t). \tag{4.1}$$

Then, if $0 \leqslant r < \infty$,

$$B_r \left\{ \sum_{m=0}^{\infty} c_m^2 \right\}^{\frac{1}{2}r} \leqslant \int_0^1 |\Phi(t)|^r\,dt \leqslant B_r \left\{ \sum_{m=0}^{\infty} c_m^2 \right\}^{\frac{1}{2}r}.$$

This result is due to Khintchine‡.

LEMMA 4. *Let f be a function defined in a field λ, and let $T(f)$ be a function defined in a field μ, whose values depend on the values of f in λ. Then $T(f)$ is described as a linear operation of the class L_λ^a, if*

(i) *the operation is distributive, that is to say, for arbitary constants p and q we have*

$$T(pf_1 + qf_2) = pT(f_1) + qT(f_2);$$

---

† Kaczmarz, **4**, Theorem II.

‡ Khintchine, **6**; see also Paley and Zygmund, **8**.

(ii) *there exists a constant M such that*

$$|T(f)| \leqslant M\left\{\int |f|^a d\lambda\right\}^{1/a}.$$

Let $M_{\alpha,\gamma}$ denote the upper bound for varying f of the ratio

$$\left\{\int |T(f)|^c d\mu\right\}^{1/c} \Big/ \left\{\int |f|^a d\lambda\right\}^{1/a}, \tag{4.2}$$

where $\alpha = 1/a$, $\gamma = 1/c$. Let (α, γ) describe a segment of a straight line, situated in the triangle

$$0 \leqslant \alpha \leqslant 1, \quad 0 \leqslant \gamma \leqslant 1, \quad \gamma \leqslant \alpha.$$

Then $\log M_{\alpha,\gamma}$ is a convex function of the points of the segment.

The result is due to M. Riesz†.

5. We now proceed to the proof of the following theorem.

THEOREM V. *Let $f(t) \epsilon L$ and have the $\{\psi\}$ expansion*

$$f(t) \sim \sum_{m=0}^{\infty} c_m \psi_m(t).$$

Let $f_n(t)$ denote the partial sum

$$f_n(t) = \sum_{m=2^n}^{2^{n+1}-1} c_m \psi_m(t) \quad (n = 0, 1, 2, \ldots). \tag{5.01}$$

For simplicity let $c_0 = 0$. Then, for $1 < k < \infty$,

$$B_k \int_0^1 \{\Sigma f_n^2(t)\}^{\frac{1}{2}k} dt \leqslant \int_0^1 |f(t)|^k dt \leqslant B_k \int_0^1 \{\Sigma f_n^2(t)\}^{\frac{1}{2}k} dt, \tag{5.1}$$

whenever either member exists.

Let $\epsilon_0, \epsilon_1, \ldots, \epsilon_n, \ldots$ denote a set of arbitrary unit factors. Let

$$f^*(t) = \sum_{n=0}^{\infty} \epsilon_n f_n(t).$$

Then

$$B_k \int_0^1 |f^*(t)|^k dt \leqslant \int_0^1 |f(t)|^k dt \leqslant B_k \int_0^1 |f^*(t)|^k dt \quad (1 < k < \infty). \tag{5.2}$$

The constants B_k are independent of the choice of the numbers ϵ_n.

† M. Riesz, **11**.

It is convenient to divide the proof up into a number of parts, which we state separately as lemmas.

LEMMA 5. *The assertions* (5.1), (5.2) *are equivalent.*

That (5.2) follows from (5.1) is immediate. For the opposite result we observe that, if (5.2) is true, and if $f_\theta^*(t)$ denotes the function

$$f_\theta^*(t) = \sum_{n=0}^{\infty} \phi_n(\theta) f_n(t),$$

$\phi_n(\theta)$ being Rademacher's functions, then, for all θ,

$$B_k \int_0^1 |f_\theta^*(t)|^k \, dt \leqslant \int_0^1 |f(t)|^k \, dt \leqslant B_k \int_0^1 |f_\theta^*(t)|^k \, dt \quad (1 < k < \infty).$$

Integration with respect to θ gives

$$B_k \int_0^1 dt \int_0^1 |f_\theta^*(i)|^k \, d\theta \leqslant \int_0^1 |f(t)|^k \, dt \leqslant B_k \int_0^1 dt \int_0^1 |f_\theta^*(t)|^k \, dt \quad (1 < k < \infty),$$

and an application of Lemma 3 gives the required result.

LEMMA 6. *Let* $m_1 > \max(m_2, m_3, \ldots, m_q)$. *Then*

$$\int_0^1 f_{m_1} f_{m_2} \ldots f_{m_q} \, dt = 0.$$

We observe that, when we expand $f_{m_2}, f_{m_3}, \ldots, f_{m_q}$ by means of (5.01), and then express the separate terms by means of Rademacher's functions, using the equation (1.1), then none of the terms contains $\phi_{m_1}(t)$ as a factor. Thus, when the product $f_{m_2} f_{m_3} \ldots f_{m_q}$ is expressed by means of Rademacher's functions, none of the terms contains $\phi_{m_1}(t)$ as a factor. On the other hand, in the expansion of f_{m_1} by means of Rademacher's functions, all of the terms contain $\phi_{m_1}(t)$ as a factor. The result of the lemma then follows from (1.4)

LEMMA 7. *For* $q \geqslant 2$,

$$\sum_{n=0}^{\infty} \int_0^1 |f_n|^q \, dt \leqslant \int_0^1 |f(t)|^q \, dt.$$

We observe first that

$$f_n(t) = \int_0^1 f(\theta) \phi_n(t) \phi_n(\theta) \prod_{m=0}^{n-1} \{1 + \phi_m(t) \phi_m(\theta)\} \, d\theta,$$

and thus

$$\max_{n,t} |f_n(t)| \leqslant \max |f(t)|.$$

Also, from elementary considerations,

$$\left\{\sum_{n=0}^{\infty}\int_0^1 |f_n|^2\,dt\right\}^{\frac{1}{2}} = \left\{\int_0^1 |f(t)|^2\,dt\right\}^{\frac{1}{2}}.$$

We now apply Lemma 4 for the line $\alpha=\gamma$, $0\leqslant\alpha\leqslant\frac{1}{2}$, λ, μ denoting respectively the fields

$$(0\leqslant t\leqslant 1) \quad \text{and} \quad n=0,\ 1,\ 2,\ \ldots \quad (0\leqslant t\leqslant 1).$$

Then the logarithm of the upper bound $M_{1/q}$ of

$$\left\{\sum_{n=0}^{\infty}\int_0^1 |f_n|^q\,dt\right\}^{1/q} \Big/ \left\{\int_0^1 |f(t)|^q\,dt\right\}^{1/q}$$

is a convex function of $1/q$ for $2\leqslant q\leqslant\infty$. Since $M_0\leqslant 1$, $M_{\frac{1}{2}}\leqslant 1$, it follows that $M_{1/q}\leqslant 1$ $(2\leqslant q\leqslant\infty)$, and the desired result follows.

LEMMA 8. *The conclusions of Theorem V are true when $k=2\nu$ is an even integer.*

Let $F_n(t)=s_{2^n}(t)$ denote the partial sum

$$F_n(t)=\sum_{m=0}^{n-1} f_m(t).$$

We have

$$(5.3)\quad \int_0^1 \{F_{n+1}^k - {F_n}^k\}\,dt = \int_0^1 \left\{k f_n F_n^{k-1} + \frac{k(k-1)}{2}\,{f_n}^2 F_n^{k-2} + \ldots + {f_n}^k\right\} dt$$

$$= \int_0^1 \left\{\frac{k(k-1)}{2}\,{f_n}^2 F_n^{k-2} + \ldots + {f_n}^k\right\} dt,$$

in virtue of Lemma 6. Now, by Hölder's inequality, we have, for $3\leqslant\mu\leqslant k-1$,

$$\left|\int_0^1 {f_n}^\mu F_n^{k-\mu}\,dt\right| \leqslant \left\{\int_0^1 {f_n}^2 F_n^{k-2}\,dt\right\}^{\vartheta} \left\{\int_0^1 {f_n}^k\,dt\right\}^{1-\vartheta},$$

where $\vartheta=\vartheta(\mu)$ lies between 0 and 1. It follows that

$$\left|\int_0^1 {f_n}^\mu F_n^{k-\mu}\,dt\right| \leqslant \int_0^1 {f_n}^2 F_n^{k-2}\,dt + \int_0^1 {f_n}^k\,dt.$$

Substitution in (5.3) gives

$$\left|\int_0^1 \{F_{n+1}^k - {F_n}^k\}\,dt\right| \leqslant B_k\left[\int_0^1 {f_n}^2 F_n^{k-2}\,dt + \int_0^1 {f_n}^k\,dt\right].$$

Summing from $n=0$ to $n=N$ we get

$$\int_0^1 F_{N+1}^k dt \leqslant B_k \int_0^1 \left\{\sum_{n=0}^N f_n^2\right\} \max_{0\leqslant n\leqslant N} \{F_n^{k-2}\}\, dt + B_k \int_0^1 \sum_{n=0}^\infty f_n{}^k\, dt$$

$$\leqslant B_k \left[\int_0^1 \left\{\sum_{n=0}^N f_n^2\right\}^{\frac{1}{2}k} dt\right]^{2/k} \left[\int_0^1 \max_{0\leqslant n\leqslant N} \{F_n{}^k\}\, dt\right]^{(k-2)/k} + B_k \left[\int_0^1 \left\{\sum_{n=0}^N f_n{}^2\right\}^{\frac{1}{2}k}\right],$$

by Hölder's inequality. Now, by Theorem I,

$$\int_0^1 \max_{0\leqslant n\leqslant N} \{F_n{}^k\}\, dt \leqslant B_k \int_0^1 F_{N+1}^k\, dt,$$

and thus it is not difficult to deduce that

$$\int_0^1 F_{N+1}^k\, dt \leqslant B_k \int_0^1 \left\{\sum_{n=0}^N f_n{}^2\right\}^{\frac{1}{2}k} dt$$

$$\leqslant B_k \int_0^1 \left\{\sum_{n=0}^\infty f_n{}^2\right\}^{\frac{1}{2}k} dt.$$

The same argument will show that, if $N_1 < N_2$,

$$\int_0^1 \{F_{N_2}(t) - F_{N_1}(t)\}^k\, dt \leqslant B_k \int_0^1 \left\{\sum_{n=N_1+1}^{N_2} f_n{}^2\right\}^{\frac{1}{2}k} dt,$$

and thus

$$\lim_{N_1\to\infty,\ N_2\to\infty} \int_0^1 \{F_{N_2}(t) - F_{N_1}(t)\}^k\, dt = 0.$$

Thus F_{N+1} tends strongly† with index k to some limit function $F(t)$, and

$$\int_0^1 F^k(t) dt \leqslant B_k \int_0^1 \left\{\sum_{n=0}^\infty f_n{}^2\right\}^{\frac{1}{2}k} dt,$$

the right-hand side being supposed to exist. Also‡

$$\int_0^1 F(t)\,\psi_n(t)\, dt = \lim_{N\to\infty} \int_0^1 F_N(t)\,\psi_n(t)\, dt = c_n,$$

so that $F(t)$ is in fact identical with $f(t)$. Thus

$$\int_0^1 f^k(t)\, dt \leqslant B_k \int_0^1 \left\{\sum_{n=0}^\infty f_n{}^2\right\}^{\frac{1}{2}k} dt.$$

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† See, *e.g.*, Hobson, **3**, 254.

‡ See, *e.g.*, Hobson, **3**, 251, equation (1).

We now go on to prove the opposite inequality. Consider the integral

$$\int_0^1 f_{n_1}^2 f_{n_2}^2 \dots f_{n_{\nu-1}}^2 F_N^2 \, dt,$$

where $\nu = \frac{1}{2}k$, and $N-1 \geqslant n_1 > n_2 > n_3 > \dots > n_{\nu-1}$. It may be written

$$\int_0^1 f_{n_1}^2 f_{n_2}^2 \dots f_{n_{\nu-1}}^2 F_{n_1+1}^2 \, dt + \sum_{n=n_1+1}^{N-1} \int_0^1 f_{n_1}^2 f_{n_2}^2 \dots f_{n_{\nu-1}}^2 f_n^2 \, dt$$

$$+ 2 \sum_{n=n_1+1}^{N-1} \int_0^1 f_{n_1}^2 f_{n_2}^2 \dots f_{n_{\nu-1}}^2 f_n F_{n_1+1} + 2 \sum_{\substack{n \neq m \\ n_1 < n,\, m < N-1}} \sum \int_0^1 f_{n_1}^2 f_{n_2}^2 \dots f_{n_{\nu-1}}^2 f_n f_m \, dt.$$

The last two summations vanish in virtue of Lemma 6. It follows that

$$\sum_{n=n_1+1}^{N-1} \int_0^1 f_{n_1}^2 f_{n_2}^2 \dots f_{n_{\nu-1}}^2 f_n^2 \, dt \leqslant \int_0^1 f_{n_1}^2 f_{n_2}^2 \dots f_{n_{\nu-1}}^2 F_N^2 \, dt.$$

Summing over all the possible combinations of the numbers

$$n_1, n_2, \dots, n_{\nu-1},$$

for which $\max(n_1, n_2, \dots, n_{\nu-1}) < N-1$, we get

$$(5.4) \qquad \sum_{\substack{n_1, n_2, \dots n_{\nu-1}, n_\nu \\ \text{all different} \\ \max(n_1, n_2, \dots, n_\nu) \leqslant N-1}} \left\{ \int_0^1 f_{n_1}^2 f_{n_2}^2 \dots f_{n_\nu}^2 \, dt \right\} \leqslant \int_0^1 F_N^2 \left\{ \sum_{n=0}^{N-1} f_n^2 \right\}^{k-2} dt.$$

Also Lemma 7 gives

$$\sum_{n=0}^{N-1} \left\{ \int_0^1 f_n^k \, dt \right\} \leqslant B_k \int_0^1 F_N^k \, dt.$$

Now, if S denotes the summation on the left-hand side of (5.4), it is not difficult to see that

$$\int_0^1 \left\{ \sum_{n=0}^{N-1} f_n^2 \right\}^{\frac{1}{2}k} dt \leqslant B_k \left[S + \sum_{n=0}^{N-1} \int_0^1 f_n^k \, dt \right]$$

$$(5.5) \qquad \leqslant B_k \left[\int_0^1 F_N^2 \left\{ \sum_{n=0}^{N-1} f_n^2 \right\}^{k-2} dt + \int_0^1 F_N^k \, dt \right].$$

From Hölder's inequality again, the right-hand side of (5.5) does not exceed

$$B_k \left[\left\{ \int_0^1 F_N^k \, dt \right\}^{2/k} \left[\int_0^1 \left\{ \sum_{n=0}^{N-1} f_n^2 \right\}^{\frac{1}{2}k} dt \right]^{(k-2)/k} + \int_0^1 F_N^k \, dt \right],$$

and thus

$$\int_0^1 \left\{\sum_{n-1}^{N-1} f_n^2\right\}^{\frac{1}{2}k} dt \leqslant B_k \int_0^1 F_N^k\, dt \leqslant B_k \int_0^1 f^k(t)\, dt,$$

from which the desired result follows.

We have proved (5.1) in the case when k is an even integer, and (5.2) follows by Lemma 5.

LEMMA 9. *The conclusions of Theorem V are true for* $k \geqslant 2$.

Let $\epsilon_0, \epsilon_1, \ldots, \epsilon_n, \ldots$ denote a fixed sequence of unit factors. Let F_N^* denote the function

$$F_N^* = \sum_{n=0}^{N-1} \epsilon_n f_n(t).$$

Then, if k is an even integer, we have

$$\left\{\int_0^1 |F_N^*(t)|^k\, dt\right\}^{1/k} \leqslant B_k \left\{\int_0^1 |F_N(t)|^k\, dt\right\}^{1/k} \leqslant B_k \left\{\int_0^1 |f(t)|^k\, dt\right\}^{1/k}, \tag{5.6}$$

by Theorem I. The conditions of Lemma 4 are satisfied, and so we may use that lemma to interpolate between $k = 2$ and any arbitrary even integer; hence the inequality (5.6) is true for all $k \geqslant 2$. Thus F_N^* tends strongly†, with index k, to a limit function f^*, whose Walsh-Kaczmarz series is obtained by expanding $\Sigma \epsilon_n f_n(t)$, and

$$\int_0^1 |f^*(t)|^k\, dt \leqslant B_k \int_0^1 |f(t)|^k\, dt.$$

Since f is obtained from f^* in the same way as f_1 is obtained from f, we have also

$$\int_0^1 |f(t)|^k\, dt \leqslant B_k \int_0^1 |f^*(t)|^k\, dt.$$

Thus (5.2) is satisfied for the particular sequence of ϵ considered, and consequently for all sequences ϵ. In virtue of Lemma 5, (5.1) is also true for $k \geqslant 2$.

LEMMA 10. *The conclusions of Theorem V are true for* $1 < k \leqslant 2$.

† See Hobson, **3**, 254, and the argument of Lemma 8.

Let $\epsilon_0, \epsilon_1, \ldots, \epsilon_n, \ldots$ denote a fixed sequence of unit factors. Let $g(t) \epsilon L^k$, where k' is defined by the equation

$$\frac{1}{k} + \frac{1}{k'} = 1.$$

Then, if g^* is formed from g in the same way as f^* is obtained from f, we have

$$\int_0^1 f_N^*(t)\, g(t)\, dt = \int_0^1 g_N^*(t) f(t)\, dt$$

$$\leqslant \left\{\int_0^1 |g_N^*(t)|^{k'} dt\right\}^{1/k'} \left\{\int_0^1 |f(t)|^k dt\right\}^{1/k},$$

by Hölder's inequality, whenever the right-hand side exists. Using (5.6),

$$\int_0^1 f_N^*(t)\, g(t)\, dt \leqslant B_k \left\{\int_0^1 |g(t)|^{k'} dt\right\}^{1/k'} \left\{\int_0^1 |f(t)|^k dt\right\}^{1/k}. \tag{5.7}$$

Since (5.7) is satisfied for all choices of $g(t)$, it follows that

$$\int_0^1 |f_N^*(t)|^k dt \leqslant B_k \int_0^1 |f(t)|^k dt,$$

where B_k is independent of N, and of the sequence $\epsilon_0, \epsilon_1, \ldots$. Thus, arguing as in the last lemma, we may show that $f_N^*(t)$ tends strongly with index k to $f^*(t)$, and that (5.2), and consequently also (5.1), is satisfied for $1 < k \leqslant 2$.

Lemmas 9 and 10 together give the result of Theorem V.

6. From Theorem V we deduce the following result, analogous to M. Riesz's well-known result† for Fourier series.

THEOREM VI. *Let n denote a fixed positive integer. We suppose that $1 < k < \infty$. Then*

$$\int_0^1 |s_n(t)|^k dt \leqslant B_k \int_0^1 |f(t)|^k dt.$$

We have

$$s_n(t) = \int_0^1 f(\theta) \sum_{m=0}^{n-1} \psi_m(t)\, \psi_m(\theta)\, d\theta. \tag{6.1}$$

† M. Riesz, 10.

Suppose that $n = 2^{n_1}+2^{n_2}+\ldots+2^{n_\lambda}$. We write

$$g(\theta) = f(\theta)\,\phi_{n_1}(\theta)\,\phi_{n_2}(\theta)\ldots\phi_{n_\lambda}(\theta) = f(\theta)\,\psi_n(\theta).$$

Suppose that

$$g(\theta) \sim \sum_{m=0}^{\infty} c_m'\,\psi_m(\theta),$$

and write

$$g_n(\theta) = \sum_{m=2^n}^{2^{n+1}-1} c_m'\,\psi_m(\theta) \quad (n = 0,\ 1,\ 2,\ \ldots).$$

From (6.1) we have

$$s_n(t)\,\psi_n(t) = \int_0^1 f(\theta)\,\psi_n(t) \sum_{m=0}^{n-1} \psi_m(t)\,\psi_m(\theta)\,d\theta$$

$$= \int_0^1 g(\theta)\,\psi_n(\theta)\,\psi_n(t) \sum_{m=0}^{n-1} \psi_m(t)\,\psi_m(\theta)\,d\theta.$$

Now the kernel

$$\psi_n(\theta)\,\psi_n(t) \sum_{m=0}^{n-1} \psi_m(t)\,\psi_m(\theta)$$

is identically equal to

$$\sum_{m=2^{n_\lambda}}^{2^{n_\lambda+1}-1} \psi_m(t)\,\psi_m(\theta) + \sum_{m=2^{n_{\lambda-1}}}^{2^{n_{\lambda-1}+1}-1} \psi_m(t)\,\psi_m(\theta) + \ldots + \sum_{m=2^{n_1}}^{2^{n_1+1}-1} \psi_m(t)\,\psi_m(\theta).$$

It follows that

$$s_n(t)\,\psi_n(t) = g_{n_\lambda}(t) + g_{n_{\lambda-1}}(t) + \ldots + g_{n_1}(t).$$

Hence, using Theorem V twice,

$$\int_0^1 |s_n(t)|^k\,dt = \int_0^1 |g_{n_\lambda}(t) + g_{n_{\lambda-1}}(t) + \ldots + g_{n_1}(t)|^k\,dt$$

$$= B_k \int_0^1 \left\{ \sum_{\nu=1}^{\lambda} g_{n_\nu}^2 \right\}^{\frac{1}{2}k} dt \leqslant B_k \int_0^1 \left\{ \sum_{n=0}^{\infty} g_n{}^2 \right\}^{\frac{1}{2}k} dt$$

$$\leqslant B_k \int_0^1 |g(t)|^k\,dt = B_k \int_0^1 |f(t)|^k\,dt,$$

the desired result.

7. We now prove the following lemmas.

LEMMA 11. *Let*

$$f(t) \sim \sum_{n=0}^{\infty} \sum_{m=0}^{n-1} a_{m,n}\,\phi_m(t)\,\phi_n(t).$$

Then $$B_r\{\Sigma a_{m,n}^2\}^{\frac{1}{2}r} \leqslant \int_0^1 |f(t)|^r\,dt \leqslant B_r\{\Sigma a_{m,n}^2\}^{\frac{1}{2}r} \quad (0 \leqslant r < \infty).$$

We first prove the lemma in the case when $r = 2\nu$ is an even integer. We observe that $f_n(t)$, defined in the usual way, is

$$\phi_n(t) \sum_{m=0}^{n-1} a_{m,n}\,\phi_m(t).$$

Now, using Theorem V, we have

$$\int_0^1 |f(t)|^r\,dt \leqslant B_r \int_0^1 \{\Sigma f_n{}^2\}^{\frac{1}{2}r}\,dt = B_r \sum_{n_1=0}^{\infty}\sum_{n_2=0}^{\infty}\cdots\sum_{n_\nu=0}^{\infty}\int_0^1 f_{n_1}^2 f_{n_2}^2 \cdots f_{n_\nu}^2\,dt.$$

Now $$\int_0^1 f_{n_1}^2 f_{n_2}^2 \cdots f_{n_\nu}^2\,dt \leqslant \left\{\int_0^1 f_{n_1}^r\right\}^{1/\nu}\left\{\int_0^1 f_{n_2}^r\right\}^{1/\nu}\cdots\left\{\int_0^1 f_{n_\nu}^r\right\}^{1/\nu}$$

$$\leqslant B_r\left\{\sum_{m=0}^{n_1-1} a_{m,n_1}^2\right\}\left\{\sum_{m=0}^{n_2-1} a_{m,n_2}^2\right\}\cdots\left\{\sum_{m=0}^{n_\nu-1} a_{m,n_\nu}^2\right\}.$$

It follows that

$$\int_0^1 |f(t)|^r\,dt \leqslant B_r\{\Sigma a_{m,n}^2\}^\nu \tag{7.1}$$

whenever r is an even integer. Since

$$\left\{\int_0^1 |f(t)|^r\,dt\right\}^{1/r} \tag{7.2}$$

increases with r, the result (7.1) is true for all values of r.

To prove the opposite inequality for $r \leqslant 2$ we need only combine the result (7.1) with the equation

$$\int_0^1 f^2(t)\,dt = \Sigma a_{m,n}^2,$$

using Hausdorff's theorem on the convexity of means. For $r > 2$, we have again to observe that the mean (7.2) increases with r.

LEMMA 12. *Let*

$$f(t, t') \sim \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} a_{m,n}\,\phi_m(t)\,\phi_n(t').$$

Then $$B_r\{\Sigma a_{m,n}^2\}^{\frac{1}{2}r} \leqslant \int_0^1\int_0^1 |f(t, t')|^r\,dt\,dt' \leqslant B_r\{\Sigma a_{m,n}^2\}^{\frac{1}{2}r} \quad (0 \leqslant r < \infty).$$

The proof is similar to that of the last lemma.

8. THEOREM VII. *Let $f(t)$ denote the function*

$$f(t) \sim \sum_{m=0}^{\infty} c_m \psi_m(t).$$

Let f_{n_1, n_2} $(n_1 > n_2)$ denote the polynomial

$$f_{n_1, n_2}(t) = \sum_{m=2^{n_1}+2^{n_2}}^{2^{n_1}+2^{n_2+1}-1} c_m \psi_m(t).$$

Then, assuming for simplicity that $c_{2^n} = 0$ $(n = 0, 1, 2, \ldots)$, $c_0 = 0$,

$$(8.1) \qquad B_k \int_0^1 \{\Sigma f_{n_1, n}^2\}^{\frac{1}{2}k} dt \leqslant \int_0^1 |f(t)|^k dt \leqslant B_k \int_0^1 \{\Sigma f_{n_1, n_2}^2\}^{\frac{1}{2}k} dt \quad (1 < k < \infty),$$

whenever either member exists.

Let $\{\epsilon_{n_1, n_2}\}$ denote a double sequence of arbitrary unit constants. Let

$$f^*(t) = \sum_{n_1=0}^{\infty} \sum_{n_2=0}^{n_1-1} \epsilon_{n_1, n_2} f_{n_1, n_2}(t).$$

Then, assuming for simplicity that $c_{2^n} = 0$ $(n = 0, 1, 2, \ldots)$, $c_0 = 0$,

$$(8.2) \qquad B_k \int_0^1 |f^*(t)|^k dt \leqslant \int_0^1 |f(t)|^k dt \leqslant B_k \int_0^1 |f^*(t)|^k dt \quad (1 < k < \infty),$$

whenever either member exists.

We write

$$f_n(t) = \sum_{m=2^n}^{2^{n+1}-1} c_m \psi_m(t) = \phi_n(t)\, \gamma_n(t);$$

$$f_{n_1, n_2}(t) = \phi_{n_1}(t)\, \phi_{n_2}(t)\, \gamma_{n_1, n_2}(t).$$

The result (5.1) may be expressed in the form

$$(8.3) \qquad \int_0^1 |f(t)|^k dt \leqslant B_k \int_0^1 dt \int_0^1 \left| \sum_{n=0}^{\infty} \gamma_n(t)\, \phi_n(t_1) \right| dt_1,$$

with the opposite inequality. Applying Theorem V again, we get

$$\int_0^1 \left| \sum_{n=0}^{\infty} \gamma_n(t)\, \phi_n(t_1) \right|^k dt \leqslant B_k \int_0^1 dt \int_0^1 \left| \sum_{n_1=0}^{\infty} \sum_{n_2=0}^{n_1-1} \gamma_{n_1, n_2}(t)\, \phi_{n_1}(t_1)\, \phi_{n_2}(t_2) \right|^k dt_2.$$

Thus (8.3) gives

$$\int_0^1 |f(t)|^k dt \leqslant B_k \int_0^1 dt \int_0^1 \int_0^1 \left| \sum_{n_1=0}^{\infty} \sum_{n_2=0}^{n_1-1} \gamma_{n_1, n_2}(t)\, \phi_{n_1}(t_1)\, \phi_{n_2}(t) \right|^k dt_1 dt_2$$

$$\leqslant B_k \int_0^1 \{\Sigma \gamma_{n_1, n_2}^2\}^{\frac{1}{2}k} dt = B_k \int_0^1 \{\Sigma f_{n_1, n_2}^2\}^{\frac{1}{2}k} dt,$$

by Lemma 12. Similarly we get

$$\int_0^1 |f(t)|^k\,dt \geqslant B_k \int_0^1 \{\Sigma f_{n_1, n_2}^2\}^{\frac{1}{2}k}\,dt.$$

This proves (8.1), and the extension to (8.2) is immediate.

The reader will see how, by further inductive stages, we may obtain further results analogous to those of Lemmas 11 and 12 and Theorem VII.

9. We now prove a generalization of Theorem V in a slightly different direction.

THEOREM VIII. *Let $\lambda_0, \lambda_1, \ldots, \lambda_n, \ldots$ denote an increasing sequence of positive integers for which $\lambda_{n+1}/\lambda_n \geqslant q > 1$. Let $\delta_n(t)$ denote the difference*

$$\delta_n(t) = s_{\lambda_n}(t) - s_{\lambda_n - 1}(t) \quad (\lambda_{-1} = 0).$$

Then

$$(9.1) \qquad B_{k,q} \int_0^1 \{\Sigma\, \delta_n^2\}^{\frac{1}{2}k}\,dt \leqslant \int_0^1 |f(t)|^k\,dt \leqslant B_{k,q} \int_0^1 \{\Sigma\, \delta_n^2\}^{\frac{1}{2}k}\,dt \quad (1 < k < \infty),$$

whenever either member exists.

We first prove the following lemma.

LEMMA 13. *Let $2^n \leqslant \lambda_n < 2^{n+1}$. Let $\rho_n(t)$ denote the partial sum $s_{\lambda_n}(t) - s_{2^n}(t)$. Then*

$$\int_0^1 \{\Sigma\, \rho_n^2\}^{\frac{1}{2}k} \leqslant B_k \int_0^1 |f(t)|^k\,dt \quad (1 < k < \infty).$$

We may assume without loss of generality that $c_0 = 0$, and that λ_n is actually greater than 2^n. Let

$$\lambda_n - 2^n = 2^{\nu_1(n)} + 2^{\nu_2(n)} + \ldots + 2^{\nu_\mu(n)} \quad [\mu = \mu(n)].$$

We define f_n, γ_n as previously. We write

$$\gamma_n^* = \gamma_n \phi_{\nu_1(n)} \phi_{\nu_2(n)} \ldots \phi_{\nu_\mu(n)}; \quad f_n^* = \phi_n \gamma_n^*,$$

and suppose that f_n^* is split up into partial sums

$$(9.2) \qquad f_n^* = \sum_{\nu=0}^{n-1} f_{n,\nu}^* + c_{2^n}^* \phi_n$$

in the usual way. As in Theorem VI, we observe that

$$\rho_n(t)\,\phi_{\nu_1(n)}(t)\,\phi_{\nu_2(n)}(t)\dots\phi_{\nu_\mu(n)}(t) = f^*_{n,\nu_\mu}(t)+f^*_{n,\nu_{\mu-1}}(t)+\dots+f^*_{n,\nu_1(n)}(t).$$

Thus

$$(9.3)\quad \int_0^1 \{\Sigma\rho_n{}^2\}^{\frac{1}{2}k}\,dt = \int_0^1 \{\Sigma[f^*_{n,\nu_\mu}+f^*_{n,\nu_{\mu-1}}+\dots+f^*_{n,\nu_1}]^2\}^{\frac{1}{2}k}\,dt$$

$$\leqslant B_k\int_0^1 \Big|\sum_n \{f^*_{n,\nu_\mu(n)}+f^*_{n,\nu_{\mu-1}(n)}+\dots+f^*_{n,\nu_1(n)}\}\Big|^k\,dt,$$

by Theorem V. Thus, by Theorem VII, the expression (9.3) does not exceed

$$B_k\int_0^1 \Big[\sum_n \{f^{*2}_{n,\nu_\mu(n)}+f^{*2}_{n,\nu_{\mu-1}(n)}+\dots+f^{*2}_{n,\nu_1(n)}\}\Big]^{\frac{1}{2}k}\,dt$$

$$(9.4)\quad \leqslant B_k\int_0^1 \Big\{\sum_{n_1=0}^{\infty}\sum_{n_2=0}^{n_1-1} f^{*2}_{n_1,n_2}\Big\}^{\frac{1}{2}k}\,dt.$$

Using Theorem VII again, we see that the expression (9.4) does not exceed

$$(9.5)\quad B_k\int_0^1 \Big|f^*(t)-\sum_{n=0}^{\infty} c^*_{2^n}\phi_n(t)\Big|^k\,dt,$$

where f^* denotes the function $\Sigma f_n{}^*(t)$. Now, by Theorem V,

$$\int_0^1 |f^*(t)|^k\,dt \leqslant B_k\int_0^1 \{\Sigma f_n^{*2}\}^{\frac{1}{2}k}\,dt$$

$$9.6)\quad = B_k\int_0^1 \{\Sigma f_n{}^2\}^{\frac{1}{2}k}\,dt \leqslant B_k\int_0^1 |f(t)|^k\,dt.$$

Also, if $h(t)=\Sigma d_n\phi_n(t)$ $(\Sigma d_n{}^2=1)$, then†, by Lemma 3,

$$\Sigma c^*_{2^n}d_n = \int_0^1 f^*(t)\,h(t)\,dt$$

$$\leqslant \Big\{\int_0^1 |f^*(t)|^k\,dt\Big\}^{1/k}\Big\{\int_0^1 |h(t)|^{k'}\,dt\Big\}^{1/k'}$$

$$\leqslant B_k\Big\{\int_0^1 |f^*(t)|^k\,dt\Big\}^{1/k} \leqslant B_k\Big\{\int_0^1 |f(t)|^k\,dt\Big\}^{1/k}.$$

† The argument is due to Zygmund. See Zygmund, **13**, Theorem G (106).

It follows that

$$\Sigma c_{2^n}^{*2} \leqslant B_k \left\{\int_0^1 |f(t)|^k dt\right\}^{2/k},$$

and thus

$$\int_0^1 \left| \sum_{n=0}^{\infty} c_{2^n}^* \phi_n(t) \right|^k dt \leqslant B_k \{\Sigma c_{2^n}^{*2}\}^{\frac{1}{2}k} \leqslant B_k \int_0^1 |f(t)|^k dt.$$

Combining this with (9.5), (9.6), we obtain the required result.

We can now prove the first half of (9.1). Suppose that $q^G > 2$. Then the number of partial sums $\delta_n(t)$ which are contained partially or entirely in the summation

$$\sum_{m=2^n}^{2^{n+1}-1} c_n \psi_n(t)$$

cannot exceed $G+1$. We divide up the set $\{\delta_n\}$ into two subsets, and write

$$\delta_n(t) = \delta_n'(t) + \delta_n''(t),$$

where $\delta_n'(t) = \delta_n(t)$, except when the range $(\lambda_{n-1}, \lambda_n)$ contains a power of 2, in which case $\delta_n'(t)$ vanishes. By Lemma 13,

$$\int_0^1 \left| \sum_{n=0}^{\infty} \delta'^2_{(G+1)n+\sigma}(t) \right|^{\frac{1}{2}k} dt \leqslant B_k \int_0^1 |f(t)|^k dt \quad (\sigma = 0, 1, \ldots, G),$$

and thus

$$\int_0^1 \left| \sum_{n=0}^{\infty} \delta_n'^2(t) \right|^{\frac{1}{2}k} dt \leqslant B_{k,\delta} \int_0^1 |f(t)|^k dt. \tag{9.7}$$

Also, if we define $\tau(n)$, $\sigma(n)$ for (not necessarily all) values of n by means of the inequalities

$$2^{n-1} \leqslant \lambda_{\tau(n)-1} < 2^n, \quad 2^{n+\sigma(n)} < \lambda_{\tau(n)} \leqslant 2^{n+\sigma(n)+1},$$

then we have

$$\int_0^1 \left| \sum_{n=0}^{\infty} \delta_n''^2(t) \right|^{\frac{1}{2}k} dt$$

$$\leqslant B_k \int_0^1 \left| \sum_{n=0}^{\infty} (s_{\lambda_{\tau(n)}} - s_{2^{n+\sigma(n)}})^2 + (s_{2^{n+\sigma(n)}} - s_{2^n})^2 + (s_{2^n} - s_{\lambda_{\tau(n)-1}})^2 \right|^{\frac{1}{2}k} dt$$

$$\leqslant B_k \int_0^1 |f(t)|^k dt.$$

Combining the last result with (9.7), we obtain

$$B_{k,\delta}\int_0^1\{\Sigma\,\delta_n{}^2\}^{\frac{1}{2}k}\,dt\leqslant\int_0^1|f(t)|^k\,dt.$$

For the proof of the opposite inequality we write

$$f^{(\sigma)}(t)=\sum_{n=0}^{\infty}\epsilon_{(2G+2)n+\sigma}\delta_{(2G+2)n+\sigma}(t)\quad(\sigma=0,\,1,\,\ldots,\,2G+1),$$

and observe that at most one of the partial sums

$$\delta_{(2G+2)n+\sigma}(t)\quad(\sigma\text{ fixed})$$

has points in common with a given interval $(2^\nu,\,2^{\nu+1})$. Thus we may apply Theorem V, and without much difficulty we obtain

$$\int_0^1|f^{(\sigma)}(t)|^k\,dt\leqslant B_k\int_0^1\left\{\sum_{n=0}^{\infty}\delta^2_{(2G+2)n+\sigma}\right\}^{\frac{1}{2}k}dt\quad(\sigma=0,\,1,\,\ldots,\,2G+1).$$

Combining these results, we obtain

$$\int_0^1|f(t)|^k\,dt\leqslant B_{k,\delta}\int_0^1\left\{\sum_{n=0}^{\infty}\delta_n{}^2\right\}^{\frac{1}{2}k}dt.$$

This completes the proof of the theorem.

10. From Theorem VIII we may obtain the following result which is the generalization of Theorem I.

THEOREM IX. *Let $\lambda_0,\,\lambda_1,\,\ldots,\,\lambda_n,\,\ldots$ denote an increasing sequence of positive integers for which $\lambda_{n+1}/\lambda_n\geqslant q>1$. Let $n(t)$ denote an arbitrary positive integer which may vary with t. Then*

$$\int_0^1|s_{\lambda_{n(t)}}(t)|^k\,dt\leqslant B_{k,q}\int_0^1|f(t)|^k\,dt\quad(1<k<\infty),\tag{10.1}$$

whenever the right-hand member exists.

Let N denote a fixed positive integer. Let $f(t)$ denote a fixed function of class L^k $(1<k<\infty)$. Then the integral

$$\int_0^1\sup_{n\geqslant N}|f(t)-s_{\lambda_n}(t)|^k\,dt\tag{10.2}$$

tends to zero as $N\to\infty$, so that

$$s_{\lambda_n}(t)\to f(t)\quad(f\in L^k)$$

for almost all t.

We prove the assertion (10.1) first for $q = 2$. Then no two members of the sequence $\{\lambda_n\}$ can be in the same interval $2^\nu \leqslant \lambda_n < 2^{\nu+1}$. We suppose that

$$2^{\nu(n)} \leqslant \lambda_n < 2^{\nu(n)+1}.$$

Then

$$\int_0^1 |s_{\lambda_{n(t)}}(t) - s_{2^{\nu[n(t)]}}(t)|^k dt \leqslant \int_0^1 \left\{ \sum_{n=0}^{\infty} (s_{\lambda_n}(t) - s_{2^{\nu(n)}})^2 \right\}^{\frac{1}{2}k} dt \leqslant B_k \int_0^1 |f(t)|^k dt,$$

by Theorem VIII. Also, by Theorem I,

$$\int_0^1 |s_{2^{\nu[n(t)]}}|^k dt \leqslant B_k \int_0^1 |f(t)|^k dt.$$

Thus

$$\int_0^1 |s_{\lambda_{n(t)}}(t)|^k dt \leqslant B_k \int_0^1 |f(t)|^k dt.$$

If $q < 2$, then we may divide the sequence $\{\lambda_n\}$ into a finite number of subsequences

$$\{\lambda_{Gn+\sigma}\} \quad (\sigma = 0, 1, \ldots, G-1) \quad G = G(\delta),$$

for each of which the theorem is satisfied. It follows at once that the theorem is satisfied for the sequence $\{\lambda_n\}$, but with a constant which now depends on G as well as on k.

Suppose that $\lambda_N > 2^\tau$. Then, by (10.1), we have

$$\int_0^1 \sup_{n \geqslant N} |f(t) - s_{\lambda_n}(t)|^k dt \leqslant B_{k,\delta} \int_0^1 \left| f(t) - \sum_{m=0}^{2^\tau - 1} c_m \psi_m(t) \right|^k dt$$

$$= B_{k,\delta} \int_0^1 \left| \sum_{n=\tau}^{\infty} f_n(t) \right|^k dt$$

$$\leqslant B_{k,\delta} \int_0^1 \left\{ \sum_{n=\tau}^{\infty} f_n^2(t) \right\}^{\frac{1}{2}k} dt,$$

by Theorem V; the last expression tends to zero in virtue of the existence of

$$\int_0^1 \left\{ \sum_{n=0}^{\infty} f_n^2(t) \right\}^{\frac{1}{2}k} dt.$$

The last assertion, that, for almost all t in (0, 1),

$$s_{\lambda_n}(t) \to f(t),$$

follows at once from (10.2).

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A REMARKABLE SERIES OF ORTHOGONAL FUNCTIONS (II)

By R. E. A. C. PALEY.

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11. We begin this half of the paper by considering the behaviour of Cesàro means of Walsh-Kaczmarz series. Let $f(t) \in L^k$ $(1 < k < \infty)$. Then, for almost all t, the $\{\psi\}$ series of f is strongly summable $(C, 1/k+\delta)$, and is summable (C, δ) to sum $f(t)$. In fact, let

$$\tau_n^{(\eta)}(t) = \frac{\sum_{m=0}^{n=1} \left(1-\frac{m}{n}\right)^{\eta-1} |s_m(t)|}{\sum_{m=0}^{n-1} \left(1-\frac{m}{n}\right)^{\eta-1}} \quad (\eta > 0) \tag{11.1}$$

denote the strong Rieszian mean, and $\sigma_n^{(\eta)}$ the weak Rieszian mean of order η. Then†, for all positive δ, we have

$$\int_0^1 \sup_{1 \leqslant n < \infty} \{\tau_n^{(1/k+\delta)}(t)\}^k dt \leqslant B_{k,\delta} \int_0^1 |f(t)|^k dt, \tag{11.2}$$

$$\int_0^1 \sup_{1 \leqslant n < \infty} |\sigma_n^{(\delta)}(t)|^k dt \leqslant B_{k,\delta} \int_0^1 |f(t)|^k dt. \tag{11.3}$$

We next consider convergence factors of $\{\psi\}$ series. If $1 < k \leqslant 2$, then $\log^{-1/k}(n+2)$ is almost everywhere a convergence factor of the $\{\psi\}$ series of $f(t) \in L^k$. In fact, if

$$s_n^*(t) = \sum_{m=0}^{n-1} c_m \log^{-1/k}(m+2)\, \psi_m(t),$$

c_m being the ψ coefficients of $f(t)$, then we have

$$\int_0^1 |s_{n(t)}^*(t)|^k dt \leqslant B_k \int_0^1 |f(t)|^k dt,$$

† The Fourier analogues of both theorems are known. That of (11.2) seems never to have been stated explicitly. For the analogue of (11.3), see Hardy and Littlewood, 3.

where $n(t)$ varies arbitrarily with t. If $f(t) \in L$, then $\log^{-1}(n+2)$ is almost everywhere a convergence factor of the $\{\psi\}$ series of f. In each case the Fourier series analogue has already been established†.

12. We next consider the ψ-analogue of an inequality of Hardy and Littlewood‡, of which proofs have also been given by Gabriel and F. Riesz§. Let $c_0, c_1, \ldots, c_n, \ldots$ denote a bounded set of real numbers, and let $c_0^*, c_1^*, \ldots, c_n^*, \ldots$ denote the set $|c_0|, |c_1|, \ldots, |c_n|, \ldots$ rearranged in descending order of magnitude. Let

$$f(t) \sim \sum_{n=0}^{\infty} c_n \psi_n(t), \quad F(t) \sim \sum_{n=0}^{\infty} c_n^* \psi_n(t).$$

We show that, if q is an even integer, then

$$\int_0^1 f^q(t)\,dt \leqslant \int_0^1 F^q(t)\,dt, \tag{12.1}$$

whenever the right-hand side exists. From the result (12.1) we may obtain results analogous to those which Hardy and Littlewood‖ deduce for Fourier series. The arguments are almost identical.

13. We begin by stating without proof the following theorem for strong means.

THEOREM X. *Let $\tau_n^{(\eta)}$ denote the strong Rieszian mean* (11.1) *of order η of the ψ series of $f(t)$. Then, for positive δ,*

$$\int_0^1 \sup_{1 \leqslant n < \infty} \{\tau_n^{(1/k+\delta)}(t)\}^k\,dt \leqslant B_{k,\delta} \int_0^1 |f(t)|^k\,dt \quad (k > 1),$$

the right-hand side being supposed to exist.

For Fourier series more is known. In fact, we may substitute δ for $1/k+\delta$ in the above theorem. The result has never been explicitly stated, but follows at once from the results of the papers Hardy and Littlewood, **3** and **6**. It seems likely that a similar result is also true for ψ-series, but I have not yet succeeded in obtaining one.

† Littlewood and Paley, **7**; Hardy, **2**.
‡ Hardy and Littlewood, **4**.
§ Gabriel, **1**, F. Riesz, **9**.
‖ Hardy and Littlewood, **5**.

For the weak Rieszian mean we obtain a stronger result, analogous to Hardy and Littlewood's theorem† for Fourier series.

THEOREM XI. *Let* $\sigma_n^{(\eta)}$ *denote the weak Rieszian mean*

$$\sigma_n^{(\eta)}(t) = \sum_{m=0}^{n-1} \left(1-\frac{m}{n}\right)^{\eta} c_m \psi_m(t) \quad (\eta > 0).$$

Then, for positive δ,

$$\int_0^1 \sup_{1 \leqslant n < \infty} |\sigma_n^{(\delta)}(t)|^k \, dt \leqslant B_{k,\delta} \int_0^1 |f(t)|^k \, dt \quad (k > 1),$$

the right-hand side being supposed to exist.

We write

$$H(t) = \sup_{1 \leqslant n < \infty} |\sigma_n(t)|.$$

Putting $\delta = 1/k'$ in Theorem X, we observe that

$$\int_0^1 H^k(t)\, dt \leqslant B_k \int_0^1 |f(t)|^k \, dt. \tag{13.1}$$

Let

$$n = 2^{n_1} + 2^{n_2} + \ldots + 2^{n_\nu}.$$

Then

$$\sum_{m=0}^{2^{n_1}-1} c_m \psi_m(t) \left(1-\frac{m}{n}\right)^{\delta} = \int_0^1 f(\theta) \left[\sum_{m=0}^{2^{n_1}-1} \psi_m(t)\, \psi_m(\theta) \left(1-\frac{m}{n}\right)^{\delta}\right] d\theta. \tag{13.2}$$

If we write

$$f'(\theta) = f(\theta)\, \phi_0(\theta)\, \phi_1(\theta) = \phi_{n_1-1}(\theta),$$

the last integral is identically equal to

$$\phi_0(t)\, \phi_1(t) \ldots \phi_{n_1-1}(t) \int_0^1 f'(\theta) \left[\sum_{m=0}^{2^{n_1}-1} \psi_m(t)\, \psi_m(\theta) \left(\frac{n-2^{n_1}+1+m}{n}\right)^{\delta}\right] d\theta.$$

If now c_m', s_m', σ_m', $H'(t)$ correspond to $f'(\theta)$ in the same way as c_m, s_m, σ_m, $H(t)$ correspond to $f(\theta)$, then the expression (13.2) has the same modulus as

$$n^{-\delta} \sum_{m=0}^{2^{n_1}-1} c_m' \psi_m(t)\, (n-2^{n_1}+1+m)^{\delta} - n^{-\delta} \left[\sum_{m=1}^{2^{n_1}-1} s_m'(t) \Delta\{(n-2^{n_1}+1+m)^{\delta}\} + s_{2^{n_1}}(t)\, n^{\delta}\right], \tag{13.3}$$

† Hardy and Littlewood, 3.

where $\Delta\{(n-2^{n_1}+1+m)^\delta\}$ denotes $(n-2^{n_1}+m)^\delta-(n-2^{n_1}+1+m)^\delta$. The second member of (13.3) is less in modulus than $2F(t)$ in virtue of Lemma 2, where $F(t)$ is the maximum average of $|f|$ at the point t. Using Abel's transformation again, we may write the first as

$$(13.4)\quad n^{-\delta}\Big[\sum_{m=1}^{2^{n_1}-2} m\sigma_m'(t)\,\Delta_2\{(n-2^{n_1}+1+m)^\delta\} + (2^{n_1}-1)\,\sigma'_{2^{n_1}-1}(t)\{(n-1)^\delta-n^\delta\}\Big],$$

$\Delta_2\{(n-2^{n_1}+1+m)^\delta\}$ denoting the double difference

$$(n-2^{n_1}+m)^\delta-2(n-2^{n_1}+1+m)^\delta+(n-2^{n_1}+2+m)^\delta.$$

Observing that $\Delta_2\{(n-2^{n_1}+1+m)^\delta\}=O(m^{\delta-2})$,

we see that the expression (13.4) is of the form

$$O(n^{-\delta})\Big[\sum_{m=1}^{2^{n_1}-2} m^{\delta-1}H'(t)+2^{n_1}n^{\delta-1}H'(t)\Big]=O\{H'(t)\}.$$

Thus we get

$$\Big|\sum_{m=0}^{2^{n_1}-1} c_m\psi_m(t)\Big(1-\frac{m}{n}\Big)^\delta\Big|\leqslant B_\delta\{H'(t)+F(t)\}.$$

Now

$$(13.5)\quad \sum_{m=2^{n_1}}^{2^{n_1}+2^{n_2}-1} c_m\psi_m(t)\Big(1-\frac{m}{n}\Big)^\delta = \int_0^1 f(\theta)\Big[\sum_{m=2^{n_1}}^{2^{n_1}+2^{n_2}-1}\psi_m(t)\psi_m(\theta)\Big(1-\frac{m}{n}\Big)^\delta\Big]d\theta.$$

If we write $f''(\theta)=f(\theta)\phi_{n_1}(\theta)\phi_0(\theta)\phi_1(\theta)\dots\phi_{n_2-1}(\theta)$,

the last integral is identically

$$\phi_{n_1}(t)\phi_0(t)\phi_1(t)\dots\phi_{n_2-1}(t)\int_0^1 f''(\theta)\Big[\sum_{m=0}^{2^{n_2}-1}\psi_m(t)\psi_m(\theta)\Big(\frac{n-2^{n_1}-2^{n_2}+1+m}{n}\Big)^\delta\Big]d\theta.$$

Arguing as before, we obtain

$$\Big|\sum_{m=2^{n_1}}^{2^{n_1}+2^{n_2}-1} c_m\psi_m(t)\Big(1-\frac{m}{n}\Big)^\delta\Big|\leqslant B_\delta\Big(\frac{n-2^{n_1}}{n}\Big)^\delta\{H''(t)+F(t)\} \leqslant 2^{-\delta}B_\delta\{H''(t)+F(t)\},$$

where $H''(t)$ is obtained from $f''(\theta)$ in the same way as $H(t)$ is obtained from $f(\theta)$. In the same way,

$$\left|\sum_{m=2^{n_1}+2^{n_2}}^{2^{n_1}+2^{n_2}+2^{n_3}-1} c_m \psi_m(t)\left(1-\frac{m}{n}\right)^{\delta}\right| \leqslant 2^{-2\delta} B_\delta\{H'''(t)+F(t)\},$$

and so on. Combining these results we get

$$|\sigma_n^{(\delta)}(t)| \leqslant B_\delta\{F(t)+H'(t)+2^{-\delta}H''(t)+2^{-2\delta}H'''(t)+\ldots\}.$$

Thus Minkowski's inequality gives

$$|\sigma_n^{(\delta)}(t)|^k \leqslant B_\delta^k(1+1+2^{-\delta}+2^{-2\delta}+\ldots)^{k-1}$$
$$\times\{F^k(t)+H'^k(t)+2^{-\delta}H''^k(t)+2^{-2\delta}H'''^k(t)+\ldots\},$$

and the desired result follows at once by (13.1),

It is not difficult to extend the proofs of the last two theorems to prove that, if $f(t) \in L^k$ $(k>1)$, then, for almost all values of t, the $\{\psi\}$ series of $f(t)$ is strongly summable $(C, 1/k+\delta)$, and summable (C, δ) for all positive δ.

14. We may prove the following theorem on convergence factors of $\{\psi\}$ series.

THEOREM XII. *Let $f(t) \in L^k$ $(1<k\leqslant 2)$, and let*

$$s_n^*(t) = \sum_{m=0}^{n-1} \frac{c_m\psi_m(t)}{\log^{1/k}(m+2)}.$$

Let $n(t)$ denote an integer which varies arbitrarily with t. Then

$$\int_0^1 |s_{n(t)}^*(t)|^k\,dt \leqslant B_k\int_0^1 |f(t)|^k\,dt,$$

where the constant B_k is independent of the choice of $n(t)$.

For almost all t, $\log^{-1/k}(n+2)$ is a convergence factor for the series $\Sigma c_n\psi_n(t)$.

The proof is almost identical with that of the corresponding result† for Fourier series.

15. When we come to investigate what happens to the above theorems in the case $k=1$, we obtain weaker results. We state without proof the following two theorems.

† Littlewood and Paley, 7.

THEOREM XIII. *Let n denote a fixed positive integer. Then, with the notation of Theorem XI,*

$$\int_0^1 |\sigma_n^{(\delta)}(t)|\,dt \leqslant B_\delta \int_0^1 |f(t)|\,dt,$$

the right-hand side being supposed existent. The constant B_δ is independent of the choice of n or $f(t)$, and depends only on δ.

THEOREM XIV. *Let*

$$f(t) \sim \sum_{n=0}^{\infty} c_n \psi_n(t) \in L. \tag{15.1}$$

Then, for almost all values of t, $\log^{-1}(n+2)$ is a convergence factor for the series (15.1).

The last result is the $\{\psi\}$ analogue of Hardy's well-known theorem† for Fourier series.

16. We now go on to prove the $\{\psi\}$ analogue of Hardy and Littlewood's‡ "star" theorem. We first have to consider what exactly the analogue is. We introduce a fresh notation. Suppose that

$$\psi_{n_1}(t)\psi_{n_2}(t)\psi_{n_3}(t)\ldots\psi_{n_\nu}(t) = \psi_m(t) \quad (0 \leqslant t < 1); \tag{16.1}$$

then we say that

$$(n_1, n_2, n_3, \ldots, n_\nu) \equiv m.$$

If the left-hand side of (16.1) is identically 1, then we have§

$$(n_1, n_2, n_3, \ldots, n_\nu) \equiv 0.$$

† Hardy, **2**.

‡ Hardy and Littlewood, **4**.

§ The game of Nim is played with matches according to the following rules. A number of matches is arranged in three heaps on a table. Two players move alternately. At each turn the player must remove one or more matches, all from the same heap. He is at liberty to decide how many matches he will remove, and from which heap. The player who removes the last match loses the game. We call (l, m, n) a winning combination, if, by leaving his opponent with heaps of l, m, and n matches, a player is in a position to force a win. The criterion that (l, m, n) should be a winning combination is either

$$(l, m, n) \equiv 0, \qquad \max(l, m, n) > 1,$$

or

$$l + m + n \equiv 1 \pmod 2, \quad \max(l, m, n) = 1.$$

The last formula may be extended to give winning combinations if the game is played with any number of heaps. For an analysis of the game see, *e.g.*, Rouse Ball, **10**.

Suppose that $a_0, a_1, \ldots, a_\lambda$ are $\lambda+1$ non-negative numbers. Suppose further that $a_0^* \geqslant a_1^* \geqslant a_2^* \ldots \geqslant a_\lambda^*$ are the same numbers rearranged in descending order of magnitude. We call this arrangement the *standard* arrangement of the a_r and say that the a_r^* are arranged *in type P*. We prove the following theorem.

THEOREM XV. *The sum*

$$S = \sum_{(r,s,t,\ldots)=0} a_r b_s c_\nu \ldots$$

is greatest when the a's, b's, c's, ... are all arranged in type P, i.e.

$$S = \Sigma a_r b_s c_t \ldots \leqslant S^* = \Sigma a_r^* b_s^* c_t^* \ldots.$$

We first need

LEMMA 14. *It is sufficient to prove the theorem in the case when all of the numbers a_r, b_s, c_t are either* 1 *or* 0.

The proof of the lemma is exactly similar to that of the analogous lemma for Fourier series, proved by Hardy and Littlewood in their paper†.

We proceed now to the proof of the theorem. We may suppose without loss of generality that the numbers of members of the sets $\{a\}$, $\{b\}$, $\{c\}$, ... are all equal, and that this number is of the form 2^ν, where ν is a positive integer. Assuming, in virtue of Lemma 14, that all the numbers a, b, c, ... are either 1 or 0, we give an inductive proof. We show that, if the theorem is assumed to be true for a given value of ν, then it may be deduced for higher values of ν.

We first establish the theorem for $\nu = 1$. Let N denote the number of sets $\{a\}$, $\{b\}$, $\{c\}$, ... which have two non-vanishing members. We assume, without loss of generality, that each set has at least one non-vanishing member. Then, if $N = 0$, we have $S \leqslant 1$, $S^* = 1$, from which the result follows. If $N > 0$, we may assume, without loss of generality, that the set $\{a\}$ contains two non-vanishing members. Now let $s, t, \ldots$ denote any set of numbers for which $b_s = c_t = \ldots = 1$. It is not difficult to see that $(s, t, \ldots) \equiv 0$ or 1, and thus‡ the corresponding member of the set $\{a\}$ is also non-vanishing. Thus we have

$$S = \sum_{(r,s,t,\ldots)\equiv 0} a_r b_s c_t \ldots = 2^{N-1}.$$

Similarly $S^* = 2^{N-1}$. This proves the required result for $\nu = 1$.

17. The proof of the result for $\nu \geqslant 2$ is best represented diagrammatically. We assume that the result has already been established for $\nu - 1$. We

† Hardy and Littlewood, **4**, 108.

‡ Clearly, if $(s, t, \ldots) \equiv r$, then $(r, s, t, \ldots) \equiv 0$.

consider first a new type of rearrangement, defined as follows. We are given 2^ν numbers $a_0, a_1, \ldots; a_{2^\nu-1}$, all equal to 1 or 0. We arrange the first $2^{\nu-1}$ numbers $a_0, a_1, \ldots, a_{2^{\nu-1}-1}$ in descending order, and we arrange the last $2^{\nu-1}$ numbers $a_{2^{\nu-1}}, a_{2^{\nu-1}+1}, \ldots, a_{2^\nu-1}$ also in descending order. We say that the numbers $\{a\}$ have been rearranged in type Q_ν. We prove the following lemma.

LEMMA 15. *The sum S is not decreased by rearrangement in type Q_ν, it being assumed that Theorem XV has already been established for sets $\{a\}, \{b\}, \{c\}, \ldots$ which contain each not more than $2^{\nu-1}$ members.*

Let

$$\sum_{n=0}^{2^\nu-1} a_n \psi_n(\theta) = f(\theta) = f_0(\theta) + \phi_{\nu-1}(\theta) f_1(\theta), \tag{17.1}$$

where $f_0(\theta)$ denotes the first $2^{\nu-1}$ terms of the polynomial (17.1) and $\phi_{\nu-1}(\theta) f_1(\theta)$ denotes the remainder. Then $f_1(\theta)$ is a polynomial of the same type as $f_0(\theta)$, containing ψ's only of suffix not exceeding $2^{\nu-1}-1$. We write similarly

$$\sum_{n=0}^{2^\nu-1} b_n \psi_n(\theta) = g(\theta) = g_0(\theta) + \phi_{\nu-1}(\theta) g_1(\theta);$$

$$\sum_{n=0}^{2^\nu-1} c_n \psi_n(\theta) = h(\theta) = h_0(\theta) + \phi_{\nu-1}(\theta) h_1(\theta);$$

$$\ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots$$

$$\ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots$$

Then

$$S = \sum_{(r,s,t,\ldots)\equiv 0} a_r b_s c_t \ldots = \int_0^1 f(\theta) g(\theta) h(\theta) \ldots d\theta$$

$$= \int_0^1 \{f_0(\theta) + \phi_{\nu-1}(\theta) f_1(\theta)\} \{g_0(\theta) + \phi_{\nu-1}(\theta) g_1(\theta)\} \ldots d\theta$$

$$= \sum_{(\rho,\sigma,\tau,\ldots)\equiv 0} \int_0^1 f_\rho(\theta) g_\sigma(\theta) h_\tau(\theta) \ldots d\theta. \tag{17.2}$$

Since the theorem is supposed to have been established for sets which contain each less than $2^{\nu-1}$ members, it follows that each term of the sum (17.2) is not decreased when we rearrange the terms of $f_\rho, g_\sigma, h_\tau, \ldots$ in star order. But this is exactly equivalent to the rearrangement Q_ν, and the result of Lemma 15 follows.

The possible distributions of the non-zero elements of any given one of the sets $\{a\}, \{b\}, \{c\}, \ldots$, after a rearrangement Q_ν, are shown in Fig. (i).

The thick lines denote that part of the range $(0, 2^{\nu}-1)$ which contains non-zero elements. The range is divided into four equal quarters.

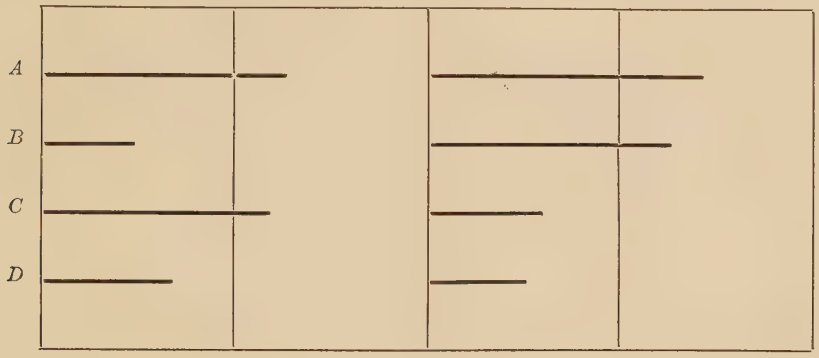

Fig. (i).

We now interchange $\phi_{\nu-1}(\theta)$ and $\phi_{\nu-2}(\theta)$, with the consequent alterations in the order of the functions ψ_n, so that

$$\psi_n(\theta) = \phi_{n_1}(\theta)\,\phi_{n_2}(\theta) \ldots \phi_{\nu-1}(\theta) \tag{17.3}$$

is changed into

$$\psi_{n'}(\theta) = \phi_{n_1}(\theta)\,\phi_{n_2}(\theta) \ldots \phi_{\nu-2}(\theta).$$

The relation between n and n' is (1, 1) and has the effect of interchanging the second and third quarters in Fig. (i). We observe that the relations

$$(r, s, t, \ldots) \equiv 0, \quad (r', s', t', \ldots) \equiv 0$$

are equivalent. It follows that S is unaltered by the change which we have carried out. The new possible distributions of the non-zero elements are shown in Fig. (ii).

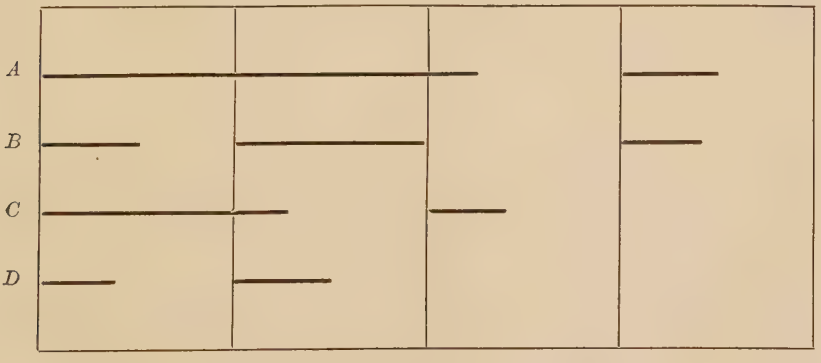

Fig. (ii).

We now apply the operation Q_ν a second time. This cannot decrease S. The new distributions of the non-zero elements are shown in Fig. (iii).

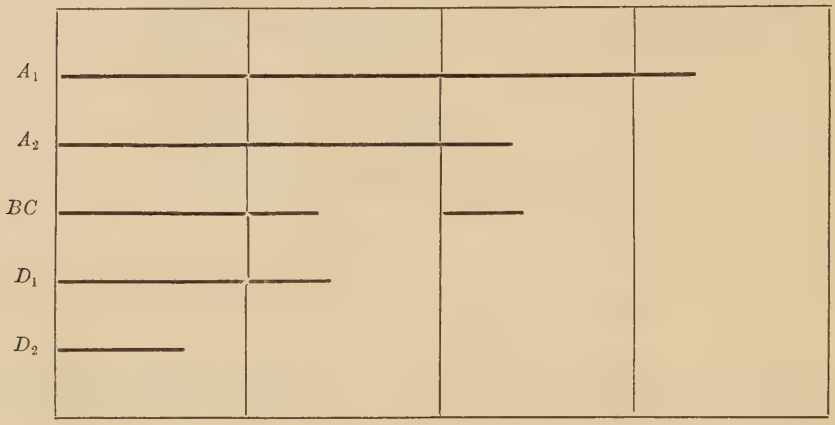

Fig. (iii).

The next step is to change $\phi_{\nu-2}(\theta)$, whenever it occurs in the expansion (17.3), into $\phi_{\nu-2}(\theta)\phi_{\nu-1}(\theta)$ and $\phi_{\nu-2}(\theta)\phi_{\nu-1}(\theta)$ into $\phi_{\nu-2}(\theta)$, leaving $\phi_{\nu-1}(\theta)$ unchanged when it occurs without $\phi_{\nu-2}(\theta)$. This again does not change S. The result is shown in Fig. (iv).

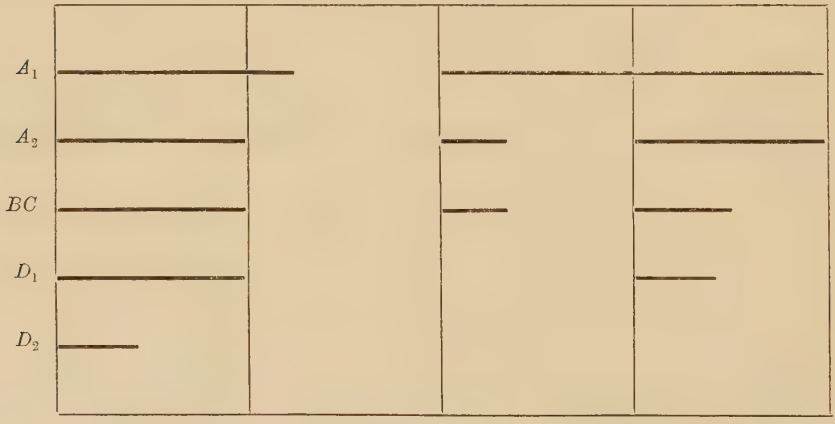

Fig. (iv).

A fresh application of Q_ν gives the distributions of Fig. (v).

We now interchange $\phi_{\nu-1}(\theta)$ and $\phi_{\nu-2}(\theta)$ once more, which can be done without altering S. The possible distributions of the non-zero elements

are shown in Fig. (vi).

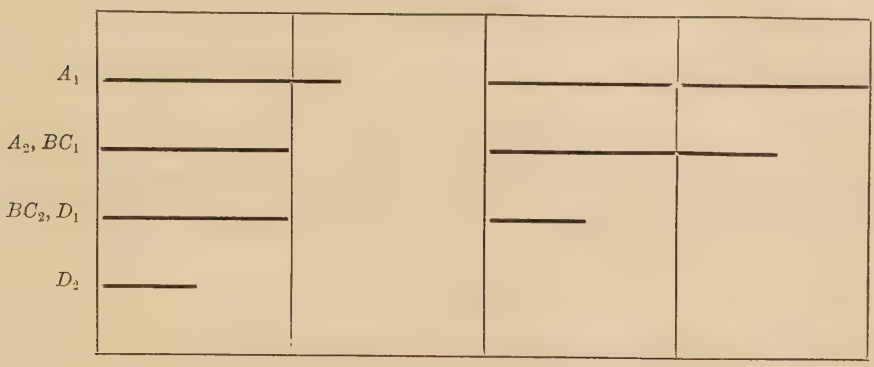

Fig. (v).

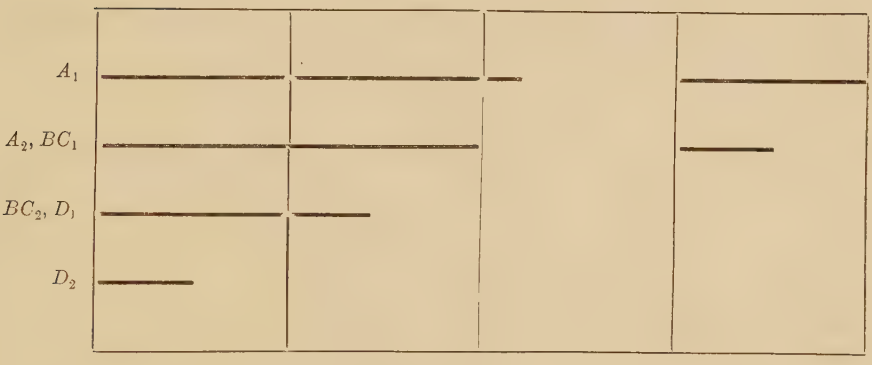

Fig. (vi).

Finally, another application of the operation Q_ν gives the standard star rearrangement of the elements. Each of the operations carried out has had the effect of increasing S or of leaving it unchanged. It follows that $S \leqslant S^*$, when the number of elements does not exceed 2^ν. The desired result now follows.

18. If we put $a_n = b_n = c_n = \ldots$ in the above theorem we obtain the following theorem, which is the $\{\psi\}$ analogue of Hardy and Littlewood's theorem† for Fourier series. Clearly, by allowing the moduli of the coefficients to vary, we do not increase the q-th mean of $f(t)$ when q is an even integer.

† Hardy and Littlewood, 5.

THEOREM XVI. *Let*

$$f(t) \sim \sum_{n=0}^{\infty} c_n \psi_n(t),$$

where $\Sigma c_n^2 < \infty$, *and let*

$$F(t) \sim \sum_{n=0}^{\infty} c_n^* \psi_n(t),$$

where the members $c_0^* \geqslant c_1^* \geqslant c_2^* \geqslant \ldots$ *denote the set* $\{|c_n|\}$ *rearranged in descending order of magnitude. If* q *is an even integer, then*

$$\int_0^1 f^q(t)\,dt \leqslant \int_0^1 F^q(t)\,dt,$$

whenever the right-hand side exists.

We need the following lemmas. The first is known†.

LEMMA 16. *Let* $\vartheta_n(t)$ $(n = 0, 1, \ldots)$ *denote a set of normalized orthogonal functions, all less in modulus than some constant* λ. *Let*

$$1 < p \leqslant 2 \leqslant q < \infty.$$

Let‡

$$f(t) \sim \sum_{m=0}^{\infty} c_m \vartheta_m(t).$$

Then

$$\sum_{m=0}^{\infty} |c_m|^p (m+1)^{p-2} \leqslant B_{p,\lambda} \int_0^1 |f(t)|^p\,dt, \tag{18.1}$$

$$\int_0^1 |f(t)|^q\,dt \leqslant B_{q,\lambda} \sum_{m=0}^{\infty} |c_m|^q (m+1)^{q-2}, \tag{18.2}$$

the right-hand side in each case being supposed existent.

In the particular case where $\vartheta_n(t)$ is identified with $\psi_n(t)$ and the coefficients c_n are in starred order, we can assert more than Lemma 16. We have, in fact,

LEMMA 17. *Let* $1 < k < \infty$, *and*

$$F(t) \sim \sum_{m=0}^{\infty} c_m^* \psi_m(t),$$

---

† See Paley, **8**.

‡ That is to say, we write

$$c_m = \int_0^1 f(t)\,\vartheta_m(t)\,dt. \tag{i}$$

The inequality (18.2) is then to be interpreted as meaning that when the sum on the right-hand side is finite, then an $f(t)$ exists satisfying (i) and (18.2).

where the coefficients c_n^* are positive and decreasing. Then

$$(18.3)\quad B_k \sum_{m=0}^{\infty} c_m^{*k}(m+1)^{k-2} \leqslant \int_0^1 |F(t)|^k\,dt \leqslant B_k \sum_{m=0}^{\infty} c_m^{*k}(m+1)^{k-2},$$

whenever either member exists†.

We consider separately the cases $k<2$ and $k>2$. When $k<2$, the first of the inequalities (18.3) follows from (18.1). For the second we observe by Theorem V that, if

$$F_n(t) = \sum_{m=2^{n-1}}^{2^n-1} c_m^*\psi_m(t),$$

then
$$\int_0^1 |F(t)|^k\,dt \leqslant B_k \int_0^1 \Big\{c_0^2 + \sum_{n=1}^{\infty} F_n^2(t)\Big\}^{\frac{1}{2}k}\,dt$$
$$\leqslant B_k \Big\{|c_0|^k + \sum_{n=1}^{\infty} \int_0^1 |F_n(t)|^k\,dt\Big\}.$$

For
$$\int_0^1 |F_n(t)^k\,dt$$

we observe first that, for $0<t<2^{-n}$, we have $|F_n(t)| \leqslant 2^{n-1}c_{2^{n-1}}^*$. Also it may easily be verified‡ that

$$\Big|\sum_{m=0}^{M} \psi_m(t)\Big| \leqslant \frac{B}{t},$$

and, since the coefficients c_m^* are decreasing, Abel's transformation formula at once gives

$$\Big|\sum_{m=2^{n-1}}^{2^n-1} c_m^*\psi_m(t)\Big| \leqslant \frac{Bc_{2^{n-1}}^*}{t}.$$

Thus
$$\int_0^1 |F_n(t)|^k\,dt \leqslant \int_0^{2^{-n}} + \int_{2^{-n}}^1$$
$$\leqslant B_k c_{2^{n-1}}^{*k}\, 2^{(k-1)n} \leqslant B_k \sum_{m=2^{n-2}}^{2^{n-1}-1} c_m^{*k}(m+1)^{k-2},$$

from which the desired result follows.

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† For the interpretation of (18.3) see the preceding foot-note.

‡ In fact, if $2^{-(N+1)} \leqslant t < 2^{-N}$, then $\phi_N(t) = -1$, and thus, for arbitrary integral l,

$$\sum_{m=0}^{l2^{N+1}-1} \psi_m(t) = 0,$$

since, when expressed by means of Rademacher's functions, the left-hand side is divisible by $1+\phi_N(t)$.

For $k>2$, the second of the inequalities (18.3) follows from (18.2). For the first we observe that, by Theorem V,

$$\int_0^1 |F_n(t)|^k\,dt \geqslant B_k \int_0^1 \Big\{c_0^2 + \sum_{n=1}^{\infty} F_n^2(t)\Big\}^{\frac{1}{2}k}\,dt$$

$$\geqslant B_k \Big\{|c_0|^k + \sum_{n=1}^{\infty} \int_0^1 |F_n(t)|^k\,dt\Big\}.$$

Also

$$\int_0^1 |F_n(t)|^k\,dt \geqslant \int_0^{2^{-n}} |F_n(t)|^k\,dt$$

$$\geqslant 2^{-n}\Big(\sum_{m=2^{n-1}}^{2^n-1} c_m^*\Big)^k \geqslant B_k\, 2^{(k-1)n}\, c_{2^n-1}^{*k}$$

$$\geqslant B_k \sum_{m=2^n}^{2^{n+1}-1} c_m^{*k}(m+1)^{k-2},$$

from which the desired result follows.

We may now continue the argument in exactly the same way as in Hardy and Littlewood's investigations, and we obtain results analogous to those which they obtain for Fourier series. Of these the most important are given in the theorems which follow, which also result immediately from Lemmas 16 and 17.

THEOREM XVII. *Let*

$$f(t) \sim \sum_{n=0}^{\infty} c_n \psi_n(t); \quad F(t) \sim \sum_{n=0}^{\infty} c_n^* \psi_n(t),$$

where c_0^*, c_1^*, c_2^*, ... denote the set $\{|c_n|\}$ rearranged in descending order. If $2 \leqslant q < \infty$, we have

$$\int_0^1 |f(t)|^q\,dt \leqslant B_q \int_0^1 |F(t)|^q\,dt,$$

whenever the right-hand side exists. B_q is a constant which depends only on q.

THEOREM XVIII. *With the notation of the last theorem we have, for $1<p\leqslant 2$,*

$$\int_0^1 |F(t)|^p\,dt \leqslant B_p \int_0^1 |f(t)|^p\,dt,$$

whenever the right-hand side exists. B_p depends only on p.

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AN ASYMPTOTIC FORMULA IN THE THEORY OF NUMBERS

By T. ESTERMANN.

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1. *Introduction.*

This paper deals with the following problem, which was suggested to me by Mr. Ingham:

Let $r(h)$ denote the number of representations of h as the sum of two squares, and let k be any given positive integer. What is the asymptotic behaviour of $\sum_{h=1}^{n} r(h)r(h+k)$ when $n \to \infty$?

I shall prove the formula

$$\sum_{h=1}^{n} r(h)r(h+k) = 8c_k k^{-1} n + O(n^{\frac{11}{12}} \log^{\frac{17}{8}+\epsilon} n) \tag{1}$$

for any positive ϵ, where

$$c_k = (-1)^k \sum_{h\ k} (-1)^h h. \tag{2}$$

The method is that of my paper* " Über die Darstellungen einer Zahl als Differenz von zwei Produkten ", hereafter quoted as I.

2. *Notation.*

The letters h, k, l, m, s, t, u, x, y, and z, without suffixes, denote positive integers only. The above letters with suffixes and the letters j and q may denote any integers.

All formulae containing ϵ are meant for any positive ϵ.

The letters A_1, A_2, ... denote positive numbers that depend at most on ϵ, and the O's are meant uniformly in all variables except ϵ, l, and k.

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* *Journal für Math.*, 164 (1931), 173–182.

$d(h)$ denotes the number of positive divisors of h, (s, t) the greatest common divisor of s and t, $\mu(h)$ Möbius's function.

If $(s, t) = 1$, then $[s, t]$ denotes the integer for which

$$s[s, t] \equiv 1 \pmod{t}, \quad 0 < [s, t] \leqslant t. \tag{3}$$

The function $\psi(w)$ is defined for all real values of w by

$$\psi(w) = w - \tfrac{1}{2} \ (0 \leqslant w < 1), \quad \psi(w+1) = \psi(w). \tag{4}$$

The numbers under the Σ's represent the following conditions of summation:

1. $(s, t) = 1$; 1a. $(s', l't) = 1$;
2. $a < s \leqslant b$;
3. $s \equiv s_0 \pmod{l}$; 3a. $s' \equiv s_0' \pmod{l'}$;
4. $sy - tu = k_1$; 4a. $s(y_0 + ly_1) - t(u_0 + lu_1) = k_1$;
5. $t \equiv t_0 \pmod{l}$;
6. $y \equiv y_0 \pmod{l}$;
7. $u \equiv u_0 \pmod{l}$;
8. $s^2 \leqslant t^2 + k_1 m^{-1}$; 8a. $s'^2 \leqslant t^2 h^{-2} + k_1 m^{-1} h^{-2}$;

8′. $s^2 < t^2 + k_1 m^{-1}$;

9. $mt^2 + k_1 < sy \leqslant n_1$; 9a. $mt^2 + k_1 < s(y_0 + ly_1) \leqslant n_1$;

9b. $mt^2 + k_1 < sy_0 + s(k_1 - sy_0 + tu_0)[s, t] + lstj \leqslant n_1$;

10. $0 < mt^2 + k_1 < n_1$; 10a. $mt^2 < nm^{-1}$; 10b. $ms^2 - km^{-1} < nm^{-1}$;

10c. $y^2 < nm^{-2}h^{-2}$; 10d. $x^2 < (k+n)m^{-2}h^{-2}$;

11. $sy_1 \equiv (k_1 - sy_0 + tu_0) l^{-1} \pmod{t}$;

12. $xy - zu = k$; 12′. $sy - tu = km^{-1}$;

13. $zu \leqslant n$; 13′. $tu \leqslant nm^{-1}$;

14. $x < y$; 14′. $ms < y$;

15. $z < u$; 15′. $mt < u$;

16. $(x, z) = m$;

17. $0 \leqslant s_0 < 4$;

18. $0 \leqslant t_0 < 4$;

19. $0 \leqslant y_0 < 4$;

20. $0 \leqslant u_0 < 4$;

21. $s_0 y_0 - t_0 u_0 \equiv km^{-1} \pmod 4$; 21a. $sy_0 - tu_0 \equiv km^{-1} \pmod 4$;

22. $s^2 \leqslant t^2 + km^{-2}$; 22a. $x^2 \leqslant y^2 + k(mh)^{-2}$;

$\overline{22}$. $s^2 > t^2 + km^{-2}$;

23. $mt^2 + km^{-1} < sy \leqslant (n+k)m^{-1}$;

24 is an abbreviation for 17, 18, 19, 20, 21;

25 ,, ,, ,, 12′, 13′, 14′, 15′, 1, 3, 5, 6, 7.

3. *Some lemmas involving $\psi(w)$ and $[s, t]$.*

LEMMA 1 (I, Lemma 2). *Let w, a, and b be any real numbers such that*

$$0 \leqslant a < b \leqslant t. \tag{5}$$

*Then**

$$\left| \sum_{1,2}^{s} \exp\{2\pi i t^{-1}(ws + q[s, t])\} \right| < 5d^{\frac{3}{4}}\left(\frac{t}{(q, t)}\right) t^{\frac{3}{4}}(q, t)^{\frac{1}{4}} \log(4t). \tag{6}$$

LEMMA 2. *If (5) holds, then*

$$\left| \sum_{1,2,3}^{s} \exp\{2\pi i t^{-1}(ws + q[s, t])\} \right| < 5d^{\frac{3}{4}}\left(\frac{t}{(q, t)}\right) t^{\frac{3}{4}}(q, t)^{\frac{1}{4}} \log(4t) \tag{7}$$

for any s_0 and l.

Proof. We have $l^{-1} \sum_{j=1}^{l} \exp[2\pi i l^{-1} j(s - s_0)] = 1$ or 0 according as 3 is or is not satisfied. Hence

$$\sum_{1,2,3}^{s} \exp\{2\pi i t^{-1}(ws + q[s, t])\} = l^{-1} \sum_{j=1}^{l} \exp(-2\pi i l^{-1} j s_0) \sum_{1,2}^{s} \exp\left\{2\pi i t^{-1}\left((w + tl^{-1}j)s + q[s, t]\right)\right\}, \tag{8}$$

which, together with Lemma 1, proves (7).

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* $d^{\frac{3}{4}}(h)$ is an abbreviation for $[d(h)]^{\frac{3}{4}}$.

LEMMA 3. *Let v, C, and D be real numbers; let*

$$t \geqslant v^{-1} > 4; \tag{9}$$

let ϑ_s be a function of s such that

$$-v < \vartheta_s < v \quad (a < s \leqslant b); \tag{10}$$

and let (5) *hold. Then*

$$\left| \sum_{\substack{s \\ 1,2,3}} \psi(C+Ds+\vartheta_s-qt^{-1}[s,t]) \right| < A_1 v(b-a)+A_2 t^{\frac{3}{4}} d^{\frac{3}{4}+\epsilon}(t)(q,t)^{\frac{1}{4}} \log^2 t. \tag{11}$$

This can be deduced from Lemma 2 in the same way as Lemma 5 of I from Lemma 2 of I.

I shall now obtain a generalization of Lemma 7 of I.

LEMMA 4. *Let $c \geqslant 0$, let c_1 be any real number, and let $0 < b \leqslant t$. Then*

$$\left| \sum_{\substack{s \\ 1,3,s \leqslant b}} \psi(cs^{-1}+c_1-qt^{-1}[s,t]) \right| \tag{12}$$

$$< A_3 c^{\frac{1}{3}} t^{\frac{1}{2}} d^{\frac{1}{2}+\epsilon}(t)(q,t)^{\frac{1}{3}} \log^{\frac{7}{3}}(2t)+A_4 t^{\frac{3}{4}} d^{\frac{3}{4}+\epsilon}(t)(q,t)^{\frac{1}{4}} \log^3(2t).$$

The proof of Lemma 7 of I requires the following slight correction: wherever $\log t$ occurs in Lemma 6 of I and in its proof and in I, (72), replace it by $\log(2t)$.

A proof of Lemma 4 can now be obtained from that of Lemma 7 of I by the following modifications:

1. Use Lemma 3 of this paper instead of Lemma 5 of I.

2. Restrict s by the condition 3 throughout.

3. Replace $\psi(cs^{-1}-qt^{-1}[s,t])$ by $\psi(cs^{-1}+c_1-qt^{-1}[s,t])$, and take $C = 2ca^{-1}+c_1$ in the proof of Lemma 6 of I (in its modified form).

4. Replace t by b in the condition of summation $s \leqslant t$ and where it first occurs in I, (71) (the trivial case $b < \frac{1}{2}$ being excluded).

5. Consider the case $0 \leqslant c < t^{-1} < \frac{1}{4}$ separately. In this case, (12) follows immediately from Lemma 3 with $a = 0$, $C = c_1$, $D = 0$, $\vartheta_s = cs^{-1}$, and $v = t^{-1}$.

4. *Evaluation of a sum.*

The investigation of $\sum_{h=1}^{n} r(h)r(h+k)$ will lead to sums of the type

$$S = S(n_1, k_1, s_0, t_0, y_0, u_0) = \sum_{\substack{s,\,t,\,y,\,u \\ 1,\,3,\,4\;\;5,\,6,\,7,\,8,\,9}} 1, \tag{13}$$

where

$$0 < n_1 \leqslant n+k, \quad mk_1 = \pm k, \tag{14}$$

and

$$s_0 y_0 - t_0 u_0 \equiv k_1 \pmod{l}. \tag{15}$$

By (13),

$$S = \sum_{\substack{s,\,t \\ 1,\,3,\,5,\,8,\,10}} \sum_{\substack{y,\,u \\ 4,\,6,\,7,\,9}} 1, \tag{16}$$

and

$$\sum_{\substack{y,\,u \\ 4,\,6,\,7,\,9}} 1 = \sum_{\substack{y_1,\,u_1 \\ 4a,\,9a}} 1 = \sum_{\substack{y_1 \\ 11,\,9a}} 1 = \sum_{\substack{j \\ 9b}} 1. \tag{17}$$

Now it follows from (4) that

$$\sum_{\substack{j \\ a<j\leqslant b}} 1 = b-a-\psi(b)+\psi(a) \quad (a \leqslant b). \tag{18}$$

Hence

$$\begin{aligned} \sum_{\substack{j \\ 9b}} 1 &= (n_1-k_1-mt^2)(lst)^{-1} \\ &\quad -\psi\{n_1(lst)^{-1}-y_0(lt)^{-1}-(k_1-sy_0+tu_0)(lt)^{-1}[s,\,t]\} \\ &\quad +\psi\{(mt^2+k_1)(lst)^{-1}-y_0(lt)^{-1}-(k_1-sy_0+tu_0)(lt)^{-1}[s,\,t]\}. \end{aligned} \tag{19}$$

Putting

$$\delta_r(t) = \sum_{\substack{s \\ 1,\,3,\,8}} \psi\{a_r s^{-1}-a_3-(k_1-sy_0+tu_0)(lt)^{-1}[s,\,t]\} \quad (r = 1,\,2), \tag{20}$$

where

$$a_1 = n_1(lt)^{-1}, \quad a_2 = (mt^2+k_1)(lt)^{-1}, \quad a_3 = y_0(lt)^{-1}, \tag{21}$$

we have, by (19),

$$\sum_{\substack{s \\ 1,3,8}} \left\{ \sum_{\substack{j \\ 9b}} 1 - (n_1 - k_1 - mt^2)(lst)^{-1} \right\} = -\delta_1(t) + \delta_2(t). \tag{22}$$

Now let s satisfy 1 and 3, and let

$$(s_0, l) = h, \quad s_0 = hs_0', \quad l = hl', \quad s = hs'. \tag{23}$$

Then

$$(s_0', l') = 1, \quad s' \equiv s_0' \pmod{l'}, \quad (s', l') = 1, \tag{24}$$

and, by (3), (4), and (15) (if t satisfies 5),

$$\begin{aligned} &\psi\{a_r s^{-1} - a_3 - (k_1 - sy_0 + tu_0)(lt)^{-1}[s, t]\} \\ &\quad = \psi\{a_r(hs')^{-1} - a_3 - [(k_1 + tu_0)h^{-1} - s'y_0](l't)^{-1}[h, t][s', l't]\} \\ &\quad = \psi\{a_r(hs')^{-1} + c_1 - q(l't)^{-1}[s', l't]\} \quad (r = 1, 2), \end{aligned} \tag{25}$$

where

$$c_1 = -a_3 + y_0(l't)^{-1}[h, t], \quad q = (k_1 + tu_0)h^{-1}[h, t]. \tag{26}$$

Now it is obvious that, if $(h, t) > 1$, then $\delta_1(t) = \delta_2(t) = 0$, and, if $(h, t) = 1$, then, by (20), (23), and (25),

$$\delta_r(t) = \sum_{\substack{s' \\ 1a, 3a, 8a}} \psi\{a_r(hs')^{-1} + c_1 - q(l't)^{-1}[s', l't]\} \quad (r = 1, 2), \tag{27}$$

and hence, if 10 holds, by (14), (21), and Lemma 4,

$$\begin{aligned} |\delta_r(t)| &\leqslant \left| \sum_{\substack{s' \\ 1a, 3a, s' \leqslant th^{-1}}} \psi\{a_r(hs')^{-1} + c_1 - q(l't)^{-1}[s', l't]\} \right| + k \\ &\leqslant k + A_3 n_1^{\frac{1}{3}} (l't)^{\frac{1}{6}} d^{\frac{1}{2}+\epsilon}(l't) \log^{\frac{7}{3}}(2l't)(q, l't)^{\frac{1}{6}} \\ &\qquad + A_4 (l't)^{\frac{3}{4}} d^{\frac{3}{4}+\epsilon}(l't)(q, l't)^{\frac{1}{4}} \log^3(2l't) \\ &= O\{n^{\frac{7}{8}} \log^{\frac{7}{3}} n d^{\frac{1}{2}+\epsilon}(t)\} \quad (r = 1, 2), \end{aligned} \tag{28}$$

for $$(q, l't) \leqslant l'(q, t) = l'(k_1 + tu_0, t) = l'(k_1, t) = O(1).$$

By (14), (16), (17), (22), and (28),

$$(29)\quad S-\sum_{\substack{s,\,t\\ 1,\,3,\,5,\,8,\,10}}(n_1-k_1-mt^2)(lst)^{-1}=\sum_{5,\,10}[-\delta_1(t)+\delta_2(t)]=O(n^{\frac{11}{12}}\log^{\frac{17}{6}+\epsilon}n).$$

Similarly, if

$$(30)\quad S'=S'(n_1,\,k_1,\,s_0,\,t_0,\,y_0,\,u_0)=\sum_{\substack{s,\,t,\,y,\,u\\ 1,\,3,\,4,\,5,\,6,\,7,\,8',\,9}}1,$$

then

$$(31)\quad S'=\sum_{\substack{s,\,t\\ 1,\,3,\,5,\,8',\,10}}(n_1-k_1-mt^2)(lst)^{-1}+O(n^{\frac{11}{12}}\log^{\frac{17}{6}+\epsilon}n).$$

5. *Application of the results of section 4.*

Henceforth let

$$(32)\quad l=4,$$

and let χ denote the non-principal character of residues mod 4. Then it is known that

$$(33)\quad r(h)=4\sum_{\substack{z,\,u\\ zu=h}}\chi(z).$$

Hence

$$(34)\quad \tfrac{1}{16}\sum_{h=1}^{n}r(h)r(h+k)=\sum_{\substack{x,\,y,\,z,\,u\\ 12,\,13}}\chi(x)\chi(z)=T+O(n^{\frac{1}{2}+\epsilon}),$$

where

$$(35)\quad T=\sum_{\substack{x,\,y,\,z,\,u\\ 12,\,13,\,x\neq y,\,z\neq u}}\chi(x)\chi(z)=\sum_{\substack{x,\,y,\,z,\,u\\ 12,\,13,\,14,\,15}}[\chi(x)+\chi(y)][\chi(z)+\chi(u)].$$

Putting

$$(36)\quad T(m)=\sum_{\substack{x,\,y,\,z,\,u\\ 12,\,13,\,14,\,15,\,16}}[\chi(x)+\chi(y)][\chi(z)+\chi(u)]$$

$$=\sum_{\substack{s,\,t,\,y,\,u\\ 12',\,13',\,14',\,15',\,1}}[\chi(ms)+\chi(y)][\chi(mt)+\chi(u)],$$

we have, by (35),

$$(37)\quad T=\sum_{m\ k}T(m).$$

Now, by (36) and (32),

$$T(m) = \sum_{\substack{s_0, t_0, y_0, u_0 \\ 24}} [\chi(ms_0)+\chi(y_0)][\chi(mt_0)+\chi(u_0)]\, U, \tag{38}$$

where

$$U = \sum_{\substack{s, t, y, u \\ 25}} 1 = V+W \tag{39}$$

and

$$V = \sum_{\substack{s, t, y, u \\ 25, 22}} 1, \quad W = \sum_{\substack{s, t, y, u \\ 25, \overline{22}}} 1. \tag{40}$$

Now, if 12′ and 22 hold, then 13′, 14′, and 15′ are together equivalent to 23. Hence, by (40) and (13),

$$V = S\,[(n+k)\,m^{-1},\ km^{-1},\ s_0,\ t_0,\ y_0,\ u_0]. \tag{41}$$

Similarly, interchanging s with t and y with u, we obtain, from (40) and (30),

$$W = S'(nm^{-1},\ -km^{-1},\ t_0,\ s_0,\ u_0,\ y_0). \tag{42}$$

Now let

$$T_1(m) = \sum_{\substack{s_0, t_0, y_0, u_0 \\ 24}} \chi(s_0)\,\chi(t_0)\, U, \tag{43}$$

$$T_2(m) = \sum_{\substack{s_0, t_0, y_0, u_0 \\ 24}} \chi(s_0)\,\chi(u_0)\, U, \tag{44}$$

$$T_3(m) = \sum_{\substack{s_0, t_0, y_0, u_0 \\ 24}} \chi(t_0)\,\chi(y_0)\, U, \tag{45}$$

and

$$T_4(m) = \sum_{\substack{s_0, t_0, y_0, u_0 \\ 24}} \chi(y_0)\,\chi(u_0)\, U. \tag{46}$$

Then, by (38),

$$T(m) = \chi^2(m)\, T_1(m)+\chi(m)\,[T_2(m)+T_3(m)]+T_4(m). \tag{47}$$

Now, by (41), (13), (29), and (32),

$$V = \tfrac{1}{4} \sum_{\substack{s, t \\ 1, 3, 5, 22, 10a}} (nm^{-1}-mt^2)(st)^{-1}+O(n^{\frac{11}{12}} \log^{\frac{17}{6}+\epsilon} n). \tag{48}$$

Similarly, again interchanging s with t, and observing that

$$\sum_{\substack{s,t \\ \overline{22},\ 10b}} (st)^{-1} = O(\log^2 n),$$

we obtain, from (42), (30), (31), and (32),

$$W = \tfrac{1}{4} \sum_{\substack{s,\ t \\ 1,\ 3,\ 5,\ \overline{22},\ 10b}} (nm^{-1} - ms^2)(st)^{-1} + O(n^{\frac{11}{12}} \log^{\frac{17}{6}+\epsilon} n).$$

Hence, by (39) and (48),

$$U = \tfrac{1}{4} nm^{-1} \sum_{\substack{s,\ t \\ 1,\ 3,\ 5,\ 10a,\ 10b}} (st)^{-1} - \tfrac{1}{4} m \sum_{\substack{s,t \\ 1,\ 3,\ 5,\ 22,\ 10a}} s^{-1}t - \tfrac{1}{4} m \sum_{\substack{s,\ t \\ 1,\ 3,\ 5,\ \overline{22},\ 10b}} st^{-1} + O(n^{\frac{11}{12}} \log^{\frac{17}{6}+\epsilon} n). \tag{49}$$

6. *Calculation of* $T_1(m)$.

By (43) and (49),

$$4T_1(m) = nm^{-1} X_1 - m(Y_1 + Z_1) + O(n^{\frac{11}{12}} \log^{\frac{17}{6}+\epsilon} n), \tag{50}$$

where

$$X_1 = \sum_{\substack{s_0,\ t_0,\ y_0,\ u_0 \\ 24}} \chi(s_0)\chi(t_0) \sum_{\substack{s,\ t \\ 1,\ 3,\ 5,\ 10a,\ 10b}} (st)^{-1} = \sum_{\substack{s,\ t \\ 1,\ 10a,\ 10b}} \chi(st)(st)^{-1} \sum_{\substack{y_0,\ u_0 \\ 19,\ 20,\ 21a}} 1, \tag{51}$$

$$Y_1 = \sum_{\substack{s,\ t \\ 1,\ 22,\ 10a}} \chi(st)\, s^{-1}t \sum_{\substack{y_0,\ u_0 \\ 19,\ 20\ 21a}} 1, \tag{52}$$

and

$$Z_1 = \sum_{\substack{s,\ t \\ 1,\ \overline{22},\ 10b}} \chi(st)\, st^{-1} \sum_{\substack{y_0,\ u_0 \\ 19,\ 20,\ 21a}} 1. \tag{53}$$

Now

$$\sum_{\substack{y_0,\ u_0 \\ 19,\ 20,\ 21a}} 1 = \sum_{\substack{u_0 \\ 20}} \sum_{\substack{y_0 \\ 19,\ 21a}} 1 = 4 \quad (s \text{ odd}). \tag{54}$$

Also, since

$$\sum_{\substack{h,\ x,\ y \\ hx=s,\ hy=t}} \mu(h) = \begin{cases} 1 & [(s,t)=1], \\ 0 & [(s,t)>1] \end{cases} \tag{55}$$

we have

$$(56)\qquad \sum_{\substack{s,\,t\\ 1,\,10a,\,10b}} \chi(st)(st)^{-1} = \sum_{h<\sqrt{nm^{-1}}} \mu(h)\chi^2(h)h^{-2} \sum_{\substack{x\\ 10d}} \chi(x)x^{-1} \sum_{\substack{y\\ 10c}} \chi(y)y^{-1}$$

$$= \sum_{h<\sqrt{nm^{-1}}} \mu(h)\chi^2(h)h^{-2}[\tfrac{1}{4}\pi + O(hn^{-\frac{1}{2}})]^2$$

$$= \tfrac{1}{16}\pi^2 \sum_{h=1}^{\infty} \mu(h)\chi^2(h)h^{-2} + O(n^{-\frac{1}{2}}\log n)$$

$$= \tfrac{1}{2} + O(n^{-\frac{1}{2}}\log n).$$

By (51), (54), and (56),

$$(57)\qquad X_1 = 2 + O(n^{-\frac{1}{2}}\log n).$$

By (55),

$$\sum_{\substack{s,\,t\\ 1,\,22,\,10a}} \chi(st)s^{-1}t = \sum_{h<\sqrt{nm^{-1}}} \mu(h)\chi^2(h) \sum_{\substack{y\\ 10c}} \chi(y)y \sum_{\substack{x\\ 22a}} \chi(x)x^{-1}$$

$$= \sum_{h<\sqrt{nm^{-1}}} \mu(h)\chi^2(h) \sum_{\substack{y\\ 10c}} \chi(y)y[\tfrac{1}{4}\pi + O(y^{-1})]$$

$$= \sum_{h<\sqrt{nm^{-1}}} O(n^{\frac{1}{2}}h^{-1}) = O(n^{\frac{1}{2}}\log n),$$

and similarly

$$\sum_{\substack{s,\,t\\ 1,\,22,\,10b}} \chi(st)st^{-1} = O(n^{\frac{1}{2}}\log n).$$

Hence, by (52), (53), and (54),

$$(58)\qquad Y_1 + Z_1 = O(n^{\frac{1}{2}}\log n).$$

By (50), (57), and (58),

$$(59)\qquad T_1(m) = \tfrac{1}{2}nm^{-1} + O(n^{\frac{11}{12}}\log^{17}{}^{+\epsilon} n).$$

7. *Inequalities for $T_2(m)$ and $T_3(m)$.*

By (44) and (49),

$$(60)\qquad 4T_2(m) = nm^{-1}X_2 - m(Y_2 + Z_2) + O(n^{\frac{11}{12}}\log^{17}{}^{+\epsilon} n),$$

where

$$X_2 = \sum_{\substack{s_0, t_0, y_0, u_0 \\ 24}} \chi(s_0)\chi(u_0) \sum_{\substack{s, t \\ 1, 3, 5, 10a, 10b}} (st)^{-1}$$

$$= \sum_{\substack{s, t \\ 1, 10a, 10b}} \chi(s)(st)^{-1} \sum_{\substack{y_0, u_0 \\ 19, 20, 21a}} \chi(u_0), \tag{61}$$

$$Y_2 = \sum_{\substack{s, t \\ 1, 22, 10a}} \chi(s)s^{-1}t \sum_{\substack{y_0, u_0 \\ 19, 20, 21a}} \chi(u_0), \tag{62}$$

and

$$Z_2 = \sum_{\substack{s, t \\ 1, 22, 10b}} \chi(s)st^{-1} \sum_{\substack{y_0, u_0 \\ 19, 20, 21a}} \chi(u_0). \tag{63}$$

Now

$$\sum_{\substack{y_0, u_0 \\ 19, 20, 21a}} \chi(u_0) = \sum_{\substack{u_0 \\ 20}} \chi(u_0) \sum_{\substack{y_0 \\ 19, 21a}} 1 = \sum_{\substack{u_0 \\ 20}} \chi(u_0) = 0 \quad (s \text{ odd}). \tag{64}$$

Hence $X_2 = Y_2 = Z_2 = 0$, and hence, by (60),

$$T_2(m) = O(n^{\frac{11}{12}} \log^{\frac{17}{6}+\epsilon} n). \tag{65}$$

Similarly

$$T_3(m) = O(n^{\frac{11}{12}} \log^{\frac{17}{6}+\epsilon} n). \tag{66}$$

8. *Calculation of* $T_4(m)$.

By (46) and (49),

$$4T_4(m) = nm^{-1}X_4 - m(Y_4 + Z_4) + O(n^{\frac{11}{12}} \log^{\frac{17}{6}+\epsilon} n), \tag{67}$$

where

$$X_4 = \sum_{\substack{s, t \\ 1, 10a, 10b}} (st)^{-1} \sum_{\substack{y_0, u_0 \\ 19, 20, 21a}} \chi(y_0)\chi(u_0), \tag{68}$$

$$Y_4 = \sum_{\substack{s, t \\ 1, 22, 10a}} s^{-1}t \sum_{\substack{y_0, u_0 \\ 19, 20, 21a}} \chi(y_0)\chi(u_0), \tag{69}$$

and

$$Z_4 = \sum_{\substack{s, t \\ 1, 22, 10b}} st^{-1} \sum_{\substack{y_0, u_0 \\ 19, 20, 21a}} \chi(y_0)\chi(u_0). \tag{70}$$

Now, if s is even, then $\sum_{\substack{y_0 \\ 19, 21a}} \chi(y_0) = 0$, and hence

$$\sum_{\substack{y_0, u_0 \\ 19, 20, 21a}} \chi(y_0)\chi(u_0) = \sum_{\substack{u_0 \\ 20}} \chi(u_0) \sum_{\substack{y_0 \\ 19, 21a}} \chi(y_0) = 0 \quad (s \text{ even}). \tag{71}$$

Similarly

$$\sum_{\substack{y_0,\ u_0 \\ 19,\ 20,\ 21a}} \chi(y_0)\chi(u_0) = 0 \quad (t \text{ even}). \tag{72}$$

Now s, t, y_0, u_0, and km^{-1} cannot all be odd and satisfy 21a. Hence, by (71) and (72),

$$\sum_{\substack{y_0,\ u_0 \\ 19,\ 20,\ 21a}} \chi(y_0)\chi(u_0) = 0 \quad (km^{-1} \text{ odd}). \tag{73}$$

and hence, by (67), (68), (69), and (70),

$$T_4(m) = O(n^{\frac{11}{12}} \log^{\frac{17}{6}+\epsilon} n) \quad (km^{-1} \text{ odd}). \tag{74}$$

Now let km^{-1} be even. Then, if s, t, and u_0 are odd, it follows from 21a that

$$\chi(y_0)\chi(u_0) = \chi(st)\chi(sy_0)\chi(tu_0) = (-1)^{k(2m)^{-1}}\chi(st).$$

Hence

$$\begin{aligned} \sum_{\substack{y_0,\ u_0 \\ 19,\ 20,\ 21a}} \chi(y_0)\chi(u_0) &= (-1)^{k(2m)^{-1}}\chi(st) \sum_{\substack{u_0 \\ 20,\ u_0 \text{ odd}}} \sum_{\substack{y_0 \\ 19,\ 21a}} 1 \\ &= \tfrac{1}{2}(-1)^{k(2m)^{-1}}\chi(st) \sum_{\substack{y_0,\ u_0 \\ 19,\ 20,\ 21a}} 1. \end{aligned} \tag{75}$$

Although it has been assumed in the proof of (75) that s and t are odd, it now follows from (71) and (72) that the formula holds without this restriction.

By (68), (75), (51), and (57),

$$X_4 = (-1)^{k(2m)^{-1}} + O(n^{-\frac{1}{2}} \log n) \quad (km^{-1} \text{ even}); \tag{76}$$

by (69), (70), (75), (52), (53), and (58),

$$Y_4 + Z_4 = O(n^{\frac{1}{2}} \log n). \tag{77}$$

By (67), (76), and (77),

$$T_4(m) = \tfrac{1}{4}nm^{-1}(-1)^{k(2m)^{-1}} + O(n^{\frac{11}{12}} \log^{\frac{17}{6}+\epsilon} n) \quad (km^{-1} \text{ even}). \tag{78}$$

Putting

$$A(m) = \begin{cases} (2m)^{-1}(-1)^{k(2m)^{-1}} & (km^{-1} \text{ even}), \\ 0 & (km^{-1} \text{ odd}), \end{cases} \tag{79}$$

we have, by (74) and (78),

$$T_4(m) = \tfrac{1}{2} n A(m) + O(n^{1\frac{1}{2}} \log^{\frac{1}{6}\frac{7}{6}+\epsilon} n). \tag{80}$$

9. *Proof of* (1).

By (47), (59), (65), (66), and (80),

$$T(m) = \tfrac{1}{2} n [\chi^2(m) m^{-1} + A(m)] + O(n^{1\frac{1}{2}} \log^{\frac{1}{6}\frac{7}{6}+\epsilon} n). \tag{81}$$

Now

$$k \sum_{m|k} \chi^2(m) m^{-1} = \sum_{\substack{h|k \\ kh^{-1} \text{ odd}}} h = (-1)^k \sum_{\substack{h|k \\ kh^{-1} \text{ odd}}} (-1)^h h, \tag{82}$$

and, by (79),

$$k \sum_{m|k} A(m) = k \sum_{\substack{r|k \\ r \text{ even}}} r^{-1}(-1)^{kr^{-1}} = (-1)^k \sum_{\substack{h|k \\ kh^{-1} \text{ even}}} (-1)^h h. \tag{83}$$

By (82), (83), and (2),

$$\sum_{m|k} [\chi^2(m) m^{-1} + A(m)] = k^{-1} c_k,$$

which, together with (34), (37), and (81), proves (1).

ON THE DYNAMICAL EQUATIONS OF THE TIDES

Part IV.—Flat seas of uniform depth.

By J. Proudman.

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1. In Parts II and III of this series of papers* the dynamical equations of the tides for a landlocked basin were transformed into an infinite set of ordinary linear differential equations in an infinite sequence of parameters which are functions only of the time. The equations are of Lagrange's type and the parameters play the part of principal coordinates of the mechanical system formed by the sea or ocean.

The present part is concerned with the solution of these equations when the basin is of uniform depth and of lateral extent so small that the curvature of the earth may be neglected. The work and results can be extended immediately to the case of an ocean of any extent in which the depth measured parallel to the earth's axis is constant.

A definite process is exhibited by which the solution of the equations can be approached as the limit of a sequence of solutions of a finite set of equations in a finite number of coordinates. Also, a proof is given of part of a general expansion-theorem which was enunciated in a previous paper† but only there proved for a particular case.

The argument consists in the application of the results of two papers‡ (by the present author and F. E. Mercer) on the oscillations of a rotating mechanical system of infinite freedom, but before this application can be

* *Proc. London Math. Soc.* (2), 18 (1916), 1–68.
† *Ibid.* (2), 29 (1928), 527–536.
‡ *Ibid.* (2), 27 (1926), 1–38; 28 (1927), 102–108.

made it is necessary to eliminate from the tidal equations the infinite set of coordinates which correspond to zero coefficients of stability.

2. *Equations of the tides.*

We shall denote by:

g the acceleration of gravity,

ω the vertical component of the earth's angular velocity,

h the depth of the sea,

x, y the Cartesian coordinates in the plane of the sea,

ξ, η the Cartesian components of the horizontal displacement of a particle of water,

ζ the elevation of the free surface of the sea,

z the "equilibrium-value" of ζ,

t the time.

It is shown in Part II that there exist two functions of position and time, ϕ and ψ, such that

$$\xi = -\frac{\partial \phi}{\partial x} - \frac{1}{h}\frac{\partial \psi}{\partial y}, \tag{2.11}$$

$$\eta = -\frac{\partial \phi}{\partial y} + \frac{1}{h}\frac{\partial \psi}{\partial x}, \tag{2.12}$$

$$\zeta = h\nabla^2 \phi, \tag{2.13}$$

where ∇^2 denotes $\partial^2/\partial x^2 + \partial^2/\partial y^2$.

Let

$$\phi_1, \phi_2, \ldots, \phi_r, \ldots,$$

$$\psi_1, \psi_2, \ldots, \psi_r, \ldots$$

denote respectively solutions of the differential equations

$$h\nabla^2 \phi_r + \lambda_r \phi_r = 0, \quad \frac{1}{h}\nabla^2 \psi_r + \mu_r \psi_r = 0,$$

with the coastal conditions

$$\frac{\partial \phi_r}{\partial n} = 0, \quad \psi_r = 0,$$

which correspond respectively to the constants

$$\lambda_1, \lambda_2, \ldots, \lambda_r, \ldots,$$

$$\mu_1, \mu_2, \ldots, \mu_r, \ldots,$$

each of these sequences being arranged in non-decreasing order of magnitude. We may take

$$\iint \phi_r \phi_s \, dS = 0 \quad (s \neq r), \quad = \frac{1}{\lambda_r} \quad (s = r),$$

$$\iint \psi_r \psi_s \, dS = 0 \quad (s \neq r), \quad = \frac{1}{\mu_r} \quad (s = r).$$

All double integrals are supposed to be taken over the whole area of the sea, dn denotes an element of the outward drawn normal to the coast, and ds an element of the coast to the left of dn.

It is further shown in Part II that ϕ and ψ may be expanded in the series

$$\phi = \sum_{r=1}^{\infty} p_r \phi_r, \qquad \psi = \sum_{r=1}^{\infty} p_{-r} \psi_r, \tag{2.2}$$

and that

$$h \iint (\dot{\xi}^2 + \dot{\eta}^2) \, dS = \sum_{r=-\infty}^{\infty} \dot{p}_r{}^2, \tag{2.31}$$

$$g \iint \zeta^2 \, dS = \sum_{r=1}^{\infty} \nu_r{}^2 p_r{}^2, \tag{2.32}$$

$$h \iint (\xi \dot{\eta} - \eta \dot{\xi}) \, dS = \sum_{r,\, s=-\infty}^{\infty} \beta_{r,s} \dot{p}_r p_s, \tag{2.33}$$

where $\nu_r{}^2 = g\lambda_r$ and

$$\left.\begin{aligned} \beta_{r,s} &= -h \int \phi_r \frac{\partial \phi_s}{\partial s} \, ds, \\ \beta_{r,-s} &= \frac{\lambda_r}{h} \iint \phi_r \psi_s \, dS, \\ \beta_{-r,-s} &= 0, \end{aligned}\right\} \tag{2.4}$$

for positive values of r and s, while the equations of motion become

$$\ddot{p}_r+2\omega \sum_{s=-\infty}^{\infty} \beta_{r,s}\dot{p}_s+\nu_r^2 p_r = P_r, \tag{2.51}$$

$$\ddot{p}_{-r}+2\omega \sum_{s=1}^{\infty} \beta_{-r,s}\dot{p}_s = 0, \tag{2.52}$$

where
$$P_r = -g\lambda_r \iint z\phi_r\, dS. \tag{2.6}$$

It follows from Part II that

$$\beta_{s,r} = -\beta_{r,s} \tag{2.7}$$

and
$$\sum_{\sigma=-\infty}^{\infty} \beta_{r,\sigma}^2 \leqslant 1, \tag{2.8}$$

for all integral values of r and s, positive and negative.

All the series of this section are absolutely convergent.

3. *Elimination of coordinates.*

From Part II we have

$$\sum_{s=1}^{\infty} \beta_{-t,s}p_s = -\iint \left(\frac{\partial\phi}{\partial x}\frac{\partial\psi_t}{\partial x}+\frac{\partial\phi}{\partial y}\frac{\partial\psi_t}{\partial y}\right) dS$$

$$= \frac{1}{h}\iint \zeta\psi_t\, dS, \tag{3.1}$$

a Fourier-coefficient with respect to the functions ψ_t, so that the series

$$\sum_{t=1}^{\infty} \left(\sum_{s=1}^{\infty} \beta_{-t,s}p_s\right)^2$$

is absolutely convergent. Also, since $\beta_{r,-t}$ is a Fourier coefficient with respect to the functions ψ_t, the series

$$\sum_{t=1}^{\infty} \beta_{r,-t}\left(\sum_{s=1}^{\infty} \beta_{-t,s}p_s\right)$$

is absolutely convergent.

We shall consider only solutions of the equations (2.52) which make

$$\dot{p}_{-r}+2\omega \sum_{s=1}^{\infty} \beta_{-r,s}\, p_s = 0, \tag{3.2}$$

for $r = 1, 2, 3, \ldots$; such integrals are obviously true for any periodic motion. Then we have

$$2\omega \sum_{\rho=1}^{\infty} \beta_{r,-\rho}\, \dot{p}_{-\rho} = -4\omega^2 \sum_{\rho=1}^{\infty} \beta_{r,-\rho} \left(\sum_{s=1}^{\infty} \beta_{-\rho,s}\, p_s \right)$$

$$= 2\omega^2 \frac{\partial}{\partial p_r} \sum_{\rho=1}^{\infty} \left(\sum_{s=1}^{\infty} \beta_{-\rho,s}\, p_s \right)^2,$$

in view of the absolute convergence of these series. The equations (2.51) may therefore be written

$$\ddot{p}_r+2\omega \sum_{s=1}^{\infty} \beta_{r,s}\, \dot{p}_s+\nu_r^2 p_r+2\omega^2 \frac{\partial}{\partial p_r} \sum_{\rho=1}^{\infty} \left(\sum_{s=1}^{\infty} \beta_{-\rho,s}\, p_s \right)^2 = P_r,$$

for $r = 1, 2, 3, \ldots$, or

$$\frac{d}{dt}\left(\frac{\partial R}{\partial \dot{p}_r}\right)-\frac{\partial R}{\partial p_r} = P_r, \tag{3.3}$$

where

$$R = T(\dot{p})+2\omega M(p, \dot{p})-V(p),$$

$$2T(\dot{p}) = \sum_{r=1}^{\infty} \dot{p}_r^2, \tag{3.41}$$

$$2M(p, \dot{p}) = \sum_{r,\, s=1}^{\infty} \beta_{r,s}\, p_s\, \dot{p}_r, \tag{3.42}$$

$$2V(p) = \sum_{r=1}^{\infty} \nu_r^2 p_r^2+4\omega^2 \sum_{r=1}^{\infty} \left(\sum_{s=1}^{\infty} \beta_{-r,s}\, p_s \right)^2. \tag{3.43}$$

The equations (3.3) only contain the coordinates p_r for positive values of r*.

4. *Certain inequalities.*

It will be convenient to admit complex values of the variables ζ, u, v, p_s; we shall denote their conjugates by $\bar{\zeta}, \bar{u}, \bar{v}, \bar{p}_s$ respectively.

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* *Cf.* Part I, pp. 13–14.

By Parseval's theorem, in view of (3.1) and (2.32), we have respectively

$$\sum_{t=1}^{\infty} \mu_t \left(\sum_{s=1}^{\infty} \beta_{-t,s} p_s \right) \left(\sum_{s=1}^{\infty} \beta_{-t,s} \bar{p}_s \right) = \frac{1}{h^2} \iint \zeta \bar{\zeta} \, dS,$$

$$\sum_{s=1}^{\infty} \nu_s^2 p_s \bar{p}_s = g \iint \zeta \bar{\zeta} \, dS,$$

so that

$$\mu_1 h^2 \sum_{t=1}^{\infty} \left(\sum_{s=1}^{\infty} \beta_{-t,s} p_s \right) \left(\sum_{s=1}^{\infty} \beta_{-t,s} \bar{p}_s \right) \leqslant \frac{1}{g} \sum_{s=1}^{\infty} \nu_s^2 |p_s|^2, \tag{4.1}$$

whatever the sequence (p) may be.

Again,

$$\left\{ h \iint (\xi \dot{\eta} - \eta \dot{\xi}) \, dS \right\}^2 \leqslant \left\{ h \iint (|\xi \dot{\eta}| + |\eta \dot{\xi}|) \, dS \right\}^2$$

$$\leqslant \left\{ h \iint (|\xi|^2 + |\eta|^2)^{\frac{1}{2}} (|\dot{\xi}|^2 + |\dot{\eta}|^2)^{\frac{1}{2}} \, dS \right\}^2$$

$$\leqslant \iint h (|\xi|^2 + |\eta|^2) \, dS \iint h (|\dot{\xi}|^2 + |\dot{\eta}|^2) \, dS,$$

on using Cauchy's and Schwarz's inequalities. Now, on using (2.11), (2.12), and (2.2), we get

$$h \iint (|\xi|^2 + |\eta|^2) \, dS = h \iint (\xi \bar{\xi} + \eta \bar{\eta}) \, dS$$

$$= h \iint \left(\frac{\partial \phi}{\partial x} \frac{\partial \bar{\phi}}{\partial x} + \frac{\partial \phi}{\partial y} \frac{\partial \bar{\phi}}{\partial y} \right) dS + \frac{1}{h} \iint \left(\frac{\partial \psi}{\partial x} \frac{\partial \bar{\psi}}{\partial x} + \frac{\partial \psi}{\partial y} \frac{\partial \bar{\psi}}{\partial y} \right) dS$$

$$= \sum_{r=-\infty}^{\infty} p_r \bar{p}_r,$$

and hence

$$\left| \sum_{r,\, s=-\infty}^{\infty} \beta_{r,s} \dot{p}_r p_s \right|^2 \leqslant \sum_{s=-\infty}^{\infty} |p_s|^2 \sum_{r=-\infty}^{\infty} |\dot{p}_r|^2. \tag{4.2}$$

This inequality is true for all sequences (p_r), $(\dot{p}_r)$.

5. *Theory of oscillations.*

By virtue of the conditions (4.1) and (4.2) the solution of the equations (3.3) comes under the theory worked out in the two papers on oscillations* referred to on page 293.

* To see this, write $x_s = \nu_s p_s$ and $\beta_{r,s}/(\nu_r \nu_s)$ for $\beta_{r,s}$. The restriction (1.22) of the first paper is rendered unnecessary by the results of the second paper.

If we neglect all coordinates p_r and forces P_r for which r is greater than an integer N, we have the equations of a mechanical system with a finite number of degrees of freedom; and when we take $P_r = 0$, for all values of r, the equations refer to the free motion of this system. In the solution of these equations, whether for free motion or for forced motion, the expression for each coordinate is a function of N, and it is a result of the theory now being quoted that the corresponding solutions of the equations (3.3) are given by the limits of these functions as $N \to \infty$.

In this way we arrive at an infinite sequence of free speeds

$$\sigma_1, \sigma_2, \ldots, \sigma_n, \ldots$$

which may be taken as positive and which are such that

$$\sigma_n \to \infty \quad \text{as} \quad n \to \infty,$$

when they are arranged in non-decreasing order of magnitude. The free tides are then given by

$$p_r = C_n a_{n,r} e^{i(\sigma_n t + \epsilon_n)}$$

for $\pm n = 1, 2, 3, \ldots$, $r = 1, 2, 3, \ldots$, where C_n and ϵ_n are arbitrary constants; and the sequence (a_n) may be chosen to satisfy

$$\left.\begin{aligned} \sigma_m \sigma_n T(a_m, a_{-n}) + V(a_m, a_{-n}) &= 0 \quad (m \neq n) \\ &= 1 \quad (m = n) \end{aligned}\right\}, \tag{5.11}$$

where

$$2T(p, p') = \sum_{r=1}^{\infty} p_r p_r', \tag{5.12}$$

and

$$2V(p, p') = \sum_{r=1}^{\infty} \nu_r^2 p_r p_r' + 4\omega^2 \sum_{r=1}^{\infty} \left(\sum_{s=1}^{\infty} \beta_{-r,s} p_s \right) \left(\sum_{s=1}^{\infty} \beta_{-r,s} p_s' \right). \tag{5.13}$$

We have also the following transformation theorem for any arbitrary sequences (p), $(\dot{p})$ which make $T(\dot{p})$ and $V(p)$ finite. If

$$\xi_n = -i\sigma_n T(a_{-n}, \dot{p}) + V(a_{-n}, p), \tag{5.2}$$

for $\pm n = 1, 2, 3, \ldots$, then

$$p_r = \sum_{n=-\infty}^{\infty} a_{n,r} \xi_n, \tag{5.31}$$

$$\dot{p}_r = \sum_{n=-\infty}^{\infty} i\sigma_n a_{n,r} \xi_n, \tag{5.32}$$

for all positive values of r, and

$$2\sum_{n=1}^{\infty}\xi_n\xi_{-n}=T(\dot{p})+V(p), \tag{5.41}$$

$$\sum_{n=-\infty}^{}|\xi_n|^2=T(\dot{p},\bar{\dot{p}})+V(p,\bar{p}). \tag{5.42}$$

In this transformation consider the special case in which

$$p_r=0,\quad \dot{p}_r=0\ (r\neq s),\quad \dot{p}_s=1,$$

where s is any positive integer. Then we have, from (5.2),

$$\xi_n=-\tfrac{1}{2}i\sigma_n\alpha_{-n,s},$$

for $\pm n=1, 2, 3, \ldots$, and on substituting into (5.31) and (5.32) we obtain identities which yield

$$\tfrac{1}{2}\sum_{n=-\infty}^{\infty}\sigma_n\alpha_{n,r}\alpha_{-n,s}=0, \tag{5.51}$$

$$\tfrac{1}{2}\sum_{n=-\infty}^{\infty}\sigma_n{}^2\alpha_{n,r}\alpha_{-n,s}=0\ (r\neq s),\quad =1\ (r=s), \tag{5.52}$$

for positive values of r and s.

On using (2.8) and (5.52) we have

$$\tfrac{1}{2}\sum_{n=-\infty}^{\infty}\sigma_n{}^2\left|\sum_{\sigma=1}^{N}\beta_{\rho,\sigma}\alpha_{n,\sigma}\right|^2=\tfrac{1}{2}\sum_{n=-\infty}^{\infty}\sigma_n{}^2\left(\sum_{r=1}^{N}\beta_{\rho,r}\alpha_{n,r}\right)\left(\sum_{s=1}^{N}\beta_{\rho,s}\alpha_{-n,s}\right)$$

$$=\sum_{\sigma=1}^{N}\beta_{\rho,\sigma}^2\leqslant 1, \tag{5.6}$$

for all integral values of ρ, positive or negative.

The equations of free motion (3.3) give

$$-\sigma_n{}^2\alpha_{n,r}+2\omega i\sigma_n\sum_{s=1}^{\infty}\beta_{r,s}\alpha_{n,s}+\nu_r{}^2\alpha_{n,r}-4\omega^2\sum_{\rho=1}^{\infty}\beta_{r,-\rho}\left(\sum_{s=1}^{\infty}\beta_{-\rho,s}\alpha_{n,s}\right)=0, \tag{5.7}$$

for $r=1, 2, \ldots, \pm n=1, 2, 3, \ldots$.

6. *Restoration of eliminated coordinates.*

From (3.2) and (5.13) we have

$$2V(p,p')=\sum_{r=1}^{\infty}{}'\nu_r{}^2p_rp_r'+\sum_{r=1}^{\infty}\dot{p}_{-r}\dot{p}'_{-r}, \tag{6.11}$$

and then
$$2V(p) = \sum_{r=1}^{\infty} \nu_r^2 p_r^2 + \sum_{r=1}^{\infty} \dot{p}_{-r}^2, \tag{6.12}$$

so that
$$T(p)+V(p) = \tfrac{1}{2} \sum_{r=-\infty}^{\infty} \dot{p}_r^2 + \tfrac{1}{2} \sum_{r=1}^{\infty} \nu_r^2 p_r^2, \tag{6.13}$$

the total energy of the system.

The equations of free motion (5.7) may be written in the form

$$-\sigma_n^2 a_{n,r} + 2\omega i \sigma_n \sum_{s=-\infty}^{\infty} \beta_{r,s} a_{n,s} + \nu_r^2 a_{n,r} = 0, \tag{6.21}$$

if we take
$$i\sigma_n a_{n,-r} + 2\omega \sum_{s=1}^{\infty} \beta_{-r,s} a_{n,s} = 0, \tag{6.22}$$

for $r = 1, 2, 3, \ldots$, $\pm n = 1, 2, 3, \ldots$.

The equation (5.11) may then be written

$$\left.\begin{aligned} \tfrac{1}{2}\sigma_m \sigma_n \sum_{r=-\infty}^{\infty} a_{m\,r} a_{-n,r} + \tfrac{1}{2} \sum_{r=1}^{\infty} \nu_r^2 a_{m,r} a_{-n,r} &= 0 \quad (m \neq n) \\ &= 1 \quad (m = n) \end{aligned}\right\}. \tag{6.3}$$

From (5.2) we have

$$\xi_n = -\tfrac{1}{2} i \sigma_n \sum_{r=-\infty}^{\infty} a_{-n,r} \dot{p}_r + \tfrac{1}{2} \sum_{r=1}^{\infty} \nu_r^2 a_{-n,r} p_r, \tag{6.4}$$

and from (5.41), (5.42),

$$2 \sum_{n=1}^{\infty} \xi_n \xi_{-n} = \tfrac{1}{2} \sum_{r=-\infty}^{\infty} \dot{p}_r^2 + \tfrac{1}{2} \sum_{r=1}^{\infty} \nu_r^2 p_r^2, \tag{6.51}$$

$$\sum_{n=-\infty}^{\infty} |\xi_n|^2 = \tfrac{1}{2} \sum_{r=-\infty}^{\infty} |\dot{p}_r|^2 + \tfrac{1}{2} \sum_{r=1}^{\infty} \nu_r^2 |p_r|^2. \tag{6.52}$$

From (5.31) we have

$$\sum_{s=1}^{N} \beta_{-r,s} p_s = \sum_{s=1}^{N} \beta_{-r,s} \sum_{n=-\infty}^{\infty} a_{n,s} \xi_n = \sum_{n=-\infty}^{\infty} \left(\xi_n \sum_{s=1}^{N} \beta_{-r,s} a_{n,s} \right). \tag{6.61}$$

In view of (5.6) and (6.52) it follows from (6.61), by a general theorem* on convergence, that

$$\sum_{s=1}^{\infty} \beta_{-r,s} p_s = \sum_{n=-\infty}^{\infty} \left(\xi_n \sum_{s=1}^{\infty} \beta_{-r,s} a_{n,s} \right), \tag{6.62}$$

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* See, for example, F. Riesz, *Les systèmes d'équations linéaires à une infinité d'inconnues*, 56.

and, on multiplication by -2ω, (6.62) is equivalent to

$$\dot{p}_{-r} = \sum_{n=-\infty}^{\infty} i\sigma_n a_{n,-r}\xi_n, \tag{6.63}$$

by (3.2) and (6.22).

The formula (6.63) is the extension of (5.32) to negative values of r. We may then immediately extend the formulae (5.51) and (5.52) to negative values of one or both of r and s. We may next extend the relation (5.6) to the form

$$\tfrac{1}{2}\sum_{n=-\infty}^{\infty}\sigma_n{}^2\left|\sum_{\sigma=-N}^{N}\beta_{r,\sigma}a_{n,\sigma}\right|^2 \leqslant \sum_{r=-N}^{N}\beta_{\rho,r}^2 \leqslant 1. \tag{6.7}$$

On using the extended form of (5.51) we have

$$\sum_{n=-\infty}^{\infty}\sigma_n\left(\sum_{\sigma=-N}^{N}\beta_{r,\sigma}a_{n,\sigma}\right)a_{-n,s} = \sum_{\sigma=-N}^{N}\beta_{r,\sigma}\sum_{n=-\infty}^{\infty}\sigma_n a_{n,\sigma}a_{-n,s} = 0,$$

from which we deduce by the general theorem already quoted, in view of the extended form of (5.52) and (6.7), that

$$\sum_{n=-\infty}^{\infty}\sigma_n\left(\sum_{\sigma=-\infty}^{\infty}\beta_{r,\sigma}a_{n,\sigma}\right)a_{-n,s} = 0. \tag{6.8}$$

From (6.21) and (6.8) we deduce that

$$\sum_{n=-\infty}^{\infty}\nu_r{}^2 a_{n,r}a_{-n,s} = \sum_{n=-\infty}^{\infty}\sigma_n{}^2 a_{n,r}a_{-n,s},$$

so that

$$\tfrac{1}{2}\sum_{n=-\infty}^{\infty}a_{n,r}a_{-n,s} = 0 \quad (r \neq s), \quad = \frac{1}{\nu_r{}^2} \quad (r = s). \tag{6.9}$$

7. *Expansion theorem.*

Let us take

$$\Phi_n = \sum_{r=1}^{\infty}a_{n,r}\phi_r, \tag{7.11}$$

$$\Psi_n = \sum_{r=1}^{\infty}a_{n,-r}\psi_r, \tag{7.12}$$

for $\pm n = 1, 2, 3, \ldots$, so that the free tides are given by

$$\phi = C_n \Phi_n e^{i(\sigma_n t + \epsilon_n)}, \quad \psi = C_n \Psi_n e^{i(\sigma_n t + \epsilon_n)}, \tag{7.13}$$

C_n and ϵ_n being again arbitrary constants.

On using (6.9) we have

$$\tfrac{1}{2} \sum_{n=-\infty}^{\infty} \left| \sum_{\rho=1}^{N} a_{n,\rho} \phi_\rho \right|^2 = \tfrac{1}{2} \sum_{n=-\infty}^{\infty} \left(\sum_{r=1}^{N} a_{n,r} \phi_r \right) \left(\sum_{s=1}^{N} a_{-n,s} \phi_s \right)$$

$$= \sum_{\rho=1}^{N} \frac{1}{\nu_\rho^2} \phi_\rho^2. \tag{7.2}$$

Now, from the theory of integral equations, the series (7.2) is known to be convergent, and hence we deduce the convergence of

$$\sum_{n=-\infty}^{\infty} |\Phi_n|^2.$$

Further, let ζ_n, u_n, v_n be defined so that, corresponding to (7.13), we have

$$\zeta = C_n \zeta_n e^{i(\sigma_n t + \epsilon_n)},$$

$$\dot{\xi} = C_n u_n e^{i(\sigma_n t + \epsilon_n)},$$

$$\dot{\eta} = C_n v_n e^{i(\sigma_n t + \epsilon_n)};$$

then, from (6.4), we have

$$\xi_n = \tfrac{1}{2} h \iint (\dot{\xi} u_{-n} + \dot{\eta} v_{-n}) \, dS + \tfrac{1}{2} g \iint \zeta \zeta_{-n} \, dS, \tag{7.31}$$

while, from (6.51) and (6.52),

$$2 \sum_{n=1}^{\infty} \xi_n \xi_{-n} = \tfrac{1}{2} h \iint (\dot{\xi}^2 + \dot{\eta}^2) \, dS + \tfrac{1}{2} g \iint \zeta^2 \, dS, \tag{7.32}$$

$$\sum_{n=-\infty}^{\infty} |\xi_n|^2 = \tfrac{1}{2} h \iint (|\dot{\xi}|^2 + |\dot{\eta}|^2) \, dS + \tfrac{1}{2} g \iint |\zeta|^2 \, dS. \tag{7.33}$$

From (5.31) we have

$$\sum_{r=1}^{N} p_r \phi_r = \sum_{r=1}^{N} \phi_r \left(\sum_{n=-\infty}^{\infty} a_{n,r} \xi_n \right) = \sum_{n=-\infty}^{\infty} \xi_n \left(\sum_{r=1}^{N} a_{n,r} \phi_r \right). \tag{7.41}$$

In view of (7.2) and (7.33) we deduce from (7.41), by the general theorem already quoted, that

$$\sum_{r=1}^{\infty} p_r \phi_r = \sum_{n=-\infty}^{\infty} \xi_n \left(\sum_{r=1}^{\infty} a_{n,r} \phi_r \right),$$

and, by (2.2) and (7.11), this is equivalent to

$$\phi = \sum_{n=-\infty}^{\infty} \xi_n \Phi_n. \qquad (7.42)$$

The equations (7.31), (7.32), (7.33), and (7.42) constitute part of the expansion theorem enunciated in the paper referred to on page 293.

A "STAR INEQUALITY" FOR HARMONIC FUNCTIONS

By R. M. GABRIEL.

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1. By $U(t)$, $U^*(t)$ we shall denote two functions defined in $-\pi \leqslant t \leqslant \pi$, and subject to the following conditions:

(i) The measure of the set of points t in $(-\pi, \pi)$ for which $H \leqslant U(t) \leqslant K$ is equal to the measure of the set of points t in $(-\pi, \pi)$ for which $H \leqslant U^*(t) \leqslant K$, and this holds for all H and all K.

(ii) $U^*(t) = U^*(-t)$ if $|t| \leqslant \pi$ and $U^*(t)$ is a decreasing function of t in $0 \leqslant t \leqslant \pi$†.

It will be seen that there exists a function $U^*(t)$ corresponding to every stretchwise constant function $U(t)$ and that $U^*(t)$, in this case, is uniquely defined save over a set of zero measure. There is no difficulty in extending these properties to the case in which $U(t)$ is measurable in $(-\pi, \pi)$.

We are now in a position to state our main result as follows.

THEOREM. *Suppose that $U(t)$ is any positive function of t, integrable in $(-\pi, \pi)$, that $\phi(x)$ is any function such that $\phi'(x) \geqslant 0$, $\phi''(x) \geqslant 0$ in $x \geqslant 0$, and that $u(r, \theta)$, $\bar{u}(r, \theta)$ are the Poisson integrals of $U(t)$ and $U^*(t)$ respectively, where $0 \leqslant r < 1$; then*

$$\int_0^{2\pi} \phi\{u(r, \theta)\}\, d\theta \leqslant \int_0^{2\pi} \phi\{\bar{u}(r, \theta)\}\, d\theta.$$

2. We shall require the following lemmas.

LEMMA 1. *If M_q, μ_{n_q}, c_n are positive numbers such that*

$$M_0 \geqslant M_1 \geqslant M_{-1} \geqslant M_2 \geqslant M_{-2} \ldots$$

$$c_0 \geqslant c_1 \geqslant c_{-1} \geqslant c_2 \geqslant c_{-2} \ldots,$$

† *Cf.* F. Riesz, "Sur une inégalité intégrale", *Journal London Math. Soc.*, 5 (1930), 162–168.

and such that

$$\sum_{q=1}^{k} \mu_{n_q} \leqslant \sum_{-\frac{1}{2}k < q \leqslant \frac{1}{2}k} M_q$$

holds for all k, then

$$\sum_{q=1}^{k} c_q \mu_{n_q} \leqslant \sum_{-\frac{1}{2}k < q \leqslant \frac{1}{2}k} c_q M_q \tag{2.1}$$

holds for all k.

We remark that the left-hand side of (2.1) takes as large a value as is possible if the μ_{n_q}'s are in the same order of magnitude as the c's. The lemma can easily be deduced by an appropriate use of Abel's lemma†.

LEMMA 2. *Let $\{a_n\}$, $\{b_n\}$, $(0 \leqslant n \leqslant N)$ be any two sequences of positive numbers such that $\{b_n\}$ is a decreasing sequence and*

$$\alpha_R = \sum_{n=0}^{R} a_n \geqslant \sum_{n=0}^{R} b_n = \beta_R$$

for all R $(0 \leqslant R \leqslant N)$; suppose further that $\phi(x)$ is any function of x such that $\phi'(x)$ is a positive increasing function of x for $x \geqslant 0$; then

$$\Sigma_1 = \sum_{n=0}^{N} \phi(a_n) \geqslant \sum_{n=0}^{N} \phi(b_n) = \Sigma_2.$$

For, if possible, suppose that there is a pair of sets of a's and b's such that $\Sigma_1 < \Sigma_2$, and let ρ be the largest number such that $\alpha_n = \beta_n$, for all n in $0 \leqslant n \leqslant \rho - 1$. It is clear that ρ exists and that $0 \leqslant \rho \leqslant N$, for otherwise we should have $\alpha_R = \beta_R$ $(0 \leqslant R \leqslant N)$, with the implication

$$a_n = b_n \quad (0 \leqslant n \leqslant N),$$

which implies

$$\Sigma_1 = \sum_{n=0}^{N} \phi(a_n) = \sum_{n=0}^{N} \phi(b_n) = \Sigma_2.$$

We note that $\alpha_R = \beta_R$ if $R \leqslant \rho - 1$, and hence we must have $a_\rho > b_\rho$ since $a_\rho \geqslant \beta_\rho$ is known to hold, and $a_\rho \neq b_\rho$.

Let us increase b_ρ by ξ, where $0 \leqslant \xi \leqslant a_\rho - b_\rho$, and let us decrease each b_n $(\rho+1 \leqslant n \leqslant N)$ by $\theta_n = \xi b_n / (\beta_N - \beta_\rho)$. It is easily verified that

$$\xi = \sum_{n=\rho+1}^{N} \theta_n \geqslant \sum_{n=\rho+1}^{R} \theta_n \quad (\rho+1 \leqslant R \leqslant N),$$

and that, in consequence, the conditions imposed upon the b's still hold if we carry out the proposed alterations in the values of the b_n's $(\rho \leqslant n \leqslant N)$.

† See R. M. Gabriel, "On the rearrangement of positive Fourier coefficients", *Proc. London Math. Soc.* (2), 33 (1931), 44, Lemma 3.

Now
$$\frac{d\Sigma_2}{d\xi} = \phi'(b_\rho+\xi) - \sum_{n=\rho+1}^{N} \frac{b_n \phi'(b_n+\theta_n)}{\beta_N-\beta_\rho}$$
$$\geqslant \phi'(b_\rho+\xi) - \frac{\phi'(b_\rho+\xi)}{\beta_N-\beta_\rho} \sum_{n=\rho+1}^{N} b_n$$
$$= 0,$$

since $b_n+\theta_n \leqslant b_\rho+\xi$ $(\rho < n \leqslant N)$, and $\phi'(x)$ is an increasing function of x. Thus, by carrying out these alterations of the values of the b's, we increase Σ_2 in the wide sense, without upsetting the conditions originally imposed. Further, if we take $\xi = a_\rho - b_\rho$, we make the new value of b_ρ equal to a_ρ, which has the effect of increasing ρ by unity. It is clear moreover that this process can be repeated without decreasing Σ_2 until we reach the stage $\rho = N+1$. This, however, implies that $a_n = b_n$ $(0 \leqslant n \leqslant N)$ and hence $\Sigma_1 = \Sigma_2$, giving us a contradiction, so that $\Sigma_2 > \Sigma_1$ is impossible. This completes the proof of the lemma.

LEMMA 3. *Suppose that we are given any h distinct points A_p, $a \exp(im_p\pi/N)$, where $p = 1, 2, \ldots, h$ $(h \leqslant 2N)$, and any k distinct points B_q, $b \exp(in_q\pi/N)$, where $q = 1, 2, \ldots, k$ $(k \leqslant 2N)$, and let us denote by $A_p{}^*$ $(1 \leqslant p \leqslant h)$ the h points $a \exp(im\pi/N)$, such that $-\frac{1}{2}h < m \leqslant \frac{1}{2}h$, and by $B_q{}^*$ $(1 \leqslant q \leqslant k)$ the k points $b \exp(in\pi/N)$, where $-\frac{1}{2}k < n \leqslant \frac{1}{2}k$. Suppose further that $f(x)$ is any decreasing function of x in $x \geqslant 0$. Then*

$$\Sigma_1 = \sum_{p=1}^{h} \sum_{q=1}^{k} f(A_p B_q) \leqslant \sum_{p=1}^{h} \sum_{q=1}^{k} f(A_p{}^* B_q{}^*) = \Sigma_2 \dagger.$$

Consider a line D passing through the origin and through points whose argument is of the form $\pm\nu\pi/(2N)$, where ν is a particular integer. The points $a \exp(im\pi/N)$ $(|m| \leqslant N)$ are placed symmetrically about D and the same is true of the points $b \exp(in\pi/N)$ $(|n| \leqslant N)$. We shall denote by A_L, A_R members of the set of points $\{A_p\}$, which are image points of each other with respect to D, all the points A_L being on one side of D and all the points A_R being on the other. By $\bar{A}_p$ we shall denote a member of $\{A_p\}$ lying on the same side of D as the points A_L, and such that $\bar{\mathcal{A}}_p$, the image of $\bar{A}_p$ in D, is not a member of $\{A_p\}$. By $A_p{}'$ we shall denote a member of $\{A_p\}$ lying on the same side of D as the points A_R, and such that $\bar{\mathcal{A}}_p{}'$, the image of $A_p{}'$ in D, is not a member of $\{A_p\}$. We have now typified all members of $\{A_p\}$, the points A_p lying on D being included under the heading A_R, A_L. A similar notation can clearly be applied to the points B_q, the points B_L lying on the same side of D as the points A_L.

† The order in which the points $A_p{}^*$ are taken is immaterial, and we do not specify it; A_pB_q means the distance between the points A_p and B_q.

We shall now investigate the effect upon Σ_1 of the modification† obtained by interchanging every pair of points $\bar{A}_p$, $\bar{\mathcal{A}}_p$, and every pair of points $\bar{B}_q$, $\bar{\mathcal{B}}_q$. The terms appearing in Σ_1 and involving $\bar{B}_q$ are typified by

(i) $$f(\bar{B}_q A_L), \quad f(\bar{B}_q A_R);$$

(ii) $$f(\bar{B}_q \bar{A}_p);$$

(iii) $$f(\bar{B}_q A_p').$$

By symmetry, $\bar{B}_q A_L = \bar{\mathcal{B}}_q A_R$, $\bar{B}_q A_R = \bar{\mathcal{B}}_q A_L$, and $\bar{B}_q \bar{A}_p = \bar{\mathcal{B}}_q \bar{\mathcal{A}}_p$, so that contributions of types (i) and (ii) are unaffected by the modification. Further, since $\mathcal{B}_q$, the image of $\bar{B}_q$, lies upon the same side of D as A_p', while $\bar{B}_q$ does not, we must have $\bar{B}_q A_p' \geqslant \bar{\mathcal{B}}_q A_p'$ and

$$f(\bar{\mathcal{B}}_q A_p') \geqslant f(\bar{B}_q A_p').$$

Thus, in terms of type (iii), the modification affects an increase. Next the terms in Σ_1 concerning B_L and B_R, if taken together, are clearly unaffected by the movement of $\{A_p\}$ to the position of its image. Finally the terms in Σ_1 involving B_q' are of types

(iv) $$f(B_q' A_L), \quad f(B_q' A_R),$$

(v) $$f(B_q' \bar{A}_p),$$

(vi) $$f(B_q' A_p').$$

Of these, (iv) and (vi) are concerned only with points that remain unaltered by the modification, and we remark that (v) can be dealt with in the manner of (iii), if we remember that the A's and B's appear upon a symmetrical footing in this lemma. Thus the proposed modification cannot decrease Σ_1.

Now in any given arrangement of the A_p's, there must be one or more arcs of the circle $|z| = a$ extending between two points A_p, containing no point A_p in their interiors, and having length greater than $a\pi/N$, unless there are $2N$ distinct points A_p, and in this case Σ_1 has only one possible value and the lemma is obviously true. If there are more than one such arc, we shall show that, by means of a sufficient number of modifications of the

† *Cf.* Hardy, Littlewood, and Pólya, "The maximum of a certain bilinear form", *Proc. London Math. Soc.* (2), 25 (1926), 265-282.

kind considered above, we can reduce this general case to the particular case in which there is only one such arc. Let us suppose that A_1 and A_2 are the members of $\{A_p\}$ at the ends of the longest arc of $|z| = a$, containing no member of $\{A_p\}$. If there are several longest arcs, this is immaterial and any one of them will suit our purpose. We shall denote the arc $A_1 A_2$ by γ, and if there are two arcs of lengths greater than $a\pi/N$, extending between two members of $\{A_p\}$ and containing no point A_p in their interiors, then there must be a point, A' say, of type $a \exp(im\pi/N)$, which is not a point of γ and which is not a member of $\{A_p\}$. Now it is possible to choose a line of type D such that A_1 and A' are symmetrical about D, and we may take the side of D on which A_1 lies to be the side of the points A_L. It will easily be seen that the points of the image of γ in D lie either on the A_R side of D or in γ itself. Hence, when the modification appropriate to D is made, no additional point A_p can be introduced into γ, while A_1 is displaced to A' so that the length of the longest arc containing no point A_p is increased by at least $a\pi/N$. We note that this operation may be repeated as many times as we please, provided that there is a point of type A' available at each stage. However, the length of γ is certainly less than or equal to $2\pi a$, and there must come a stage at which increase in the length of γ is impossible, *i.e.* at which there is no point A' available. At this stage all the h points A_p are included as consecutive points $a \exp(im\pi/N)$ in an arc α of length $a(h-1)\pi/N$. A similar process can now be carried out with regard to the points B_q, which will occur in the last stage as consecutive points $b \exp(in\pi/N)$ in an arc β of length $b(k-1)\pi/N$. Such a process will not disturb the consecutive occurrence of the points A_p, and all these modifications will have been carried out without decreasing Σ_1. Finally we remark that it is easily shown that the maximum value of Σ_1 occurs if the middle points of the arcs α and β are as close together as possible, in which case $\Sigma_1 = \Sigma_2$. This establishes our lemma.

3. We are now in a position to prove our main theorem. We shall regard r as fixed throughout and we write $u(r, \theta) = u(\theta)$, $\bar{u}(r, \theta) = \bar{u}(\theta)$. In view of the continuity of $u(\theta)$, ${}^{u}(\theta)$, and $\phi(x)$, and since $U(t)$, $U^*(t)$ are integrable in $(-\pi, \pi)$, it is sufficient to discuss the case in which $U(t)$ and $U^*(t)$ are stretchwise constant functions, whose points of discontinuity are $(2n+1)\pi/(2N)$, where $-N < n \leqslant N$. It is clear that the values of $U^*(t)$ may be taken to be the rearranged values of $U(t)$. Further, for any particular pair of stretchwise constant functions $U(t)$, $U^*(t)$, we may increase N by any multiple we please, since this merely introduces points of discontinuity whose oscillation is zero. Next, by the continuity of

$\phi\{u(\theta)\}$, $\phi\{\bar{u}(\theta)\}$ in θ, we can approximate as closely as we please to the integrals

$$\int_{-\pi}^{\pi} \phi\{u(\theta)\}\,d\theta, \quad \int_{-\pi}^{\pi} \phi\{\bar{u}(\theta)\}\,d\theta$$

by means of the sums

$$\frac{1}{N} \sum_{m=-(N-1)}^{N} \phi\{u(m\pi/N)\}, \quad \frac{1}{N} \sum_{m=-(N-1)}^{N} \phi\{\bar{u}(m\pi/N)\}$$

by taking N sufficiently large. Thus it will be sufficient to prove that

$$(3.1) \qquad \frac{1}{N} \sum_{m=-(N-1)}^{N} \phi\{u(m\pi/N)\} \leqslant \frac{1}{N} \sum_{m=-(N-1)}^{N} \phi\{\bar{u}(m\pi/N)\},$$

and, by Lemma 2, we may establish this by proving that

$$(3.2) \qquad \sum_{p=1}^{h} u(m_p \pi/N) \leqslant \sum_{-\frac{1}{2}h < p \leqslant \frac{1}{2}h} \bar{u}(p\pi/N)$$

is true for all h, if $\theta = m_p \pi/N$ $(p = 1, 2, \ldots, h)$ gives the h largest values of $u(m\pi/N)$. Now

$$\begin{aligned} u(m\pi/N) &= \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{U(t)\,(1-r^2)\,dt}{1-2r\cos\{t-(m\pi/N)\}+r^2} \\ &= \sum_{n=-N}^{N} c_n \int_{(2n-1)\pi/(2N)}^{(2n+1)\pi/(2N)} \frac{(1-r^2)\,dt}{1-2r\cos\{t-(m\pi/N)\}+r^2} \\ &= \sum_{n=-N}^{N} c_n \int_{-\pi/(2N)}^{\pi/(2N)} \frac{(1-r^2)\,dt}{1-2r\cos[t+\{(n-m)\,\pi/N\}]+r^2} \\ &= \sum_{n=-N}^{N} c_n\, Q(n, m), \end{aligned}$$

where $c_n = U(t)$, $(2n-1)\,\pi/(2N) \leqslant t < (2n+1)\,\pi/(2N)$. Similarly

$$\bar{u}(m\pi/N) = \sum_{n=-N}^{N} c_n{}^{+}\, Q(n, m),$$

where the $c_n{}^{+}$'s are the c's arranged so that

$$c_0{}^{+} \geqslant c_1{}^{+} \geqslant c_{-1}^{+} \geqslant c_2{}^{+} \geqslant c_{-2}^{+} \geqslant \ldots.$$

We shall call the point $\exp(in\pi/N)$ B_n, and the point $r \exp(im\pi/N)$ A_m. It is easily seen geometrically that

$$Q(n, m) = \int_{-\pi/(2N)}^{\pi/(2N)} \frac{(1-r^2)\,dt}{1-2r\cos[t+\{(n-m)\,\pi/N\}]+r^2} = Q(m, n)$$

is a decreasing function of the distance $A_m B_n$, and depends on $|m-n|$ only. Now, with the notation of (3.2),

$$\sum_{p=1}^{h} u(m_p \pi/N) = \sum_{p=1}^{h} \sum_{n=-N}^{N} c_n Q(m_p, n)$$

$$= \sum_{n=-N}^{N} c_n \left\{ \sum_{p=1}^{h} Q(m_p, n) \right\}$$

$$= \sum_{n=-N}^{N} c_n \mu_n \quad \text{(say)},$$

while

$$\sum_{-\frac{1}{2}h < p \leqslant \frac{1}{2}h} \bar{u}(m\pi/N) = \sum_{n=-N}^{N} c_n^{+} \left\{ \sum_{-\frac{1}{2}h < p \leqslant \frac{1}{2}h} Q(p, n) \right\}$$

$$= \sum_{n=-N}^{N} c_n^{+} M_n \quad \text{(say)}.$$

We proceed to discuss some properties of the μ's and M's.

Since $Q(m, n)$ is a decreasing function of $A_m B_n$, by Lemma 3 we have

$$\sum_{p=1}^{h} \sum_{q=1}^{h} Q(m_p, n_q) \leqslant \sum_{-\frac{1}{2}h < p \leqslant \frac{1}{2}h} \sum_{-\frac{1}{2}k < q \leqslant \frac{1}{2}k} Q(p, q),$$

i.e.

$$\sum_{q=1}^{k} \mu_{n_q} \leqslant \sum_{-\frac{1}{2}k < q \leqslant \frac{1}{2}k} M_q, \tag{3.3}$$

true for all k.

Also, if $0 \leqslant n \leqslant N$, since $Q(p, -n) = Q(-p, n)$,

$$M_n - M_{-n} = \sum_{-\frac{1}{2}h < p \leqslant \frac{1}{2}h} Q(p, n) - \sum_{-\frac{1}{2}h < p \leqslant \frac{1}{2}h} Q(p, -n)$$

$$= \sum_{-\frac{1}{2}h < p \leqslant \frac{1}{2}h} Q(p, n) - \sum_{-\frac{1}{2}h < p \leqslant \frac{1}{2}h} Q(-p, n)$$

$$= \begin{cases} 0 & \text{if } h \text{ is odd}, \\ Q(\frac{1}{2}h, n) - Q(-\frac{1}{2}h, n) & \text{if } h \text{ is even}. \end{cases}$$

Now we know that $0 \leqslant \frac{1}{2}h \leqslant N$ and $0 \leqslant n \leqslant N$, and we have therefore the following inequalities:

$$\tfrac{1}{2}h + n \geqslant |\tfrac{1}{2}h - n| \tag{3.4}$$

and

$$[2N - (\tfrac{1}{2}h + n)] \pm (\tfrac{1}{2}h - n) \geqslant 0,$$

so that

$$2N - (\tfrac{1}{2}h + n) \geqslant |\tfrac{1}{2}h - n|. \tag{3.5}$$

From (3.4) and (3.5) it follows at once that $A_{\frac{1}{2}h}B_n \leqslant A_{-\frac{1}{2}h}B_n$ if h is even, so that

$$Q(\tfrac{1}{2}h, n)-Q(-\tfrac{1}{2}h, n) \geqslant 0$$

if h is even. Thus $M_n \geqslant M_{-n}$ in any case if $n \geqslant 0$.

Again, if $0 \leqslant n < N$, since $Q(p, n+1) = Q(p-1, n)$,

$$\begin{aligned} M_{-n}-M_{n+1} &= \sum_{-\frac{1}{2}h < p \leqslant \frac{1}{2}h} Q(p, -n) - \sum_{-\frac{1}{2}h < p \leqslant \frac{1}{2}h} Q(p, n+1) \\ &= \sum_{-\frac{1}{2}h < p \leqslant \frac{1}{2}h} Q(-p, n) - \sum_{-\frac{1}{2}h < p \leqslant \frac{1}{2}h} Q(p-1, n) \\ &= \sum_{-\frac{1}{2}h \leqslant p < \frac{1}{2}h} Q(p, n) - \sum_{-\frac{1}{2}h < p \leqslant \frac{1}{2}h} Q(p-1, n) \\ &= \begin{cases} 0 & \text{if } h \text{ is even,} \\ Q\left(\dfrac{h-1}{2}, n\right) - Q\left(-\dfrac{h+1}{2}, n\right) & \text{if } h \text{ is odd.} \end{cases} \end{aligned}$$

Now we know that $0 \leqslant \frac{1}{2}h \leqslant N$ and $0 \leqslant n < N$, and we have the following inequalities:

$$\frac{h+1}{2}+n \geqslant \left|\frac{h-1}{2}-n\right| \tag{3.6}$$

and

$$\left[2N-\left(\frac{h+1}{2}-n\right)\right] \pm \left(\frac{h-1}{2}-n\right) \geqslant 0,$$

since $0 \leqslant n < N$, so that

$$2N-\left(\frac{h+1}{2}-n\right) \geqslant \left|\frac{h-1}{2}-n\right|. \tag{3.7}$$

From (3.6) and (3.7) it follows as before that

$$Q\left(\frac{h-1}{2}, n\right)-Q\left(-\frac{h+1}{2}, n\right) \geqslant 0$$

if h is odd. Thus $M_{-n} \geqslant M_{n+1}$ if $0 \leqslant n < N$, and we see therefore that

$$M_0 \geqslant M_1 \geqslant M_{-1} \geqslant M_2 \geqslant M_{-2} \ldots \geqslant M_N \geqslant M_{-N}.$$

We denote this by writing $M_n = M_n{}^+$ in accordance with our previous notation. Hence we have

$$\begin{aligned} \sum_{p=1}^{h} u(m_p \pi/N) &= \sum_{n=-N}^{N} c_n \mu_n \\ &\leqslant \sum_{n=-N}^{N} c_n{}^+ \mu_n{}^+ \leqslant \sum_{n=-N}^{N} c_n{}^+ M_n{}^+ \\ &= \sum_{-\frac{1}{2}h < p \leqslant \frac{1}{2}h} \bar{u}(p\pi/N), \end{aligned}$$

by Lemma 1, whose appropriate conditions are known to be satisfied by (3:3). Thus (3.2) is established and our theorem is therefore established for stretchwise constant functions of the type considered. The general theorem follows at once, as we have already indicated.

We conclude by noting that the theorem remains true if we replace $U(t)$ by a function $f(t)$, which may be positive or negative, and $U^*(t)$ by a function $f^*(t)$ which bears the same relation to $|f(t)|$ that $U^*(t)$ bore to $U(t)$.

An appendix concerning Hardy's inequality.

Lemma 2 of the foregoing paper has an origin outside this paper and it may be of interest to give one other result† that arises from it.

If $a_n \geqslant 0$ *(all* n*),* $\alpha_\nu = (a_1+a_2+\ldots+a_\nu)/\nu$, $k>1$, *and* $\phi(x)$ *satisfies the conditions of Lemma* 2, *then*

$$\sum_{n=1}^{N} \phi(\alpha_n{}^k) \leqslant \sum_{n=1}^{N} \phi(B^k a_n{}^k), \tag{4.1}$$

where $B = k/(k-1.)$

This is an extension of Hardy's inequality and results from the application of Lemma 2 to the well-known form of Hardy's inequality.

If the a's are given save in order of arrangement, it is clear that the maximum value of each α_n is obtained when the a's are in descending order. Since $\phi(x)$ is an increasing function it is therefore sufficient to prove (4.1) subject to the assumption that the a's, and therefore the α's, are in descending order. By Hardy's inequality,

$$\sum_{n=1}^{m} \alpha_n{}^k \leqslant B^k \sum_{n=1}^{m} a_n{}^k = \sum_{n=1}^{m} (Ba_n)^k \quad (1 \leqslant m \leqslant N).$$

Since the α's are in descending order, (4.1) follows at once by Lemma 2.

† Our result will give an immediate extension of the theory set out by Hardy and Littlewood, "A maximal theorem with function-theoretic applications", *Acta Math.*, 54 (1930), 81–116.

THE CESÀRO-PERRON INTEGRAL

By J. C. BURKILL.

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1. *Introduction.*

Perron constructed the integral $F(x)$ of a function $f(x)$ as the section between classes of major and minor functions. He defined a major function as a continuous function whose lower derivate is greater than or equal to $f(x)$ and is never equal to $-\infty$. There is a corresponding definition for a minor function. The scope of this integral—which we may call the ordinary Perron integral—was proved by Alexandroff and Looman to be the same as that of the (special) Denjoy integral.

The scope of the ordinary Perron integral may be widened by replacing the derivates occurring in the definitions of major and minor functions by some kind of generalized derivates. It then becomes natural to make a corresponding relaxation of the condition that the major and minor functions are to be continuous. I have recently set out the properties of the integral which is evolved by extending the idea of ordinary derivate to that of approximate derivate. This integral is an approximately continuous function, and I called it the approximately continuous Perron integral or AP-integral†.

In the present note I introduce another generalization of differentiation and investigate the integral which is correlated with it by Perron's method. In forming a derivative or derivate we shall take for the numerator of the incrementary ratio, not the actual increment in the function, but the arithmetic mean of its increment. It seems appropriate to call such a derivate a Cesàro-derivate or C-derivate. The corresponding integral, which is a C-continuous function, we shall call the Cesàro-Perron (CP) integral. I leave aside for the present any discussion of a Cesàro mean other than that of order one.

† J. C. Burkill, "The approximately continuous Perron integral", *Math. Zeitschrift*, 34 (1932), 270–278. This paper will be referred to as AP.

Much of the detail necessary for a complete account of the CP-integral is omitted, or stated very briefly, in this paper. The establishment of many of the fundamental properties is a piece of routine which the reader will immediately see how to carry out; while some of the more characteristic properties are analogous to those which have been proved for the AP-integral, and the sketch given here can be supplemented, where necessary, by reference to AP.

It was observed in AP that the AP-integral figures as the natural development of the special Denjoy integral in the direction already marked by the general Denjoy-Khintchine integral. The CP-integral is an extension in a different direction.

We do not in this paper enter into any of the applications of the CP-integral which suggest themselves, such as the Fourier series of CP-integrable functions. We may observe, however, that the CP-integral appears to be somewhat easier to work with than the AP-integral; for one thing, it is more amenable to integration by parts.

2. *Cesàro continuity and Cesàro derivates.*

We shall write $C(f, a, b)$, the arithmetic mean of $f(x)$ in (a, b), as an abbreviation for

$$\frac{1}{b-a}\int_a^b f(x)\,dx,$$

where the sense in which it is appropriate to understand the integral will be considered in a moment.

We say that $f(x)$ is C-continuous (Cesàro-continuous) at x_0 if $C(f, x_0, x_0+h)$ tends to $f(x_0)$ as h tends to 0.

If $f(x)$ is finite and C-continuous at every point of an interval, we see from the definition that $f(x)$ is at every point of the interval the derivative of its indefinite integral (or what is called by some writers a Duhamelian function). Now an everywhere-finite derivative can be integrated by the (special) Denjoy process. Hence this is an appropriate sense in which to interpret the integral expressing the mean value of $f(x)$.

We define $CD^*f(x)$, the (two-sided) upper Cesàro derivate of $f(x)$ for a given value of x, to be

$$\overline{\lim_{h\to 0}}\,\frac{C(f, x, x+h)-f(x)}{\frac{1}{2}h}.$$

The lower C-derivate, $CD_*f(x)$, has a corresponding definition. When $CD^*f = CD_*f$, f has a C-derivative CDf.

It is easy to prove that the C-derivates so defined are measurable functions.

These C-derivates were constructed by Dr. Haslam-Jones and myself in the course of investigations into the differential properties of real functions.

The following lemma will be needed.

LEMMA. *If, for* $a \leqslant x \leqslant b$,

(1) $$f(x) \text{ is } C\text{-continuous},$$

(2) $$CD_* f(x) \geqslant 0,$$

then $$f(b) \geqslant f(a).$$

Take ϵ (small).

Let l be the line through the point $\{a, f(a)\}$ with gradient $-\epsilon$. Since $CD_* f(a) > -\epsilon$, the centroid of points of the curve $y = f(x)$ with abscissae between a and x lies above l for all x sufficiently near to a. It is clear then that there are points of $y = f(x)$ above l for values of x arbitrarily near to a. Hence we can find x_1 such that

$$f(x_1) - f(a) \geqslant -\epsilon(x_1 - a)$$

and $$C(f, a, x_1) - f(a) \geqslant -\tfrac{1}{2}\epsilon(x_1 - a).$$

Similarly, starting from x_1, define x_2 such that

$$f(x_2) - f(x_1) \geqslant -\epsilon(x_2 - x_1)$$

and $$C(f, x_1, x_2) - f(x_1) \geqslant -\tfrac{1}{2}\epsilon(x_2 - x_1).$$

We thus generate a Lebesgue chain of intervals and we obtain, by combining the inequalities,

$$C(f, x_m, x_n) - f(a) \geqslant -\epsilon(x_m - a) - \tfrac{1}{2}\epsilon(x_n - x_m).$$

Suppose that the sequence x_n tends to a limit-point ξ. Then, from the last inequality, keeping x_m fixed, we have

$$C(f, x_m, \xi) - f(a) \geqslant -\epsilon(x_m - a) - \tfrac{1}{2}\epsilon(\xi - x_m).$$

Now let x_m tend to ξ and remember that $f(x)$ is C-continuous at ξ. This gives

$$f(\xi) - f(a) \geqslant -\epsilon(\xi - a).$$

Covering (a, b) with the Lebesgue chain we have

$$f(b)-f(a) \geqslant -\epsilon(b-a),$$

and, since ϵ is arbitrary,

$$f(b) \geqslant f(a).$$

It is easy to see, further, that a C-continuous function which is monotonic must be continuous, so that $f(x)$ is continuous for $a < x < b$.

3. *Major and minor functions and definition of the integral.*

Suppose that $f(x)$ is measurable and finite almost everywhere in (a, b). We say that $M(x)$ is a major function of $f(x)$ in (a, b) if

(1) $$M(x) \text{ is } C\text{-continuous in } (a, b),$$

(2) $$M(a) = 0,$$

and, for all x in $a \leqslant x \leqslant b$,

(3) $$CD_* M(x) > -\infty,$$

(4) $$CD_* M(x) \geqslant f(x).$$

A minor function $m(x)$ is defined in a corresponding way.

It follows from the lemma of 2 that, if $M(x)$ is any major function and $m(x)$ any minor function of $f(x)$, then $M(x)-m(x)$ is an increasing function, and, in particular, $M(b) \geqslant m(b)$.

If $f(x)$ has major and minor functions, then

$$I(b) = \text{lower bound of all } M(b),$$

$$J(b) = \text{upper bound of all } m(b)$$

are finite and $I(b) \geqslant J(b)$.

If $I(b) = J(b)$ we say that $f(x)$ has a Cesàro-Perron integral

$$(CP)\int_a^b f(x)\,dx$$

equal to their common value.

We shall usually be able without ambiguity to omit the sign (CP), on the understanding that, in the absence of a special remark, an integral is to be interpreted in the CP sense, with the natural exception that an auxiliary integral serving to express a mean value is a Denjoy integral.

It may be proved as in AP that the scope of the CP-integral is unaltered if the inequality (4) for major and minor functions is assumed to hold only

almost everywhere in (a, b). It is generally convenient to adopt condition (4) in this less stringent form.

It is worth while to construct an example of a function which is integrable by the CP process, but not by the ordinary Perron process. We start with a suitable function $\Phi(x)$ which has an everywhere-finite derivative $F(x)$. We may take

$$\Phi(x) = x^2 \sin \frac{1}{x^2} \quad (x \neq 0),$$

$$\Phi(0) = 0,$$

which gives

$$F(x) = 2x \sin \frac{1}{x^2} - \frac{2}{x} \cos \frac{1}{x^2} \quad (x \neq 0),$$

$$F(0) = 0.$$

$F(x)$ is continuous at every point except $x = 0$, where it is C-continuous.

Consider now the function

$$f(x) = 2 \sin \frac{1}{x^2} - \frac{2}{x^2} \cos \frac{1}{x^2} - \frac{4}{x^4} \sin \frac{1}{x^2} \quad (x \neq 0),$$

$$f(0) = 0.$$

For $x \neq 0$, $F'(x) = f(x)$, while, for the exceptional value $x = 0$, we find that $CD^* F(x) = 2$ and $CD_{*} F(x) = -2$.

If we take any interval (a, b) containing the origin, $F(x) - F(a)$ will serve as a major function for $f(x)$ and also as a minor function. Therefore $(CP) \int_a^b f(x)\,dx$ exists and has the value $F(b) - F(a)$.

Further, considering separately the three terms in $f(x)$, we see for the first that $\sin(1/x^2)$ is bounded and for the second that

$$\lim_{\epsilon \to 0} \int_\epsilon^b \frac{1}{x^2} \cos \frac{1}{x^2}\,dx$$

exists as the limit of a Lebesgue integral and *a fortiori* as a Denjoy integral. It is the third term which requires a powerful integral process, and we have as our example of a CP-integrable function

$$\phi(x) = \frac{1}{x^4} \sin \frac{1}{x^2} \quad (x \neq 0),$$

$$\phi(0) = 0.$$

4. *Properties of the integral.*

Omitting the most obvious properties†, we observe first that the CP-integral is a generalization of the ordinary Perron integral.

The integral has the further properties expressed by the following theorems.

THEOREM I. *$F(x) = \int_a^x f(x)\,dx$ is a C-continuous function of x.*

THEOREM II. *If, at all points of (a, b), $G(x)$ has a finite C-derivative $g(x)$, then $g(x)$ is integrable and*

$$\int_a^b g(x)\,dx = G(b) - G(a).$$

THEOREM III. *If $F(x) = \int_a^x f(x)\,dx$, then $CDF(x) = f(x)$ almost everywhere.*

THEOREM IV. *If in (a, b) $f(x) \geqslant 0$ and is integrable (CP), then $f(x)$ is integrable (Lebesgue) and the integrals in the two senses are equal.*

THEOREM V. (*Limit under integral sign.*) *If*

(1) *for each n, $g(x) \leqslant f_n(x) \leqslant h(x)$ in (a, b), where $g(x)$, $f_n(x)$, $h(x)$ are integrable,*

(2) *$f_n(x) \to f(x)$,*

then $f(x)$ is integrable and

$$\int_a^b f_n(x)\,dx \to \int_a^b f(x)\,dx.$$

The proofs of Theorems I, II, III, and V follow the proofs given in AP, with a change in the argument in the lemma for I. This and the proof of Theorem IV are given below.

LEMMA FOR I. *If in (a, b) each $s_n(x)$ is C-continuous and $s_n(x)$ tends uniformly to $s(x)$, then $s(x)$ is C-continuous.*

Let c be a point of (a, b).

Given ϵ, choose n such that, for all x in (a, b),

$$|s(x) - s_n(x)| < \tfrac{1}{3}\epsilon. \tag{A}$$

† See AP, 275.

Choose δ such that

$$|C(s_n, c, c+h)-s_n(c)| < \tfrac{1}{3}\epsilon \quad \text{if} \quad |h| < \delta.$$

It follows from (A) that

$$|C(s, c, c+h)-C(s_n, c, c+h)| < \tfrac{1}{3}\epsilon.$$

These inequalities give what we require, namely that

$$|C(s, c, c+h)-s(c)| < \epsilon \quad \text{if} \quad |h| < \delta.$$

Proof of IV. Let $M(x)$ be a major function of $f(x)$ according to the CP definition. The lemma of 2 shows that $M(x)$ is a continuous increasing (non-decreasing) function, and the CP-integral, being a limit of M's, is clearly also a continuous increasing function.

Now $M(x)$ is the sum of an absolutely continuous function $M_0(x)$ [with $M_0(a) = 0$] and a function having derivative zero p.p. We have $M'(x) = M_0'(x)$ p.p. and hence

$$M_0'(x) \geqslant f(x) \quad \text{p.p.}, \tag{B}$$

so that $M_0(x)$ satisfies the conditions for a CP major function of $f(x)$.

Since $M_0'(x)$ is L-integrable, the inequality (B) shows that $f(x)$ is also L-integrable, and we have

$$(L)\int_a^b f(x)\,dx \leqslant (L)\int_a^b M_0'(x)\,dx = M_0(b) \leqslant M(b).$$

Since this is true for any major function $M(x)$, it follows that

$$(L)\int_a^b f(x)\,dx \leqslant (CP)\int_a^b f(x)\,dx.$$

To prove that the equality sign must be taken, we observe that, by a theorem of de la Vallée Poussin, the Lebesgue integral can be approximated to arbitrarily closely by an absolutely continuous major function $M_0(x)$, and $M_0(x)$ serves as a major function for the CP integral.

5. *Integration by parts.*

We need to know sufficient conditions for the product of two functions to be (*a*) C-continuous, (*b*) C-differentiable.

LEMMA A. *If $\Phi(x)$ is finite and C-continuous at $x = x_0$ and $\Psi'(x)$ is bounded near $x = x_0$, then $\Phi(x)\Psi(x)$ is C-continuous at $x = x_0$.*

LEMMA B. *If* $CD\Phi(x_0) = \phi(x_0)$ *and* $\Psi'(x) = \psi(x)$, *bounded near* $x = x_0$, *then, for* $x = x_0$,

$$CD(\Phi\Psi) = \Phi\psi + \phi\Psi.$$

We give the proof of Lemma B. The proof of Lemma A is similar but easier.

For brevity we take $x_0 = 0$.

If $\Psi(x)$ is a constant, the conclusion clearly holds. We may therefore suppose in proving the lemma that $\Psi(x)$ has been adjusted by the addition of a constant to make $\Psi(0) = 0$. We have then to prove that $\Phi(0)\psi(0)$ is the limit, as h tends to 0, of

$$\frac{2}{h^2}\int_0^h \Phi(x)\Psi(x)\,dx,$$

the integral existing in Denjoy's sense for sufficiently small h, since $\Psi(x)$ is of bounded variation.

Integrating by parts and writing $\Phi_1(x) = \int_0^x \Phi(x)\,dx$, we obtain

$$\frac{2}{h^2}\Phi_1(h)\Psi(h) - \frac{2}{h^2}\int_0^h \Phi_1(x)\psi(x)\,dx.$$

The integrated part tends to $2\Phi(0)\psi(0)$. Since $\Phi_1(x) = x\Phi(0) + o(x)$, the integral term is equal to

$$-\frac{2}{h^2}\Phi(0)\int_0^h x\psi(x)\,dx - \frac{2}{h^2}\Phi(0)\int_0^h o(x)\psi(x)\,dx.$$

As h tends to 0, the second of these terms is $o(1)$, since $\psi(x)$ is bounded. The first, after integration by parts, becomes

$$-\frac{2}{h^2}\Phi(0)\{h\Psi(h)\} + \frac{2}{h^2}\Phi(0)\int_0^h \Psi(x)\,dx.$$

As h tends to 0, this gives

$$-2\Phi(0)\psi(0) + \Phi(0)\lim\{\Psi(h)/h\} = -\Phi(0)\psi(0).$$

Collecting the limits of the various terms, we have proved the lemma.

The argument can be adapted to establish this corollary to Lemma B.

COROLLARY. *If* $CD_*\Phi(x_0) = \phi(x_0)$ *and* $\Psi'(x) = \psi(x)$, *bounded near* $x = x_0$, *then, for* $x = x_0$, $\Phi\psi + \phi\Psi$ *is equal to* $CD_*(\Phi\Psi)$ *or* $CD^*(\Phi\Psi)$ *according as* $\Psi(x_0)$ *is positive or negative.*

THEOREM VI. (*Integration by parts.*) *If, for* $a \leqslant x \leqslant b$,

$$F(x) = (CP)\int_a^x f(x)\,dx$$

and

$$G(x) = \int_a^x g(x)\,dx,$$

where $g(x)$ *has bounded variation, then* $f(x)\,G(x)$ *is CP-integrable in* (a, b) *and*

$$\int_a^b fG\,dx = [FG]_a^b - \int_a^b Fg\,dx.$$

We observe first that the theorem may be reduced to the case in which $G(x)$ has constant sign in (a, b). For $g(x)$ can be expressed as the difference between two positive functions $g_1(x)$, $g_2(x)$, both positive in (a, b). Then $G(x)$ is the difference between two positive functions $G_1(x)$, $G_2(x)$, and if the theorem is true for the pair $g_1(x)$, $G_1(x)$ and for $g_2(x)$, $G_2(x)$, it will be true for the pair $g(x)$, $G(x)$. We shall, therefore, suppose that $g(x)$ and $G(x)$ are positive.

We now prove that $Fg+fG$ is CP-integrable and that

$$\int_a^b (Fg+fG)\,dx = [FG]_a^b.$$

Let $M(x)$ be a major function of $f(x)$. Then our lemmas show that MG is a major function of $Fg+fG$. For condition (1) for a major function is ensured by Lemma A; and the corollary to Lemma B gives

$$\begin{aligned} CD_*(MG) &= Mg+(CD_*\,M)\,G \\ &\geqslant Fg+(CD_*\,M)\,G, \end{aligned}$$

which shows that, since M satisfies conditions (3) and (4) for a major function of f, MG satisfies the corresponding conditions for $Fg+fG$.

Similarly, if $m(x)$ is a minor function of $f(x)$, mG is a minor function of $Fg+fG$. Moreover, MG and mG may be made to differ by arbitrarily little by taking M and m near enough together. This shows that

$$(CP)\int_a^b (Fg+fG)\,dx = [FG]_a^b.$$

Now Fg is D-integrable (and *a fortiori* CP-integrable) since F is D-integrable and g has bounded variation.

Therefore fG, being the difference of two CP-integrable functions, is CP-integrable and the theorem follows.

ON CERTAIN RELATIONS BETWEEN HYPERGEOMETRIC SERIES OF HIGHER ORDERS

By H. B. C. DARLING.

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The main object of this paper is to establish relations between hypergeometric series of higher orders, corresponding to certain known relations between ordinary hypergeometric series. Examples are equation (7), namely that, if $|z| < 1$,

$$F\left(\begin{matrix}\alpha, \beta, \gamma; \\ \delta, \epsilon\end{matrix}\; z\right) F\left(\begin{matrix}1-\alpha, 1-\beta, 1-\gamma; \\ 2-\delta, 2-\epsilon\end{matrix}\; z\right)$$

$$= \frac{\epsilon-1}{\epsilon-\delta} F\left(\begin{matrix}\alpha+1-\delta, \beta+1-\delta, \gamma+1-\delta; \\ 2-\delta, \epsilon+1-\delta\end{matrix}\; z\right) F\left(\begin{matrix}\delta-\alpha, \delta-\beta, \delta-\gamma; \\ \delta, \delta+1-\epsilon\end{matrix}\; z\right)$$

$$+ \frac{\delta-1}{\delta-\epsilon} F\left(\begin{matrix}\alpha+1-\epsilon, \beta+1-\epsilon, \gamma+1-\epsilon; \\ 2-\epsilon, \delta+1-\epsilon\end{matrix}\; z\right) F\left(\begin{matrix}\epsilon-\alpha, \epsilon-\beta, \epsilon-\gamma; \\ \epsilon, \epsilon+1-\delta\end{matrix}\; z\right),$$

which, when $\gamma = \epsilon \to \infty$, becomes the well-known relation

$$F\left(\begin{matrix}\alpha, \beta; \\ \delta\end{matrix}\; z\right) F\left(\begin{matrix}1-\alpha, 1-\beta; \\ 2-\delta\end{matrix}\; z\right)$$

$$= F\left(\begin{matrix}\alpha+1-\delta, \beta+1-\delta; \\ 2-\delta\end{matrix}\; z\right) F\left(\begin{matrix}\delta-\alpha, \delta-\beta; \\ \delta\end{matrix}\; z\right);$$

and equation (11), viz.

$$(1-z)^{\alpha+\beta+\gamma-\delta-\epsilon} F\left(\begin{matrix}\alpha, \beta, \gamma; \\ \delta, \epsilon\end{matrix}\; z\right)$$

$$= \frac{\epsilon-1}{\epsilon-\delta} F\left(\begin{matrix}\delta-\alpha, \delta-\beta, \delta-\gamma; \\ \delta, \delta+1-\epsilon\end{matrix}\; z\right) F\left(\begin{matrix}\epsilon-\alpha, \epsilon-\beta, \epsilon-\gamma; \\ \epsilon-1, \epsilon+1-\delta\end{matrix}\; z\right)$$

$$+ \frac{\delta-1}{\delta-\epsilon} F\left(\begin{matrix}\epsilon-\alpha, \epsilon-\beta, \epsilon-\gamma; \\ \epsilon, \epsilon+1-\delta\end{matrix}\; z\right) F\left(\begin{matrix}\delta-\alpha, \delta-\beta, \delta-\gamma; \\ \delta-1, \delta+1-\epsilon\end{matrix}\; z\right);$$

which, when $\gamma = \epsilon \to \infty$, becomes

$$(1-z)^{\alpha+\beta-\delta} F\left(\begin{matrix}\alpha, \beta; \\ \delta\end{matrix} z\right) = F\left(\begin{matrix}\delta-\alpha, \delta-\beta; \\ \delta\end{matrix} z\right).$$

The modulus of the variable z is assumed to be always less than unity; but, by making $z \to 1$, we are able, in general, to deduce formulae in which z is replaced by unity. In the latter part of the paper the difference equations satisfied by certain hypergeometric series are considered.

The differential equation

$$z^2(1-z)\frac{d^3y}{dz^3} + \{(\delta+\epsilon+1)z - (\alpha+\beta+\gamma+3)z^2\}\frac{d^2y}{dz^2}$$

$$+\{\delta\epsilon - (1+\alpha+\beta+\gamma+\alpha\beta+\alpha\gamma+\beta\gamma)z\}\frac{dy}{dz} - \alpha\beta\gamma\, y = 0, \quad (1)$$

or, for brevity,

$$z^2(1-z)\frac{d^3y}{dz^3} + P\frac{d^2y}{dz^2} + Q\frac{dy}{dz} - \alpha\beta\gamma\, y = 0, \quad (2)$$

(where $|z| < 1$) is satisfied by*

$$\left.\begin{aligned} A = F\left(\begin{matrix}\alpha, \beta, \gamma; \\ \delta, \epsilon\end{matrix} z\right), \quad B = z^{1-\delta} F\left(\begin{matrix}\alpha+1-\delta, \beta+1-\delta, \gamma+1-\delta; \\ 2-\delta, \epsilon+1-\delta\end{matrix} z\right), \\ \text{and} \qquad C = z^{1-\epsilon} F\left(\begin{matrix}\alpha+1-\epsilon, \beta+1-\epsilon, \gamma+1-\epsilon; \\ 2-\epsilon, \delta+1-\epsilon\end{matrix} z\right). \end{aligned}\right\} \quad (3)$$

We shall require also the three expressions

$$\left.\begin{aligned} A' = F\left(\begin{matrix}1-\alpha, 1-\beta, 1-\gamma; \\ 2-\delta, 2-\epsilon\end{matrix} z\right), \quad B' = z^{\delta-1} F\left(\begin{matrix}\delta-\alpha, \delta-\beta, \delta-\gamma; \\ \delta, \delta+1-\epsilon\end{matrix} z\right), \\ \text{and} \qquad C' = z^{\epsilon-1} F\left(\begin{matrix}\epsilon-\alpha, \epsilon-\beta, \epsilon-\gamma; \\ \epsilon, \epsilon+1-\delta\end{matrix} z\right). \end{aligned}\right\} \quad (4)$$

From (2) we find without difficulty that

$$u = C\frac{dB}{dz} - B\frac{dC}{dz}$$

* A. R. Forsyth, "On linear differential equations", *Quart. J. of Math.*, 19 (1883), 292–337 (295); and *Treatise on differential equations* (1885), 213.

satisfies the differential equation

$$z^4(1-z)^2\frac{d^3u}{dz^3}+2z^2(1-z)P\frac{d^2u}{dz^2}+\left\{P^2-(2z-3z^2)P+z^2(1-z)\left(\frac{dP}{dz}+Q\right)\right\}\frac{du}{dz}$$

$$+\left\{PQ-(2z-3z^2)Q+z^2(1-z)\left(\frac{dQ}{dz}+\alpha\beta\gamma\right)\right\}u=0; \quad (5)$$

and appertains to the root $\rho=1-\delta-\epsilon$ of its indicial equation

$$(\rho+\delta)(\rho+\epsilon)(\rho+\delta+\epsilon-1)=0.$$

But, making the necessary modifications in (2), we see that

$$v=z^{1-\delta-\epsilon}(1-z)^{\delta+\epsilon-1-\alpha-\beta-\gamma}A',$$

where A' is given by (4), also satisfies (5) and appertains to the same root of the indicial equation. Hence, comparing the first terms of u and v, we have

$$C\frac{dB}{dz}-B\frac{dC}{dz}=(\epsilon-\delta)z^{1-\delta-\epsilon}(1-z)^{\delta+\epsilon-1-\alpha-\beta-\gamma}A'. \quad (6)$$

Again, writing $\alpha+1-\delta$ for α, $\beta+1-\delta$ for β, $\gamma+1-\delta$ for γ, $2-\delta$ for δ, and $\epsilon+1-\delta$ for ϵ, in (3) and (4), we find that B becomes $z^{\delta-1}A$, C becomes $z^{\delta-1}C$, and A' becomes $z^{1-\delta}B'$; so that (6) becomes

$$z^{2\delta-2}\left(C\frac{dA}{dz}-A\frac{dC}{dz}\right)=(\epsilon-1)z^{2\delta-\epsilon-2}(1-z)^{\delta+\epsilon-1-\alpha-\beta-\gamma}z^{1-\delta}B';$$

that is,

$$C\frac{dA}{dz}-A\frac{dC}{dz}=(\epsilon-1)z^{1-\delta-\epsilon}(1-z)^{\delta+\epsilon-1-\alpha-\beta-\gamma}B';$$

whence, interchanging δ and ϵ,

$$B\frac{dA}{dz}-A\frac{dB}{dz}=(\delta-1)z^{1-\delta-\epsilon}(1-z)^{\delta+\epsilon-1-\alpha-\beta-\gamma}C'.$$

But

$$A\left(C\frac{dB}{dz}-B\frac{dC}{dz}\right)-B\left(C\frac{dA}{dz}-A\frac{dC}{dz}\right)+C\left(B\frac{dA}{dz}-A\frac{dB}{dz}\right)=0$$

identically; and therefore, by the above equations,

$$AA'=\frac{\epsilon-1}{\epsilon-\delta}BB'+\frac{\delta-1}{\delta-\epsilon}CC';$$

that is, by (3) and (4),

$$F\left(\begin{matrix}\alpha,\ \beta,\ \gamma;\\ \delta,\ \epsilon\end{matrix}\ z\right) F\left(\begin{matrix}1-\alpha,\ 1-\beta,\ 1-\gamma;\\ 2-\delta,\ 2-\epsilon\end{matrix}\ z\right)$$

$$= \frac{\epsilon-1}{\epsilon-\delta} F\left(\begin{matrix}\alpha+1-\delta,\ \beta+1-\delta,\ \gamma+1-\delta;\\ 2-\delta,\ \epsilon+1-\delta\end{matrix}\ z\right) F\left(\begin{matrix}\delta-\alpha,\ \delta-\beta,\ \delta-\gamma;\\ \delta,\ \delta+1-\epsilon\end{matrix}\ z\right)$$

$$+ \frac{\delta-1}{\delta-\epsilon} F\left(\begin{matrix}\alpha+1-\epsilon,\ \beta+1-\epsilon,\ \gamma+1-\epsilon;\\ 2-\epsilon,\ \delta+1-\epsilon\end{matrix}\ z\right) F\left(\begin{matrix}\epsilon-\alpha,\ \epsilon-\beta,\ \epsilon-\gamma;\\ \epsilon,\ \epsilon+1-\delta\end{matrix}\ z\right), \quad (7)$$

where $|z| < 1$.

Clausen's* formula is

$$F\left(\begin{matrix}2\alpha,\ \alpha+\beta,\ 2\beta;\\ \alpha+\beta+\frac{1}{2},\ 2\alpha+2\beta\end{matrix}\ z\right) = \left\{F\left(\begin{matrix}\alpha,\ \beta;\\ \alpha+\beta+\frac{1}{2}\end{matrix}\ z\right)\right\}^2,$$

and therefore

$$F\left(\begin{matrix}1-2\alpha,\ 1-\alpha-\beta,\ 1-2\beta;\\ \frac{3}{2}-\alpha-\beta,\ 2-2\alpha-2\beta\end{matrix}\ z\right) = \left\{F\left(\begin{matrix}\frac{1}{2}-\alpha,\ \frac{1}{2}-\beta;\\ \frac{3}{2}-\alpha-\beta\end{matrix}\ z\right)\right\}^2;$$

whence, by (7),

$$F\left(\begin{matrix}\alpha-\beta+\frac{1}{2},\ \beta-\alpha+\frac{1}{2},\ \frac{1}{2};\\ \alpha+\beta+\frac{1}{2},\ \frac{3}{2}-\alpha-\beta\end{matrix}\ z\right) = F\left(\begin{matrix}\alpha,\ \beta;\\ \alpha+\beta+\frac{1}{2}\end{matrix}\ z\right) F\left(\begin{matrix}\frac{1}{2}-\alpha,\ \frac{1}{2}-\beta;\\ \frac{3}{2}-\alpha-\beta\end{matrix}\ z\right)$$

$$= (1-z)^{\frac{1}{2}} F\left(\begin{matrix}\alpha,\ \beta;\\ \alpha+\beta+\frac{1}{2}\end{matrix}\ z\right) F\left(\begin{matrix}1-\alpha,\ 1-\beta;\\ \frac{3}{2}-\alpha-\beta\end{matrix}\ z\right),$$

a result due to Orr†. Several relations between ${}_4F_3$ series may be derived from this, by equating coefficients of like powers of z.

The result in (7) is the second member of a sequence of formulae, of which the first is the well-known relation

$$F\left(\begin{matrix}\alpha,\ \beta;\\ \gamma\end{matrix}\ z\right) F\left(\begin{matrix}1-\alpha,\ 1-\beta;\\ 2-\gamma\end{matrix}\ z\right)$$

$$= F\left(\begin{matrix}\gamma-\alpha,\ \gamma-\beta;\\ \gamma\end{matrix}\ z\right) F\left(\begin{matrix}\alpha+1-\gamma,\ \beta+1-\gamma;\\ 2-\gamma\end{matrix}\ z\right).$$

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* *Cf.* G. N. Watson, "The theorems of Clausen and Cayley", *Proc. London Math. Soc.* (2), 22 (1923), 163–170 (163); F. J. W. Whipple, "Algebraical proofs of the theorems of Cayley and Orr", *Journal London Math. Soc.*, 2 (1927), 85–90, and "On a formula implied in Orr's theorems", *ibid.*, 4 (1929), 48–50.

† W. McF. Orr, "Theorems relating to the product of two hypergeometric series", *Trans. Camb. Phil. Soc.*, 17 (1899), 1–15, formula 55.

If the last formula was not already known, then, to establish it by equating coefficients, we should require the relation

$$(1-\alpha)_n(1-\beta)_n F\left(\begin{matrix}\alpha, \beta, \gamma-n-1, -n\\ \alpha-n, \beta-n, \gamma\end{matrix}\right)$$

$$= (\alpha+1-\gamma)_n(\beta+1-\gamma)_n F\left(\begin{matrix}\gamma-\alpha, \gamma-\beta, \gamma-n-1, -n\\ \gamma-\alpha-n, \gamma-\beta-n, \gamma\end{matrix}\right), \quad (8)$$

and this has already been obtained by Whipple* in a different and more general form. The corresponding relation in the case of (7) is

$$\frac{(1-\alpha)_n(1-\beta)_n(1-\gamma)_n}{(1-\delta)_{n+1}(1-\epsilon)_{n+1}} F\left(\begin{matrix}\alpha, \beta, \gamma, \delta-n-1, \epsilon-n-1, -n\\ \alpha-n, \beta-n, \gamma-n, \delta, \epsilon\end{matrix}\right)$$

$$+\frac{(\delta-\alpha)_n(\delta-\beta)_n(\delta-\gamma)_n}{(\delta-1)_{n+1}(\delta-\epsilon)_{n+1}}$$

$$\times F\left(\begin{matrix}\alpha+1-\delta, \beta+1-\delta, \gamma+1-\delta, 1-\delta-n, \epsilon-\delta-n, -n\\ \alpha+1-\delta-n, \beta+1-\delta-n, \gamma+1-\delta-n, \epsilon+1-\delta, 2-\delta\end{matrix}\right)$$

$$+\frac{(\alpha+1-\epsilon)_n(\beta+1-\epsilon)_n(\gamma+1-\epsilon)_n}{(1-\epsilon)_{n+1}(\delta-\epsilon)_{n+1}}$$

$$\times F\left(\begin{matrix}\epsilon-\alpha, \epsilon-\beta, \epsilon-\gamma, \epsilon-n-1, \epsilon-\delta-n, -n\\ \epsilon-\alpha-n, \epsilon-\beta-n, \epsilon-\gamma-n, \epsilon+1-\delta, \epsilon\end{matrix}\right) = 0, \quad (9)$$

in which all the series terminate; and this formula is therefore proved by (7). Multiplying (9) by $(\delta-\epsilon)(1-\epsilon)_{n+1}$, we see that, when $\delta = \gamma \to \infty$, this reduces to Whipple's result (8).

It may be mentioned that, if $0 < R(\delta+\epsilon-\alpha-\beta-\gamma) < 1$, we may obtain (7), when $z = 1$, from Thomae's formulae†

$$F\left(\begin{matrix}\alpha, \beta, \gamma\\ \delta, \epsilon\end{matrix}\right) = \frac{\Gamma(\delta)\,\Gamma(\delta+\epsilon-\alpha-\beta-\gamma)}{\Gamma(\delta-\alpha)\,\Gamma(\delta+\epsilon-\beta-\gamma)} F\left(\begin{matrix}\alpha, \epsilon-\beta, \epsilon-\gamma\\ \delta+\epsilon-\beta-\gamma, \epsilon\end{matrix}\right)$$

(Thomae I)

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* F. J. W. Whipple, "Well-poised series . . .", *Proc. London Math. Soc.* (2), 25 (1926), 525–544, equation (10.1).

† *Cf.* G. N. Watson, *loc. cit.*, 166.

and

$$F\left(\begin{matrix}\alpha, \beta, \gamma\\ \delta, \epsilon\end{matrix}\right) = \frac{\Gamma(\delta)\,\Gamma(\delta-\alpha-\beta)}{\Gamma(\delta-\alpha)\,\Gamma(\delta-\beta)}\,F\left(\begin{matrix}\alpha, \beta, \epsilon-\gamma\\ \alpha+\beta+1-\delta, \epsilon\end{matrix}\right)$$

$$+\frac{\Gamma(\delta)\,\Gamma(\epsilon)\,\Gamma(\alpha+\beta-\delta)\,\Gamma(\delta+\epsilon-\alpha-\beta-\gamma)}{\Gamma(\alpha)\,\Gamma(\beta)\,\Gamma(\epsilon-\gamma)\,\Gamma(\delta+\epsilon-\alpha-\beta)}$$

$$\times F\left(\begin{matrix}\delta-\alpha, \delta-\beta, \delta+\epsilon-\alpha-\beta-\gamma\\ \delta+1-\alpha-\beta, \delta+\epsilon-\alpha-\beta\end{matrix}\right). \quad \text{(Thomae II)}$$

In manipulating the gamma functions which occur in this derivation, the following trigonometrical identity (true for all a, b, ...) is required:

$$\sin(d-e)\{\sin d \sin e \sin(a+b+c-d-e) - \sin a \sin b \sin c\}$$

$$= \sin d \sin(e-a) \sin(e-b) \sin(e-c) - \sin e \sin(d-a) \sin(d-b) \sin(d-c).$$

If $R(\delta+\epsilon-\alpha-\beta-\gamma) > 1$, then, when $z = 1$, the second in each of the three pairs of series in (7) is divergent. Dividing (7) by

$$F\left(\begin{matrix}1-\alpha, 1-\beta, 1-\gamma;\ z\\ 2-\delta, 2-\epsilon\end{matrix}\right),$$

and then making $z \to 1$, it follows from Cesàro's theorem* of comparison of divergent series that

$$F\left(\begin{matrix}\alpha, \beta, \gamma\\ \delta, \epsilon\end{matrix}\right) = \frac{\Gamma(1-\alpha)\,\Gamma(1-\beta)\,\Gamma(1-\gamma)\,\Gamma(\delta)\,\Gamma(\delta-\epsilon)}{\Gamma(2-\delta)\,\Gamma(1-\epsilon)\,\Gamma(\delta-\alpha)\,\Gamma(\delta-\beta)\,\Gamma(\delta-\gamma)}$$

$$\times F\left(\begin{matrix}\alpha+1-\delta, \beta+1-\delta, \gamma+1-\delta\\ 2-\delta, \epsilon+1-\delta\end{matrix}\right)$$

$$+\frac{\Gamma(1-\alpha)\,\Gamma(1-\beta)\,\Gamma(1-\gamma)\,\Gamma(\epsilon)\,\Gamma(\epsilon-\delta)}{\Gamma(2-\epsilon)\,\Gamma(1-\delta)\,\Gamma(\epsilon-\alpha)\,\Gamma(\epsilon-\beta)\,\Gamma(\epsilon-\gamma)}$$

$$\times F\left(\begin{matrix}\alpha+1-\epsilon, \beta+1-\epsilon, \gamma+1-\epsilon\\ 2-\epsilon, \delta+1-\epsilon\end{matrix}\right), \quad (10)$$

a Thomae formula, belonging to the class II above. The condition may now be reduced to $R(\delta+\epsilon-\alpha-\beta-\gamma) > 0$, for which all the series are convergent.

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* T. J. I'A. Bromwich, *Theory of infinite series* (1926), § 51, 149–150.

From (6) we see that

$$(1-z)^{\alpha+\beta+\gamma-\delta-\epsilon} F\left(\begin{matrix}\alpha, \beta, \gamma; \\ \delta, \epsilon\end{matrix}\; z\right)$$

$$= \frac{\delta-1}{\delta-\epsilon} F\left(\begin{matrix}\delta-\alpha, \delta-\beta, \delta-\gamma; \\ \delta-1, \delta+1-\epsilon\end{matrix}\; z\right) F\left(\begin{matrix}\epsilon-\alpha, \epsilon-\beta, \epsilon-\gamma; \\ \epsilon, \epsilon+1-\delta\end{matrix}\; z\right)$$

$$+ \frac{\epsilon-1}{\epsilon-\delta} F\left(\begin{matrix}\epsilon-\alpha, \epsilon-\beta, \epsilon-\gamma; \\ \epsilon-1, \epsilon+1-\delta\end{matrix}\; z\right) F\left(\begin{matrix}\delta-\alpha, \delta-\beta, \delta-\gamma; \\ \delta, \delta+1-\epsilon\end{matrix}\; z\right). \quad (11)$$

Writing γ for ϵ, and then interchanging γ and δ, we get

$$(1-z)^{\alpha+\beta-\gamma} F\left(\begin{matrix}\alpha, \beta; \\ \gamma\end{matrix}\; z\right)$$

$$= \frac{\gamma-1}{\gamma-\delta} F\left(\begin{matrix}\gamma-\alpha, \gamma-\beta, \gamma-\delta; \\ \gamma-1, \gamma+1-\delta\end{matrix}\; z\right) + \frac{\delta-1}{\delta-\gamma} F\left(\begin{matrix}\gamma-\alpha, \gamma-\beta, \gamma-\delta; \\ \gamma, \gamma+1-\delta\end{matrix}\; z\right),$$

where δ is arbitrary. By taking δ equal to unity, the right-hand side reduces to

$$F\left(\begin{matrix}\gamma-\alpha, \gamma-\beta; \\ \gamma\end{matrix}\; z\right),$$

and we have a well-known formula of which (11) is the ${}_3F_2$ analogue.

It may be noticed that (11) is in agreement with the result that, when $R(\delta+\epsilon-\alpha-\beta-\gamma) < 0$,

$$\lim_{z\to 1} \left\{(1-z)^{\alpha+\beta+\gamma-\delta-\epsilon} F\left(\begin{matrix}\alpha, \beta, \gamma; \\ \delta, \epsilon\end{matrix}\; z\right)\right\}$$

is finite. Thus, if $\delta+\epsilon-\alpha-\beta-\gamma = 0$, we have from (11), with the help of Cesàro's theorem of comparison of divergent series,

$$\frac{\Gamma(\alpha)\Gamma(\beta)\Gamma(\gamma)\Gamma(\epsilon-\delta)}{\Gamma(\epsilon-\alpha)\Gamma(\epsilon-\beta)\Gamma(\epsilon-\gamma)\Gamma(\delta)} F\left(\begin{matrix}\delta-\alpha, \delta-\beta, \delta-\gamma \\ \delta, \delta+1-\epsilon\end{matrix}\right)$$

$$+ \frac{\Gamma(\alpha)\Gamma(\beta)\Gamma(\gamma)\Gamma(\delta-\epsilon)}{\Gamma(\delta-\alpha)\Gamma(\delta-\beta)\Gamma(\delta-\gamma)\Gamma(\epsilon)} F\left(\begin{matrix}\epsilon-\alpha, \epsilon-\beta, \epsilon-\gamma \\ \epsilon, \epsilon+1-\delta\end{matrix}\right) = 1, \quad (12)$$

where both series are convergent. It is interesting to compare (12) with

the corresponding Thomae II result, in the case $0 < R(\delta+\epsilon-\alpha-\beta-\gamma) < 1$; viz.:

$$\frac{\Gamma(\epsilon)\,\Gamma(\epsilon-\delta)\,\Gamma(\delta+\epsilon-\alpha-\beta-\gamma)}{\Gamma(\epsilon-\alpha)\,\Gamma(\epsilon-\beta)\,\Gamma(\epsilon-\gamma)}\,F\left(\begin{matrix}\delta-\alpha,\ \delta-\beta,\ \delta-\gamma\\ \delta,\ \delta+1-\epsilon\end{matrix}\right)$$

$$+\frac{\Gamma(\delta)\,\Gamma(\delta-\epsilon)\,\Gamma(\delta+\epsilon-\alpha-\beta-\gamma)}{\Gamma(\delta-\alpha)\,\Gamma(\delta-\beta)\,\Gamma(\delta-\gamma)}\,F\left(\begin{matrix}\epsilon-\alpha,\ \epsilon-\beta,\ \epsilon-\gamma\\ \epsilon,\ \epsilon+1-\delta\end{matrix}\right)=F\left(\begin{matrix}\alpha,\ \beta,\ \gamma\\ \delta,\ \epsilon\end{matrix}\right).$$

We next obtain the third member of the sequence of formulae referred to above The equation*

$$z^3(1-z)\frac{d^4y}{dz^4}+\{(3+\epsilon+\zeta+\theta)z^2-(6+\alpha+\beta+\gamma+\delta)z^3\}\frac{d^3y}{dz^3}$$

$$+\{(1+\epsilon+\zeta+\theta+\epsilon\zeta+\epsilon\theta+\zeta\theta)z-(7+3\Sigma\alpha+\Sigma\alpha\beta)z^2\}\frac{d^2y}{dz^2}$$

$$+[\epsilon\zeta\theta-\{(1+\alpha)(1+\beta)(1+\gamma)(1+\delta)-\alpha\beta\gamma\delta\}z]\frac{dy}{dz}-\alpha\beta\gamma\delta\,y=0$$

(where the Σ's apply to $\alpha, \beta, \gamma, \delta$) or, for brevity,

$$z^3(1-z)\frac{d^4y}{dz^4}+P\frac{d^3y}{dz^3}+Q\frac{d^2y}{dz^2}+R\frac{dy}{dz}-\alpha\beta\gamma\delta\,y=0, \tag{13}$$

is satisfied by

$$A=F\left(\begin{matrix}\alpha,\ \beta,\ \gamma,\ \delta;\ z\\ \epsilon,\ \zeta,\ \theta\end{matrix}\right),$$

$$B=z^{1-\epsilon}F\left(\begin{matrix}\alpha+1-\epsilon,\ \beta+1-\epsilon,\ \gamma+1-\epsilon,\ \delta+1-\epsilon;\ z\\ 2-\epsilon,\ \zeta+1-\epsilon,\ \theta+1-\epsilon\end{matrix}\right),$$

$$C=z^{1-\zeta}F\left(\begin{matrix}\alpha+1-\zeta,\ \beta+1-\zeta,\ \gamma+1-\zeta,\ \delta+1-\zeta;\ z\\ 2-\zeta,\ \epsilon+1-\zeta,\ \theta+1-\zeta\end{matrix}\right),$$

$$D=z^{1-\theta}F\left(\begin{matrix}\alpha+1-\theta,\ \beta+1-\theta,\ \gamma+1-\theta,\ \delta+1-\theta;\ z\\ 2-\theta,\ \epsilon+1-\theta,\ \zeta+1-\theta\end{matrix}\right).$$

* Forsyth, "On linear differential equations", *loc. cit.*, 302.

We shall need also

$$A' = F\left(\begin{matrix}1-\alpha,\ 1-\beta,\ 1-\gamma,\ 1-\delta\ ;\\ 2-\epsilon,\ 2-\zeta,\ 2-\theta\end{matrix}\ z\right),$$

$$B' = z^{\epsilon-1}F\left(\begin{matrix}\epsilon-\alpha,\ \epsilon-\beta,\ \epsilon-\gamma,\ \epsilon-\delta\ ;\\ \epsilon,\ \epsilon+1-\zeta,\ \epsilon+1-\theta\end{matrix}\ z\right),$$

$$C' = z^{\zeta-1}F\left(\begin{matrix}\zeta-\alpha,\ \zeta-\beta,\ \zeta-\gamma,\ \zeta-\delta\ ;\\ \zeta,\ \zeta+1-\epsilon,\ \zeta+1-\theta\end{matrix}\ z\right),$$

$$D' = z^{\theta-1}F\left(\begin{matrix}\theta-\alpha,\ \theta-\beta,\ \theta-\gamma,\ \theta-\delta\ ;\\ \theta,\ \theta+1-\epsilon,\ \theta+1-\zeta\end{matrix}\ z\right).$$

With the help of (13), we find that the equation satisfied by

$$\Delta = \begin{vmatrix} \dfrac{d^2B}{dz^2}, & \dfrac{d^2C}{dz^2}, & \dfrac{d^2D}{dz^2} \\ \dfrac{dB}{dz}, & \dfrac{dC}{dz}, & \dfrac{dD}{dz} \\ B, & C, & D \end{vmatrix} \tag{14}$$

is

$$\begin{aligned} &z^9(1-z)^3\frac{d^4y}{dz^4}+3z^6(1-z)^2P\frac{d^3y}{dz^3} \\ &\quad+z^3(1-z)\left\{z^3(1-z)\left(3\frac{dP}{dz}+Q\right)+3P(P-3z^2+4z^3)\right\}\frac{d^2y}{dz^2} \\ &\quad+\left[z^6(1-z)^2\left(\frac{d^2P}{dz^2}+2\frac{dQ}{dz}-R\right)\right. \\ &\qquad\qquad+z^3(1-z)\left\{(3P-6z^2+8z^3)\frac{dP}{dz}+2Q(P-3z^2+4z^3)\right\} \\ &\qquad\qquad\left.+P\{P^2-(9z^2-12z^3)P+12z^4-30z^5+20z^6\}\right]\frac{dy}{dz} \\ &\quad+\left[z^6(1-z)^2\left(\frac{d^2Q}{dz^2}-\frac{dR}{dz}-\alpha\beta\gamma\delta\right)\right. \\ &\qquad\qquad+z^3(1-z)\left\{Q\frac{dP}{dz}+(P-3z^2+4z^3)\left(2\frac{dQ}{dz}-R\right)\right\} \\ &\qquad\qquad\left.+Q\{P^2-(9z^2-12z^3)P+12z^4-30z^5+20z^6\}\right]y=0. \end{aligned} \tag{15}$$

Again, making the necessary substitutions in (13), we find that

$$z^{-\epsilon-\zeta-\theta}(1-z)^{\epsilon+\zeta+\theta-2-\alpha-\beta-\gamma-\delta}A'$$

also satisfies (15), and appertains to the same root, $\rho = -\epsilon-\zeta-\theta$, of its indicial equation

$$(\rho+\epsilon+\zeta+\theta)(\rho+\epsilon+\zeta+1)(\rho+\zeta+\theta+1)(\rho+\theta+\epsilon+1) = 0.$$

Hence, by comparing the first terms, we have

$$\Delta = (\epsilon-\zeta)(\zeta-\theta)(\theta-\epsilon)z^{-\epsilon-\zeta-\theta}(1-z)^{\epsilon+\zeta+\theta-2-\alpha-\beta-\gamma-\delta}A'. \qquad (16)$$

In the notation of the determinants of functions*, the determinant Δ in (14) is written $-\mathbf{D}(B, C, D)$; so that, by (16),

$$-A\,\mathbf{D}(B, C, D) = (\epsilon-\zeta)(\zeta-\theta)(\theta-\epsilon)z^{-\epsilon-\zeta-\theta}(1-z)^{\epsilon+\zeta+\theta-2-\alpha-\beta-\gamma-\delta}AA'.$$

In this equation, write $\alpha+1-\epsilon$ for α, $\beta+1-\epsilon$ for β, $\gamma+1-\epsilon$ for γ, $\delta+1-\epsilon$ for δ, $2-\epsilon$ for ϵ, $\zeta+1-\epsilon$ for ζ, and $\theta+1-\epsilon$ for θ; then A becomes $z^{\epsilon-1}B$, B becomes $z^{\epsilon-1}A$, C becomes $z^{\epsilon-1}C$, D becomes $z^{\epsilon-1}D$, and A' becomes $z^{1-\epsilon}B'$. Accordingly, the left-hand side of the equation becomes† $-z^{\epsilon-1}Bz^{3\epsilon-3}\mathbf{D}(A, C, D)$, and the right-hand side becomes

$$(1-\zeta)(\zeta-\theta)(\theta-1)z^{3\epsilon-4-\zeta-\theta}(1-z)^{\epsilon+\zeta+\theta-2-\alpha-\beta-\gamma-\delta}z^{\epsilon-1}Bz^{1-\epsilon}B';$$

and thus,

$$-B\,\mathbf{D}(A, C, D) = (1-\zeta)(\zeta-\theta)(\theta-1)z^{-\epsilon-\zeta-\theta}(1-z)^{\epsilon+\zeta+\theta-2-\alpha-\beta-\gamma-\delta}BB'.$$

Interchanging ϵ and ζ in this equation, we have

$$-C\,\mathbf{D}(A, B, D) = (1-\epsilon)(\epsilon-\theta)(\theta-1)z^{-\epsilon-\zeta-\theta}(1-z)^{\epsilon+\zeta+\theta-2-\alpha-\beta-\gamma-\delta}CC';$$

which, on interchanging ζ and θ, becomes

$$-D\,\mathbf{D}(A, B, C) = (1-\epsilon)(\epsilon-\zeta)(\zeta-1)z^{-\epsilon-\zeta-\theta}(1-z)^{\epsilon+\zeta+\theta-2-\alpha-\beta-\gamma-\delta}DD'.$$

But

$$A\,\mathbf{D}(B, C, D)-B\,\mathbf{D}(A, C, D)+C\,\mathbf{D}(A, B, D)-D\,\mathbf{D}(A, B, C) = 0,$$

since the left-hand side forms a determinant, of which the first and second rows are the same. It follows from the above equations that

$$AA' = \frac{(\zeta-1)(\theta-1)}{(\zeta-\epsilon)(\theta-\epsilon)}BB'+\frac{(\theta-1)(\epsilon-1)}{(\theta-\zeta)(\epsilon-\zeta)}CC'+\frac{(\epsilon-1)(\zeta-1)}{(\epsilon-\theta)(\zeta-\theta)}DD'; \qquad (17)$$

* R. F. Scott and G. B. Mathews, *Theory of determinants* (1904), 203.

† Scott and Mathews, *loc. cit.*

that is,

$$F\left(\begin{matrix}\alpha, \beta, \gamma, \delta; \\ \epsilon, \zeta, \theta\end{matrix} z\right) F\left(\begin{matrix}1-\alpha, 1-\beta, 1-\gamma, 1-\delta; \\ 2-\epsilon, 2-\zeta, 2-\theta\end{matrix} z\right)$$

$$= \frac{(\zeta-1)(\theta-1)}{(\zeta-\epsilon)(\theta-\epsilon)} F\left(\begin{matrix}\alpha+1-\epsilon, \beta+1-\epsilon, \gamma+1-\epsilon, \delta+1-\epsilon; \\ 2-\epsilon, \zeta+1-\epsilon, \theta+1-\epsilon\end{matrix} z\right)$$

$$\times F\left(\begin{matrix}\epsilon-\alpha, \epsilon-\beta, \epsilon-\gamma, \epsilon-\delta; \\ \epsilon, \epsilon+1-\zeta, \epsilon+1-\theta\end{matrix} z\right)$$

$$+ \frac{(\theta-1)(\epsilon-1)}{(\theta-\zeta)(\epsilon-\zeta)} F\left(\begin{matrix}\alpha+1-\zeta, \beta+1-\zeta, \gamma+1-\zeta, \delta+1-\zeta; \\ 2-\zeta, \theta+1-\zeta, \epsilon+1-\zeta\end{matrix} z\right)$$

$$\times F\left(\begin{matrix}\zeta-\alpha, \zeta-\beta, \zeta-\gamma, \zeta-\delta; \\ \zeta, \zeta+1-\theta, \zeta+1-\epsilon\end{matrix} z\right)$$

$$+ \frac{(\epsilon-1)(\zeta-1)}{(\epsilon-\theta)(\zeta-\theta)} F\left(\begin{matrix}\alpha+1-\theta, \beta+1-\theta, \gamma+1-\theta, \delta+1-\theta; \\ 2-\theta, \epsilon+1-\theta, \zeta+1-\theta\end{matrix} z\right)$$

$$\times F\left(\begin{matrix}\theta-\alpha, \theta-\beta, \theta-\gamma, \theta-\delta; \\ \theta, \theta+1-\epsilon, \theta+1-\zeta\end{matrix} z\right), \quad (18)$$

where $|z| < 1$. This result is the third member of the sequence of formulae. The relation corresponding in this case to (8) and (9) could now be written down. It involves four ${}_8F_7$ series, and is already proved by (18).

In the way in which (10) was derived from (7), we obtain at once from (18)

$$F\left(\begin{matrix}\alpha, \beta, \gamma, \delta \\ \epsilon, \zeta, \theta\end{matrix}\right) = \frac{\Gamma(1-\alpha)\Gamma(1-\beta)\Gamma(1-\gamma)\Gamma(1-\delta)\Gamma(\epsilon)\Gamma(\epsilon-\zeta)\Gamma(\epsilon-\theta)}{\Gamma(2-\epsilon)\Gamma(1-\zeta)\Gamma(1-\theta)\Gamma(\epsilon-\alpha)\Gamma(\epsilon-\beta)\Gamma(\epsilon-\gamma)\Gamma(\epsilon-\delta)}$$

$$\times F\left(\begin{matrix}\alpha+1-\epsilon, \beta+1-\epsilon, \gamma+1-\epsilon, \delta+1-\epsilon \\ 2-\epsilon, \zeta+1-\epsilon, \theta+1-\epsilon\end{matrix}\right)$$

$$+ \frac{\Gamma(1-\alpha)\Gamma(1-\beta)\Gamma(1-\gamma)\Gamma(1-\delta)\Gamma(\zeta)\Gamma(\zeta-\theta)\Gamma(\zeta-\epsilon)}{\Gamma(2-\zeta)\Gamma(1-\theta)\Gamma(1-\epsilon)\Gamma(\zeta-\alpha)\Gamma(\zeta-\beta)\Gamma(\zeta-\gamma)\Gamma(\zeta-\delta)}$$

$$\times F\left(\begin{matrix}\alpha+1-\zeta, \beta+1-\zeta, \gamma+1-\zeta, \delta+1-\zeta \\ 2-\zeta, \theta+1-\zeta, \epsilon+1-\zeta\end{matrix}\right)$$

$$+ \frac{\Gamma(1-\alpha)\Gamma(1-\beta)\Gamma(1-\gamma)\Gamma(1-\delta)\Gamma(\theta)\Gamma(\theta-\epsilon)\Gamma(\theta-\zeta)}{\Gamma(2-\theta)\Gamma(1-\epsilon)\Gamma(1-\zeta)\Gamma(\theta-\alpha)\Gamma(\theta-\beta)\Gamma(\theta-\gamma)\Gamma(\theta-\delta)}$$

$$\times F\left(\begin{matrix}\alpha+1-\theta, \beta+1-\theta, \gamma+1-\theta, \delta+1-\theta \\ 2-\theta, \epsilon+1-\theta, \zeta+1-\theta\end{matrix}\right), \quad (19)$$

where $R(\epsilon+\zeta+\theta-\alpha-\beta-\gamma-\delta)>0$; for which all the series are convergent. When $\theta=\delta$, this reduces to the Thomae formula (10).

Comparing Forsyth's* formula with (17), we see that

$$z^{-3-\epsilon-\zeta-\theta}(1-z)^{\epsilon+\zeta+\theta-3-\alpha-\beta-\gamma-\delta}\left(D\frac{dC}{dz}-C\frac{dD}{dz}\right)=K\Delta^2\frac{d}{dz}\left(\frac{B'}{A'}\right),$$

where K is independent of z. Hence, by (16),

$$D\frac{dC}{dz}-C\frac{dD}{dz}=\frac{\theta-\zeta}{\epsilon-1}z^{3-\epsilon-\zeta-\theta}(1-z)^{\epsilon+\zeta+\theta-1-\alpha-\beta-\gamma-\delta}\left(A'\frac{dB'}{dz}-B'\frac{dA'}{dz}\right);$$

which may be regarded as the ${}_4F_3$ analogue of (11).

It may be mentioned that, by partial integration, we may express each of the three quantities†

$$u_n=\frac{z^n}{n}F\left(\begin{matrix}1-\alpha,\ 1-\beta,\ 1-\gamma,\ n\\ 2-\epsilon,\ 2-\zeta,\ n+1\end{matrix};\ z\right),$$

$$v_n=\frac{z^{\epsilon+n-1}}{\epsilon+n-1}F\left(\begin{matrix}\epsilon-\alpha,\ \epsilon-\beta,\ \epsilon-\gamma,\ \epsilon+n-1\\ \epsilon,\ \epsilon+1-\zeta,\ \epsilon+n\end{matrix};\ z\right),$$

$$w_n=\frac{z^{\zeta+n-1}}{\zeta+n-1}F\left(\begin{matrix}\zeta-\alpha,\ \zeta-\beta,\ \zeta-\gamma,\ \zeta+n-1\\ \zeta,\ \zeta+1-\epsilon,\ \zeta+n\end{matrix};\ z\right),$$

in terms of S_0, S_1, S_2, where

$$S_0=\text{sum of the first } n \text{ terms of } F\left(\begin{matrix}\alpha,\ \beta,\ \gamma\\ \epsilon,\ \zeta\end{matrix};\ z\right)$$

$$S_1=\quad ,,\quad ,,\quad n-1\quad ,,\quad F\left(\begin{matrix}\alpha+1,\ \beta+1,\ \gamma+1\\ \epsilon+1,\ \zeta+1\end{matrix};\ z\right),$$

$$S_2=\quad ,,\quad ,,\quad n-2\quad ,,\quad F\left(\begin{matrix}\alpha+2,\ \beta+2,\ \gamma+2\\ \epsilon+2,\ \zeta+2\end{matrix};\ z\right);$$

and so, by eliminating S_1 and S_2, we can obtain a relation equivalent to (18) in the case $\delta=1$. The additional (sixth) general-valued parameter replaces n, which need no longer be integral. In the course of this

* Forsyth, *Differential equations, loc. cit.*, 118.

† *Cf.* J. Hodgkinson, "Note on one of Ramanujan's theorems", *Journal London Math. Soc.*, 6 (1931), 42–43.

investigation we meet with two results which seem worth recording, viz.:

$$\text{(i)}\quad \epsilon-\zeta=(\epsilon-\zeta)\,F\left(\begin{matrix}\alpha,\ \beta,\ \gamma;\\ \epsilon,\ \zeta\end{matrix}\ z\right)F\left(\begin{matrix}-\alpha,\ -\beta,\ -\gamma;\\ 1-\epsilon,\ 1-\zeta\end{matrix}\ z\right)$$

$$-\frac{\alpha\beta\gamma}{\epsilon(\epsilon-1)}\,zF\left(\begin{matrix}\alpha+1-\epsilon,\ \beta+1-\epsilon,\ \gamma+1-\epsilon;\\ 2-\epsilon,\ \zeta+1-\epsilon\end{matrix}\ z\right)$$

$$\times F\left(\begin{matrix}\epsilon-\alpha,\ \epsilon-\beta,\ \epsilon-\gamma;\\ \epsilon+1,\ \epsilon+1-\zeta\end{matrix}\ z\right)$$

$$+\frac{\alpha\beta\gamma}{\zeta(\zeta-1)}\,zF\left(\begin{matrix}\alpha+1-\zeta,\ \beta+1-\zeta,\ \gamma+1-\zeta;\\ 2-\zeta,\ \epsilon+1-\zeta\end{matrix}\ z\right)$$

$$\times F\left(\begin{matrix}\zeta-\alpha,\ \zeta-\beta,\ \zeta-\gamma;\\ \zeta+1,\ \zeta+1-\epsilon\end{matrix}\ z\right),$$

whence (10) again follows by aid of Cesàro's theorem of comparison of divergent series.

(ii) When $\epsilon+\zeta-\alpha-\beta-\gamma=1$, and n is finite,

The sum of the first n terms of $F\left(\begin{matrix}\alpha,\ \beta,\ \gamma\\ \epsilon,\ \zeta\end{matrix}\right)$ is

$$\frac{\Gamma(\alpha+n)\,\Gamma(\beta+n)\,\Gamma(\gamma+n)}{\Gamma(n)\,\Gamma(\epsilon+n-1)\,\Gamma(\zeta+n-1)}$$

$$\times\left\{\frac{\Gamma(\zeta)\,\Gamma(\zeta-\epsilon)}{(\epsilon+n-1)\,\Gamma(\zeta-\alpha)\,\Gamma(\zeta-\beta)\,\Gamma(\zeta-\gamma)}\,F\left(\begin{matrix}\epsilon-\alpha,\ \epsilon-\beta,\ \epsilon-\gamma,\ \epsilon+n-1\\ \epsilon,\ \epsilon+1-\zeta,\ \epsilon+n\end{matrix}\right)\right.$$

$$\left.+\frac{\Gamma(\epsilon)\,\Gamma(\epsilon-\zeta)}{(\zeta+n-1)\,\Gamma(\epsilon-\alpha)\,\Gamma(\epsilon-\beta)\,\Gamma(\epsilon-\gamma)}\,F\left(\begin{matrix}\zeta-\alpha,\ \zeta-\beta,\ \zeta-\gamma,\ \zeta+n-1\\ \zeta,\ \zeta+1-\epsilon,\ \zeta+n\end{matrix}\right)\right\},$$

which may be compared with Bailey's* value, true also for n infinite, in terms of a ${}_7F_6$ series. If $\zeta=\gamma$, the above formula reduces to the result that, when $\epsilon-\alpha-\beta=1$, the sum of the first n terms of $F\left(\begin{matrix}\alpha,\ \beta\\ \epsilon\end{matrix}\right)$ is

$$\frac{\Gamma(\alpha+n)\,\Gamma(\beta+n)\,\Gamma(\epsilon)}{\Gamma(n)\,\Gamma(\epsilon-\alpha)\,\Gamma(\epsilon-\beta)\,\Gamma(\epsilon+n-1)},$$

as should be the case.

* W. N. Bailey, "The partial sum of the coefficients of the hypergeometric series", *Journal London Math. Soc.*, 6 (1931), 40–41.

Similarly, from

$$u_n' = \frac{z^n}{n} F\left(\begin{matrix}\alpha, \beta, n; \\ \gamma, n+1\end{matrix}\; z\right) \quad \text{and} \quad v_n' = \frac{z^{\gamma+n-1}}{\gamma+n-1} F\left(\begin{matrix}\gamma-\alpha, \gamma-\beta, \gamma+n-1; \\ \gamma, \gamma+n\end{matrix}\; z\right)$$

(so that u_n' is the function considered by Hodgkinson), we find that

$$F\left(\begin{matrix}\alpha+\delta, \beta+\delta, 1; \\ \gamma+\delta, \delta+1\end{matrix}\; z\right) = \frac{\gamma+\delta-1}{\gamma-1} F\left(\begin{matrix}\alpha, \beta; \\ \gamma\end{matrix}\; z\right) F\left(\begin{matrix}1-\alpha, 1-\beta, \delta; \\ 2-\gamma, \delta+1\end{matrix}\; z\right)$$

$$+ \frac{\delta}{1-\gamma} F\left(\begin{matrix}\alpha+1-\gamma, \beta+1-\gamma; \\ 2-\gamma\end{matrix}\; z\right)$$

$$\times F\left(\begin{matrix}\gamma-\alpha, \gamma-\beta, \gamma+\delta-1; \\ \gamma, \gamma+\delta\end{matrix}\; z\right),$$

and so, by writing n for δ, we obtain at once an alternative expression for the remainder after n terms of $F\left(\begin{matrix}\alpha, \beta; \\ \gamma\end{matrix}\; z\right)$. In the same way, we may deduce from u_n' and v_n' a simple form of integral representing under certain conditions the remainder after n terms of the Bessel function $J_m(x)$, namely

$$\frac{(-)^n \Gamma(m) \Gamma(1-m)}{2^{m+2n-1} \Gamma(n) \Gamma(m+n)} \int_0^x z^{m+2n-1} \{J_m(x) J_{-m}(z) - J_{-m}(x) J_m(z)\} dz.$$

By obtaining the differential equation of the fifth order corresponding to (15), the author has verified that the relation analogous to (16) holds for the series $F\left(\begin{matrix}\alpha, \beta, \gamma, \delta, \epsilon; \\ \zeta, \eta, \theta, i\end{matrix}\; z\right)$; this relation is

$$\mathbf{D}(B, C, D, E)$$

$$= (\zeta-\eta)(\zeta-\theta)(\zeta-i)\,(\eta-\theta)(\eta-i)\,(\theta-i)\, z^{-\zeta-\eta-\theta-i-2}(1-z)^{\zeta+\eta+\theta+i-3-\Sigma\alpha} A',$$

so that the relations for this series corresponding to (18), (19), and (9) could be written down. To establish the general case, that is to obtain the n-th member of the sequence of identities, it would be necessary to prove, for ${}_{n+1}F_n\left(\begin{matrix}\alpha, \beta, \gamma, \delta, \ldots, \kappa; \\ \lambda, \mu, \nu, \ldots, \upsilon\end{matrix}\; z\right)$, that

$$\mathbf{D}(B, C, D, \ldots, K) = M z^{-\frac{1}{2}n(n-3)-\Sigma\lambda} (1-z)^{1-n+\Sigma\lambda\,-\,\Sigma\alpha} A',$$

where

$$A = F\begin{pmatrix}\alpha, \beta, \gamma, \ldots, \kappa; & z\\ \lambda, \mu, \ldots, \upsilon & \end{pmatrix}, \quad B = z^{1-\lambda} F\begin{pmatrix}\alpha+1-\lambda, \ldots, \kappa+1-\lambda; & z\\ 2-\lambda, \mu+1-\lambda, \ldots, \upsilon+1-\lambda & \end{pmatrix}, \quad \ldots$$

and A', B', ... are obtained from A, B, ... by writing $1-\alpha$ for α, $1-\beta$ for β, ..., $1-\kappa$ for κ, $2-\lambda$ for λ, ..., $2-\upsilon$ for υ; and where

$$M = (\lambda-\mu)(\lambda-\nu)\ldots(\lambda-\upsilon)\,(\mu-\nu)\ldots(\mu-\upsilon)\ldots(\sigma-\tau)(\sigma-\upsilon)\,(\tau-\upsilon);$$

a well-known algebraical expression, involving $\frac{1}{2}n(n-1)$ factors. The proof would be a lengthy but not insuperable task.

Thomae was able to obtain numerous equivalent expressions for $F\begin{pmatrix}\alpha, \beta, \gamma\\ \delta, \epsilon\end{pmatrix}$ by means of the known relation

$$\frac{\Gamma(c-a)\,\Gamma(c-b)}{\Gamma(c)\,\Gamma(c-a-b)} F\begin{pmatrix}a, b\\ c\end{pmatrix} = 1. \tag{20}$$

The same artifice cannot be applied to $F\begin{pmatrix}\alpha, \beta, \gamma, \delta\\ \epsilon, \zeta, \theta\end{pmatrix}$, since there is no relation similar to (20) for $F\begin{pmatrix}a, b, c\\ d, e\end{pmatrix}$. We must, therefore, have recourse to difference equations for information concerning the existence of equivalent expressions other than (19).

The difference equation satisfied by $u_\alpha = F\begin{pmatrix}\alpha, \beta, \gamma\\ \delta, \epsilon\end{pmatrix}$ is given in a very general form by Thomae*, who obtained it by the theory of functions. A simpler method may be used for finding the difference equations satisfied by series of the form ${}_{n+1}F_n$. For example, consider $U_\epsilon = F\begin{pmatrix}\alpha, \beta, \gamma; & z\\ \delta, \epsilon & \end{pmatrix}$, where $R(\delta+\epsilon-\alpha-\beta-\gamma) > 0$. By subtraction, we have at once

$$(\epsilon-1)\,(U_{\epsilon-1}-U_\epsilon) = z\,\frac{dU_\epsilon}{dz};$$

whence

$$z^2\frac{d^2U_\epsilon}{dz^2} = \epsilon(\epsilon-1)\,U_\epsilon - 2(\epsilon-1)^2\,U_{\epsilon-1} + (\epsilon-1)\,(\epsilon-2)\,U_{\epsilon-2}.$$

Substituting for dU_ϵ/dz and d^2U_ϵ/dz^2 in the equation (1) satisfied by U_ϵ,

* J. Thomae, "Ueber die Funktionen . . .", *Journal für Math.*, 87 (1879), 26-73 (equation 19).

making z unity, and writing u_ϵ for U_ϵ $(z=1)$, we find that

$$(\epsilon+1-\alpha)(\epsilon+1-\beta)(\epsilon+1-\gamma)u_{\epsilon+2}+(\epsilon+1)Q_\epsilon u_{\epsilon+1}$$
$$+\epsilon(\epsilon+1)(\delta+\epsilon-\alpha-\beta-\gamma)u_\epsilon=0; \quad (21)$$

where

$$-Q_\epsilon=2\epsilon^2-\epsilon(2\alpha+2\beta+2\gamma-2-\delta)+1-\alpha-\beta-\gamma+\alpha\beta+\alpha\gamma+\beta\gamma,$$

so that the sum of the coefficients of (21) is $-\alpha\beta\gamma$. By Thomae II we know that $v_\epsilon=F\left(\begin{matrix}\alpha,\ \beta,\ \epsilon-\gamma\\ \alpha+\beta+1-\delta,\ \epsilon\end{matrix}\right)$, where $R(\gamma+1-\delta)>0$, also satisfies (21). Eliminating Q_ϵ between the two equations, we have

$$\epsilon(\epsilon+1)(\delta+\epsilon-\alpha-\beta-\gamma)(u_\epsilon v_{\epsilon+1}-v_\epsilon u_{\epsilon+1})$$
$$=(\epsilon+1-\alpha)(\epsilon+1-\beta)(\epsilon+1-\gamma)(u_{\epsilon+1}v_{\epsilon+2}-v_{\epsilon+1}u_{\epsilon+2});$$

whence, for ϵ finite,

$$u_\epsilon v_{\epsilon+1}-v_\epsilon u_{\epsilon+1}=K\frac{\Gamma(\epsilon)\Gamma(\epsilon+1)\Gamma(\delta+\epsilon-\alpha-\beta-\gamma)}{\Gamma(\epsilon+1-\alpha)\Gamma(\epsilon+1-\beta)\Gamma(\epsilon+1-\gamma)}.$$

Dividing this equation, and also Thomae II, by u_ϵ and comparing the results when $\epsilon\to\alpha+\beta+\gamma-\delta$, and therefore $u_\epsilon\to\infty$, we determine K, and thus obtain

$$F\left(\begin{matrix}\alpha,\ \beta,\ \gamma\\ \delta,\ \epsilon\end{matrix}\right)F\left(\begin{matrix}\alpha,\ \beta,\ \epsilon+1-\gamma\\ \alpha+\beta+1-\delta,\ \epsilon+1\end{matrix}\right)-F\left(\begin{matrix}\alpha,\ \beta,\ \gamma\\ \delta,\ \epsilon+1\end{matrix}\right)F\left(\begin{matrix}\alpha,\ \beta,\ \epsilon-\gamma\\ \alpha+\beta+1-\delta,\ \epsilon\end{matrix}\right)$$
$$=\frac{\Gamma(\delta)\Gamma(\epsilon)\Gamma(\epsilon+1)\Gamma(\alpha+\beta+1-\delta)\Gamma(\gamma+1-\delta)\Gamma(\delta+\epsilon-\alpha-\beta-\gamma)}{\Gamma(\alpha)\Gamma(\beta)\Gamma(\gamma)\Gamma(\epsilon+1-\alpha)\Gamma(\epsilon+1-\beta)\Gamma(\epsilon+1-\gamma)},$$

of which a particular case $(\gamma=\epsilon)$ is equivalent to a relation quoted by Whipple* as due to Dougall. A still more general form of this equivalent formula is (19), which is reduced by the substitutions $\theta=\delta+1$, $\zeta=\gamma$.

In the same way we may obtain the difference equations of the third order satisfied by

$$u_\alpha=F\left(\begin{matrix}\alpha,\ \beta,\ \gamma,\ \delta\\ \epsilon,\ \zeta,\ \theta\end{matrix}\right) \quad \text{and} \quad u_\epsilon=F\left(\begin{matrix}\alpha,\ \beta,\ \gamma,\ \delta\\ \epsilon,\ \zeta,\ \theta\end{matrix}\right).$$

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* F. J. W. Whipple, "A group of generalized hypergeometric series", *Proc. London Math. Soc.* (2), 23 (1924), 111 (equation 20).

Also, starting from

$$e(e-d)(V_e-V_{e+1}) = (e-d)z\frac{dV_{e+1}}{dz} - ez\frac{dV_e}{dz},$$

where $V_e = F\left(\begin{matrix}a, b, c, e-d\\ e, f, g\end{matrix}; z\right)$, we find the difference equation satisfied by $v_e = F\left(\begin{matrix}a, b, c, e-d\\ e, f, g\end{matrix}\right)$. Comparing the difference equations for u_ϵ and v_e, we are able to deduce that, if $\alpha, \beta, \gamma, \delta, \zeta, \theta$ have general values and are independent of ϵ, the difference equation in ϵ for $F\left(\begin{matrix}\alpha, \beta, \gamma, \delta\\ \epsilon, \zeta, \theta\end{matrix}\right)$ is not also satisfied by any function

$$\phi(\epsilon) F\left(\begin{matrix}a, b, c, \epsilon-d\\ \epsilon, f, g\end{matrix}\right),$$

where a, b, c, d, f, g are independent of ϵ, and $\phi(\epsilon)$ is a product of gamma functions.

ON SPECIAL CREMONA INVOLUTIONS AND TRANSFORMATIONS

By J. W. ARCHBOLD.

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1. In this paper we begin by finding all the Cremona involutions in [4] which are determined by linear systems of quadrics. One of these is discussed in detail. Examples show how to derive, by projection, from these involutions, involutions in [3] determined by linear systems of sextic, quartic, and cubic surfaces.

We show that Geiser involutions of [3] can be obtained from some of the involutions in [4]; this suggests that the involutions in [4] may be derivable from involutions in [5] in the same way. This leads us to discuss the involutions in [5] determined by linear systems of quadrics with specified base surfaces. The method also gives the homaloidal systems of quadrics. Incidentally, it is shown that the quintic del Pezzo surface and the normal rational ruled quartic surface each have one apparent double point, and that the normal rational sextic surface with prime sections of genus 2 has two apparent double points*.

SECTION I.

2. In finding linear systems of quadrics in [4] of grade 2 and freedom 4 two methods are open; they apply equally to linear systems of any order n. First we may see how the freedom and grade are affected when we impose certain base points, curves, etc. The disadvantage here is that it is a "trial" method and involves a knowledge of formulae of postulation and equivalence in [4]. No complete set of such formulae

* Segre, *Encyk. d. Math. Wiss.*, III C 7, 908, Note 417. The first two results are mentioned there as being proved by Severi, but the third result appears to be new.

exists; Severi* gives formulae for special cases. The second method depends on the free surface of intersection of two general primals of the system being rational, which is always the case when the primals are quadrics. This method is adopted here as it is systematic. Briefly we proceed thus : two general primals Φ, Ψ of a linear system meet in a free surface F which is rational; F is mapped on a plane ϖ, and the curves cut on F by the residual linear system are then mapped by a linear system of curves Λ. From Λ we pick out all the nets of grade 2; this is equivalent to imposing base loci and contact conditions on the Φ, and each net gives rise to a linear system of Φ of freedom 4 and grade 2 with F as typical free surface. We call the free intersection of three primals of such a system a ϕ-curve.

The extension to spaces of higher dimension is obvious. For all but small values of n both methods soon become laborious, but unless we deduce the involutions from higher space no other method seems available.

The spade-work when $n = 2$ has been done by Semple† in finding the quadric Cremona transformations in [4]. His method differs from ours only in that we seek systems of different grade. It will, therefore, be convenient and save much repetition to assume his paper as read. A full account of the method is given in Semple's paper.

3. We may remark at once that, whenever a Cremona transformation is determined by a linear system of primals with a simple fixed point, an involution can be derived by replacing the condition imposed by the fixed point by any linear condition which does not affect the grade of the system; and *vice versa*. For quadrics such a linear condition is obtained by expressing that two arbitrary points should be conjugate. By referring to Semple's paper we thus obtain the involutions numbered 1, 4, 8, 9, 10, 13–17, 20–25, 28–31 in Table I.

4. There still remain some involutions which we obtain from the system of curves Λ by adding extra base elements. We take in turn the cases where F is (i) I, the general surface of intersection of two quadrics; (ii) I_1, the surface of intersection of two quadrics which touch a fixed prime Π at a fixed point O; (iii) I_2, the surface of intersection of two quadrics which touch a given plane σ along a line λ; (iv) I_3, the surface

* Severi, *Mem. Acc. Torino* (2), 52 (1903), 61–118, and *Rend. di Palermo*, 28 (1909), 33.

† Semple, *Phil. Trans. Royal Soc.* (A), 228 (1929), 331–376.

TABLE I.

Quadric Involutions in [4].

No.	Base Surface	Base Curves	Base Points	No. of linear conditions	Contact Conditions	Curve ϕ		$\phi . c_m$	$\phi . c_n$	Related Cremona Transformation	Remarks
						Order	Genus				
1		Rational c_4		1		4	0	6		$\{2, 4, 4\}$	
2		Elliptic c_4	$Y_1 Y_2$			4	1	4			$\phi \equiv Y_1 Y_2$
3		Rational c_3	$Y_1 Y_2 Y_3$			5	1	5			$\phi \equiv Y_1 Y_2 Y_3$
4		c_1		1	Fixed point of osculation Y_1	7	0	3		$\{2, 4, 7\}$	$\phi \equiv Y_1^3$
5		c_1	Y_1		Touch planes σ_1 at Y_2 and σ_2 at Y_3	7	1	3			$\phi \equiv Y_1 Y_2^2 Y_3^2$, the branches through Y_2, Y_3 touching σ_1, σ_2 there
6		c_2	$Y_1 Y_2$		Touch plane σ at Y_3	6	1	4			$\phi \equiv Y_1 Y_2 Y_3^2$, each branch through Y_3 touching σ
7		c_1, c_1' skew	$Y_1 Y_2 Y_3 Y_4$			6	1	3	3		$\phi \equiv Y_1 Y_2 Y_3 Y_4$
8		c_1, c_1' skew		1	Touch plane σ at Y_1	6	0	3	3	$\{2, 4, 6\}_1$	$\phi \equiv Y_1^2$, each branch at Y_1 touching σ
9		c_1	$Y_1 Y_2$	1	Quadrics touch a fixed surface I along c_1, two of them intersecting in I	6	0	4		$\{2, 4, 6\}_2$	$\phi \equiv Y_1 Y_2$ and touches I at four points of c_1
10		c_1, c_2 skew	Y_1	1		5	0	3	4	$\{2, 4, 5\}_2$	$\phi \equiv Y_1$
11			Y_1		Fixed point of osculation at Y_2; touch plane σ at Y_3	8	1				$\phi \equiv Y_1 Y_2^3 Y_3^2$, each branch at Y_3 touching σ
12					Touch planes $\sigma_1, \sigma_2, \sigma_3$ at Y_1, Y_2, Y_3	8	2				$\phi \equiv Y_1^2 Y_2^2 Y_3^2$, each branch at Y_i touching σ_i
13		$c_4 \equiv O^2$		1	Touch prime Π at O	4	0	2		$\{2, 4(1), 4\}$	$\phi \equiv O^2$
14		$c_1 \equiv O$ $c_2 \equiv O$	Y_1	1	Touch Π at O	5	0	1		$\{2, 4(1), 5\}_2$	$\phi \equiv O^2 Y_1$
15		$c_2 \equiv O$	$Y_1 Y_2$	1	Touch Π at O	6	0	2		$\{2, 4(1), 6\}_1$	$\phi \equiv O^3 Y_1 Y_2$
16		$c_1 \equiv O$ and lying in Π	Y_1	1	Touch Π at O and plane σ at Y_2	7	0	1		$\{2, 4(1), 7\}$	$\phi \equiv O^3 Y_1 Y_2^2$

17			$Y_1 Y_2$	1	Touch Π at O and plane σ at Y_3	8	0			$\{2, 4(1), 8\}$	$\phi \equiv O^4 Y_1 Y_2 Y_3^2$
18		$c_1 \equiv O$ and lying in Π	$Y_1 \dots Y_5$		Touch Π at O	7	1	1			$\phi \equiv O^3 Y_1 \dots Y_5$
19			$Y_1 \dots Y_6$		Touch Π at O	8	1				$\phi \equiv O^4 Y_1 \dots Y_6$
20		λ^2	$Y_1 Y$	1	Touch Π_1 at D_1 and Π_2 at D_2	6	0			III, 12	$\phi \equiv D_1^2 D_2^2 Y_1 Y_2$. σ is the plane common to Π_1, Π_2, and λ is the line $D_1 D_2$
21		$c_1 \equiv D_1$ and lying in Π_1. Also λ^2	Y_1	1	Touch Π_1 at D_1 and Π_2 at D_2	5	0			III, 11	$\phi \equiv D_1 D_2^2 Y_1$
22		$c_1 \equiv D_1$ lying in Π_1; $c_1' \equiv D_2$ lying in Π_2. Also λ^2		1	Touch Π_1 at D_1 and Π_2 at D_2	4	0			III, 9	$\phi \equiv D_1 D_2$
23		$c_1 \equiv D_1$, $c_1' \equiv D_1$ both in Π_1. Also λ^2		1	Touch Π_1 at D_1 and Π_2 at D_2	4	0			III, 10	$\phi \equiv D_2^2$
24		c_2 touching Π_1 at D_1. Also λ^2		1	Touch σ along λ and Π_1 at D_1	4	0	2		II, 4	$\phi \equiv D_1^2$. σ lies in Π_1 and λ goes through D_1
25		λ^2		1	Touch σ along λ, Π_1 at D_1, and a plane ρ at Y_1	6	0			II, 6	$\phi \equiv D_1^3 Y_1^2$
26		λ^2	$Y_1 Y_2 Y_3 Y_4$		Touch σ along λ and Π_1 at D_1	6	1				$\phi \equiv D_1^3 Y_1 \dots Y_4$
27		λ^2	$Y_1 Y_2$		Touch σ along λ and plane ρ at Y_3	6	1				$\phi \equiv Y_1 Y_2 Y_3^2$
28		l	$Y_1 Y_2$	1	Touch prime Π along a line l	4	0			$\{2, 4(3), 4\}$	$\phi \equiv Y_1 Y_2$ and touches Π at two points of l
29	Plane σ		$Y_1 Y_2 Y_3$	1		4	0			$\{2, 3, 4\}$	$\phi \equiv Y_1 Y_2 Y_3$ and meets σ twice. The quadrics are cones
30	Plane σ	c_1 meeting σ	Y_1	1		3	0	1		$\{2, 3, 3\}$	$\phi \equiv Y_1$ and meets σ twice. The quadrics are cones
31	Quadric surface			1		2	0			$\{2, 2, 2\}$	ϕ meets the quadric surface twice

Here $\phi . c_n$ denotes the number of variable intersections with the base curve; where there are two base curves, c_n refers to the second in the order given in the "base curve" column.

The Cremona transformation related to an involution is the one in Semple's paper from which the involution was derived.

of intersection of two quadric cones touching a prime Π along a line l; (v) K, the intersection residual to a plane, namely a cubic ruled surface meeting the plane in a conic.

(i) I is mapped on ϖ by $k_3 \equiv A_1 \dots A_5$* and Λ is the system $k_6 \equiv A_1^2 \dots A_5^2$. All the involutory nets picked out from Λ do not give different involutions in [4]; however, all the distinct involutions can be obtained from the following nets:

$$(a) \qquad \bar{k}_3 \equiv A_1 \dots A_5 + k_3 \equiv A_1 \dots A_5 X_1 X_2 \dagger.$$

$$(b) \qquad \bar{k}_3 \equiv A_1^2 A_2 \dots A_5 + k_3 \equiv A_2 \dots A_5 X_1 X_2 X_3.$$

$$(c) \qquad \bar{k}_2 \equiv A_1 \dots A_5 + k_4 \equiv A_1 \dots A_5 X_1^2 X_2^2 X_3.$$

$$(d) \qquad \bar{k}_2 \equiv A_1 \dots A_4 + k_4 \equiv A_1 \dots A_4 A_5^2 X_1^2 X_2 X_3.$$

$$(e) \qquad \bar{k}_1 \equiv A_1 A_2 + k_1' \equiv A_1 A_3 + k_4 \equiv A_2 A_3 A_4^2 A_5^2 X_1 X_2 X_3 X_4.$$

$$(f) \qquad k_6 \equiv A_1^2 \dots A_5^2 X_1^3 X_2^2 X_3.$$

$$(g) \qquad k_6 \equiv A_1^2 \dots A_5^2 X_1^2 X_2^2 X_3^2.$$

These give rise to the involutions numbered respectively 2, 3, 5, 6, 7, 11, 12.

Number 3 has formed the subject of two memoirs by Aprile‡. The system (g) is the well known Bertini involution.

The rest of the involutions with F of type I arise from the above cases when the base curve breaks up; when this happens the corresponding part in ϖ also breaks up. We thus obtain the following special cases. (We deal only with those involutions *not* derived from a Cremona transformation.)

(2) α. c_4§ becomes a pair of conics intersecting in two points; ϕ meets each twice.

β. One of the conics in α becomes a pair of intersecting lines; ϕ meets each line once.

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* $k_n \equiv A_1^{\alpha_1} \dots A_r^{\alpha_r}$ denotes the complete system of curves of order n with the fixed points A_i to multiplicity not less than α_i.

† This denotes an involutory net of sextics, each curve consisting of the fixed part $\bar{k}_3$ together with a variable part k_3.

‡ Aprile, "Di una trasformazione doppia nello spazio a quattro dimensioni", *Giornale di Matematiche*, 56 (1918), 90–112; 57 (1919), 129–150.

§ See Table I.

γ. Both conics in α break up; c_4 becomes four lines, each meeting two others; ϕ meets each line once.

(3) α. c_3 becomes a conic and a line meeting it; ϕ meets the conic three times and the line twice.

β. c_3 becomes a line and two skew lines meeting the first; ϕ meets the first line once and the others twice each.

(6) α. c_2 becomes a pair of intersecting lines; ϕ meets each twice.

(ii) I_1 is mapped by $k_3 \equiv A_1 \dots A_5$, where A_3, A_4, A_5 lie on a line d which represents the point O. Λ consists of the $k_4 \equiv A_1^2 A_2^2$ augmented by the line d counted twice. Two new involutions arise.

$$(a) \qquad d^2 + \bar{k}_1 \equiv A_1 A_2 + k_3 \equiv A_1 A_2 X_1 \dots X_5.$$

$$(b) \qquad d^2 + k_4 \equiv A_1^2 A_2^2 X_1 \dots X_6.$$

The corresponding involutions in [4] are numbers 18, 19 respectively. These have no special cases.

(iii) At two points D_1, D_2 of λ the two quadrics have the same tangent primes Π_1, Π_2 which intersect in σ. I_2 has therefore nodes at D_1, D_2; it is mapped on ϖ by $k_3 \equiv A_1 \dots A_5$, where A_1, A_2, A_3 lie on a line d_1 which represents D_1, and A_1, A_4, A_5 lie on a line d_2 which represents D_2. Λ consists of the $k_4 \equiv A_1^2 A_2 \dots A_5$ augmented by d_1 and d_2. The involutory nets are:

$$(a) \qquad d_1^2 + d_2 + k_3 \equiv A_1 A_4 A_5 X_1 \dots X_4.$$

$$(b) \qquad d_1 + d_2 + k_4 \equiv A_1^2 A_2 \dots A_5 X_1^2 X_2 X_3.$$

These give numbers 26, 27 respectively.

(iv) and (v). It is easily verified that I_3 and K give no involutions apart from those already obtained.

SECTION II.

5. *The involution by quadrics through an elliptic c_4 and two points Y_1, Y_2.* [*Case* (2).]

Denote this system of quadrics by Σ and the system of quadrics through c_4 and Y_1 only by Σ'. The quadrics of Σ' are ∞^5, of grade 3, and

therefore represent the [4] on a V_4^3 in [5]. Quadrics of Σ correspond to prime sections of V through the point O on V which corresponds to Y_2. A pair of points P, P', conjugate in the involution, will be represented on V by points Q, Q' collinear with O.

Loci on V corresponding to the base system of Σ'. A general prime m meets ϕ in four points and is represented by a quartic primal M on V; but if we make m go through Y_1 it will have only three free intersections with ϕ. M breaks up into a cubic primal corresponding to m and a fixed prime y_1 which corresponds to the point Y_1. A plane section of V meets y_1 in a single point which corresponds to the direction of the line tangent at Y_1 to the corresponding ϕ. The ∞^3 points of y_1 thus correspond to the ∞^3 points in the neighbourhood of Y_1.

Thus it follows that a curve with s branches through Y_1 will correspond to a curve on V meeting y_1 s times. The ∞^1 points near Y_1 on a Φ correspond to a line in y_1; if a surface has an s-ple point at Y_1 it will correspond to a surface meeting y_1 in a curve of order s.

The ∞^2 points near Y_1 on a quadric of Σ' correspond to a plane of y_1; a primal with Y_1 as s-ple point will correspond to a threefold meeting y_1 in a surface of order s.

A general plane u meets Φ in four points and is represented by a quartic surface U on V; u contains ∞^2 lines to each of which corresponds a conic on U; U is a Veronese surface. If u meets c_4 in a point A, one intersection with Φ is fixed; U breaks up into a ruled cubic surface and a fixed plane α which corresponds to the point A. There are ∞^2 points A and therefore ∞^2 planes α forming a threefold whose order is the number of times a ϕ meets c_4, *i.e.* four. The planes α therefore generate a V_3^4 on V.

A solid section of V meets α in one point which corresponds to the ∞^1 points near A on the corresponding Φ; each of these points presents the same condition to quadrics of Σ. A surface with c_4 as s-ple line corresponds to a surface on V meeting each α s times; a primal with c_4 as s-ple line corresponds to a threefold on V meeting each α in a curve of order s.

Now consider the quadric surfaces q through c_4 lying in the prime θ containing c_4; each of these presents a single condition to quadrics of Σ' and such a quadric contains only one q; hence to the q's there correspond on V the points r of a line R. This line meets the line l, through O and meeting y_1, which corresponds to Y_1Y_2. Since the generators of a q are chords of c_4 it follows that V_3^4 is a cone with the line R as vertex. The line R represents the prime θ.

The double primal of the involution. This is the locus of P when P', P coincide. Q' then coincides with Q, and therefore the double

primal corresponds to the intersection of V with the polar quadric of O. The polar quadric meets V in a V_3^6 with a node at O, y_1 in a quadric surface, each a in a conic, contains R and therefore l. *The double primal therefore is quartic and has* $Y_1 Y_2$ *as double line and the* q *which meets* $Y_1 Y_2$ *as double quadric surface. The prime* θ *must also be regarded as a double primal.*

The Jacobian of Σ. Suppose that P is at the vertex of a quadric point-cone of Σ. A curve ϕ through P meets the cone twice at P. Hence P' coincides with P. Now take P on θ; all quadrics through P contain the q through P; there are thus ∞^2 points P' corresponding to P and one coincides with P. *The Jacobian of* Σ *is therefore made up by* θ *and the double primal, the latter being the locus of the vertices of all the point-cones of* Σ.

The Cremona transformation associated with the involution. Let P move on a general line x; P' will then trace out a rational curve x' which meets x in the four intersections of x with the double primal. The line x corresponds to a conic z on V; x' corresponds to the residual intersection z' of V with the conical quadric surface Oz. Now x does not meet c_4 but meets one q; z therefore meets R once and no a elsewhere; the eight intersections of Oz with V_3^4 therefore lie four on z' and four where z meets R. Further, z' has a node at O and meets y_1 in two points. Hence x' *is a rational curve of order* 5 *with double points at* Y_1, Y_2 *and meeting* c_4 *in four points.*

To get the locus π' traced by P' when P moves on a prime π we consider the cone $O(M_3^4)$, where M_3^4 corresponds to π and contains R. This meets V residually in an M_3^8 with a 4-ple point at O and meeting each a in a cubic curve apart from R and y_1 in a quartic surface. Hence π' *is of order* 5 *and has* c_4 *as triple curve and* Y_1, Y_2 *as* 4*-ple points.*

The associated Cremona transformation is, therefore, symmetric and of order 5.

6. *Special Geiser involutions in* [3] *deduced from the involution in* [4]. *The congruence* Ξ.

There are ∞^2 primes β through the points Y_1, Y_2. Each meets c_4 in four coplanar points $\beta . c_4$ which lie on a conic δ passing through the fixed point D in which their plane meets the line $Y_1 Y_2$. The quadrics of Σ through a general point of δ contain the whole curve and also the line $Y_1 Y_2$; in fact, they contain the quadric surface through c_4 and D. The conics δ are the ∞^2 plane sections through D of this quadric surface.

Let P be any point of a prime β. There is one line ξ through P which meets δ and $Y_1 Y_2$; ∞^2 quadrics of Σ contain δ, $Y_1 Y_2$, and P, and therefore contain the whole line ξ. It follows that *P' lies on ξ and in β.*

One quadric of Σ through P is made up by θ and β. Hence Σ meets β in an ∞^3 linear system σ of quadric surfaces with six base points, namely Y_1, Y_2, and $\beta . c_4$; those surfaces of σ which pass through P pass also through P'. Thus *σ determines an involution in the solid β.* It is a Geiser involution with four of its base points coplanar; we shall refer to it as G_β. It is evident that we can use our previous work to obtain properties of the Geiser involution, and *vice versa.*

The double surface of G_β. This is the section by β of the double primal in [4]. It is therefore a quartic surface with δ as double conic and $Y_1 Y_2$ as double line. This surface is Cayley's seventh type of quartic scroll; any plane through $Y_1 Y_2$ meets the surface in a pair of lines intersecting on δ. The double primal is thus generated by the ∞^2 such quartic scrolls.

The lines ξ which lie in β form the congruence of lines joining $Y_1 Y_2$ to δ. This congruence is (1, 2); hence Ξ, the congruence of all the lines ξ in [4], is of class 3.

If P moves on a line x in β, then P' lies on the cubic scroll with $Y_1 Y_2$ double and passing through x and δ simply, and also on the one quadric surface which contains x. These two surfaces meet in x and in a curve x' of order 5 which is the locus of P'. This meets x four times and passes through each of the points Y_1, Y_2 twice and $\beta . c_4$ once. Now primes β are ∞^2; we can therefore regard x as a general line of [4]. Thus *x' corresponds to x both in G_β and in the involution in* [4]. This confirms the result previously obtained, but shows also that *the four points in which x' meets c_4 are coplanar.*

Since a general plane in the space [4] does not lie in any solid β, we cannot extend the method to find what surfaces correspond to any but particular planes of [4].

Further properties of Ξ. We have seen that every ξ meets $Y_1 Y_2$. Let H be any point on a ξ and let P, P' be the points determining ξ. Then ∞^2 quadrics of Σ contain H, P, and therefore also P', the whole line ξ, and the line $Y_1 Y_2$.

Thus *every ξ forms part of a degenerate curve ϕ*; the remainder consists of the line $Y_1 Y_2$ and a conic meeting both lines.

The lines of Ξ which are chords of a general ϕ form a ruled surface. Since ϕ is elliptic it lies in a prime β; moreover, every ξ which is a chord of ϕ lies in β and meets $Y_1 Y_2$. The ruled surface is thus the surface of

chords of ϕ which meet $Y_1 Y_2$. There is one quadric surface through ϕ and $Y_1 Y_2$; hence *the ruled surface is formed by that regulus of the quadric surface which does not contain* $Y_1 Y_2$.

Now consider the lines ξ which are chords of Φ; these form a V_3 which is a cone with $Y_1 Y_2$ as line-vertex. A prime through $Y_1 Y_2$ meets Φ in a ϕ whose ξ-chords are the section of V_3 by the prime. Thus V_3 *is a quadric line-cone,* $Y_1 Y_2$ *being the vertex.*

We have seen above that the lines ξ which meet a general line x form a cubic scroll with $Y_1 Y_2$ as double line and the conic δ in the β through x simple. This scroll is the section by β of the V_3 of lines ξ meeting a general plane; hence *this* V_3 *is a ruled cubic primal through the plane, having* $Y_1 Y_2$ *as double line and containing the* q *meeting* $Y_1 Y_2$.

The quadrics of Σ not containing a line ξ form a pencil and cut on the line a g_2^1 of pairs P, P'. The two coincidences of the g_2^1 arise from the two quadrics of the pencil which touch ξ, and they lie at the two intersections of ξ with the double primal, not lying on $Y_1 Y_2$.

The general Geiser involution. It is clear that the general case of the Geiser involution, where the six base points are generally situated, arises from cases (3) and (7). A discussion similar to that just given holds for each*.

SECTION III.

1. *Three involutions in ordinary space associated with cases* (3), (2), (7).

Case (3). In space [4] quadrics through c_3 and four points $Y_1, \ldots, Y_4$ are ∞^3 and have all a fifth point Y_4' in common, where Y_4' is conjugate in the involution to Y_4. Hence we can always draw an elliptic c_5 to meet c_3 in five points and to pass through $Y_1, \ldots, Y_4$ and an arbitrary point R.

A quartic primal, with c_3 as double curve and $Y_1, \ldots, Y_4$ as double points, meets this c_5 in two further points P, P' either of which, since c_5 is elliptic, determines the other. Draw then any definite quadric Ω through c_3 and $Y_1, \ldots, Y_4$; the quartic primal meets it in a surface of order 8 with $c_3, Y_1, \ldots, Y_4$ double and meets every generator of the quadric in four points. If then we project Ω on to a prime from Y_4, we get *a sextic surface with a double rational cubic curve, a double conic, and three double points. The conic meets the cubic curve three times. There are* ∞^3 *such sextic surfaces with the same fixed double elements and they determine an involution in the*

* Aprile, *loc. cit.*, discusses the case (3) in this connection.

prime. The pairs of points P, P' on c_5 project into pairs of points on an elliptic quartic curve which meets the double cubic in five points, the double conic in one point (on the tangent at Y_4 to c_5), and passes through the three double points.

Case (2). Proceeding as above, we obtain *an involution in* [3] *by sextic surfaces with a double elliptic quartic curve, a double conic* (*meeting the quartic curve in four points*), *and two double points*.

Pairs of conjugate points lie on a (plane) elliptic cubic curve which meets the double quartic in four points, the double conic once, and passes through the double points.

Case (7). We obtain *an involution in* [3] *by sextic surfaces with two double skew lines, a double conic* (*meeting each line once*), *and four double points*.

Pairs of conjugate points lie on an elliptic quintic curve which meets each double line three times, the double conic once, and passes through the double points.

Similarly we may obtain other involutions by sextic surfaces with a double conic. If we project from different points on Ω, other involutions will arise.

Other involutions in [3] *derived from cases* (3), (2), (7).

Case (2). Let Ω be a definite quadric of the system defining the involution in [4]. The quadrics residual to Ω will cut Ω in a system of quartic surfaces meeting each generator twice and through c_4, Y_1, Y_2. Conjugate pairs of points lie on an elliptic quartic curve meeting c_4 four times and passing through Y_1, Y_2.

Projecting Ω from a point O of itself on to a prime, we get three different kinds of involution as follows:

(i) O a general point on Ω.

Quartic surfaces through a fixed elliptic quartic curve, a double conic (*meeting the quartic in four points*), *and passing through two fixed points*.

Pairs of conjugate points lie on an elliptic quartic curve meeting the fixed quartic four times, the conic once, and passing through the fixed points.

(ii) O at Y_1.

Cubic surfaces through a fixed elliptic quartic curve, a conic (*meeting the quartic in four points*), *and one fixed point*.

Conjugate points lie on a (plane) elliptic cubic meeting the quartic four times, the conic once, and passing through the fixed point.

(iii) O on c_4.

Cubic surfaces through a fixed elliptic cubic curve, a conic (meeting the cubic twice), and two points.

Conjugate points lie on an elliptic quartic meeting the cubic four times, the conic once, and passing through the fixed points.

Case (3). We obtain the following involutions:

(i) *Quartic surfaces with a fixed double conic, through a rational cubic (meeting the conic three times), and three points.*

(ii) *Cubic surfaces through a rational cubic, a conic (meeting the cubic three times), and two points.*

(iii) *Cubic surfaces through two fixed conics, intersecting twice, and three points.*

Case (7). (i) *Quartic surfaces through two skew lines, a double conic (meeting each line once), and four points.*

(ii) *Cubic surfaces through two skew lines, a conic (meeting each line once), and three points.*

(iii) *Cubic surfaces through a line, a conic (meeting the line once), and five points.*

The other involutions similarly give rise to involutions in [3]. The quartic surfaces will all have a double conic and the cubic surfaces a simple conic as part of their base. By suitably choosing the prime of projection the conic can be made to break up into two intersecting lines.

SECTION IV.

8. *Involutions and Cremona transformations in* [5] *by means of quadrics.*

We saw above how Geiser involutions can be obtained from some of the involutions which we obtained in [4]. This suggests that these involutions in [4] may arise as sections of quadric involutions in [5]. Thus

case (2) suggests an involution in [5] by quadrics through a Segre quartic surface and two points, and case (3) suggests an involution by quadrics through a rational normal cubic ruled surface and three points; this is verified below. Clearly the process can be extended to as many dimensions as we wish.

9. Let F be a rational normal ruled cubic surface; it lies in a prime θ. The plane representation shows that ∞^8 quadrics pass through F. Three quadrics Ω, Ω', Ω'' through F meet further in a quintic surface F'.

Take a section of the space by a prime π and let π meet Ω in ω, and so on. Then ω meets ω' in a Segre quartic surface Φ which is met by ω'' in f and f'. If we represent Φ on a plane by $k_3 \equiv A_1 \dots A_5$, it is easy to see that $f+f'$ is mapped by a composite $k_6 \equiv A_1^2 \dots A_5^2$. f will be mapped by a $k_3' \equiv A_1^2 A_2 \dots A_5$; therefore f' will be mapped by a $k_3'' \equiv A_2 \dots A_5$ meeting k_3' in five points apart from the A_i. Hence *F' is quintic and has elliptic prime sections; it is the normal quintic del Pezzo surface.* Further, *F' meets F in a curve C of order* 5.

To find how C lies on F, we take π to contain the directrix d and two generators g_1, g_2 of F. These are mapped respectively by curves $k_2 \equiv A_4 A_5$, $k_1^{(1)} \equiv A_1 A_2$, $k_1^{(2)} \equiv A_1 A_3$, which are met, apart from the A_i, by k_3'' in 1, 2, 2 points respectively. Hence *C meets the directrix of F once and every generator twice.* The plane representation of F shows that *C is elliptic*, which is otherwise obvious, since C is clearly a prime section of F'.

The plane representation of F' shows that ∞^4 quadrics pass through F'; we cannot, therefore, obtain Cremona transformations or involutions by quadrics through F', but one interesting result appears. We can always put three quadrics through F' and a general point P. These will meet in an F through P. One generator of F passes through P and is a chord of C, and any other quadric of the system through P will contain this chord. We have thus the well known result that *the normal quintic del Pezzo surface has one apparent double point.* Also *the linear system of quadrics through the surface is compounded from the linear congruence of its chords.*

Now map F' on a plane by $k_3 \equiv B_1 B_2 B_3 B_4$; C is mapped by one of these curves; hence sections of F' by quadrics through F are represented by this system of cubic curves. There is only one involutory net, namely $k_3 \equiv B_1 \dots B_4 X_1 X_2 X_3$. We have thus an involution determined by

Quadrics through F and three points Y_1, Y_2, Y_3. The free curves of intersection of four quadrics are the prime sections of F's through Y_1, Y_2, Y_3.

There is one homaloidal net, with a sub-case, namely $k_2 \equiv B_1 \dots B_4 + k_1$. The Cremona transformation is by

Quadrics through F and a conic c_2 meeting F twice. The free curves common to four quadrics are rational cubic curves meeting F three times and c_2 twice. *The reverse transformation is therefore cubic.*

The sub-case is when k_2 becomes the two lines $B_1 B_2$ and $B_3 B_4$; c_2 then becomes two intersecting lines each meeting F once, and each is met by a free curve once.

10. Now take F to be a Segre quartic surface lying in a prime θ. Proceeding as before, we find that f is mapped by a $k_3' \equiv A_1 \dots A_5$ and f' by another $k_3'' \equiv A_1 \dots A_5$. Thus *$F'$ is also a Segre quartic surface and C is quartic.* Taking π to contain two intersecting lines of F and a conic meeting each once, we find that *C meets each line of F once and therefore is elliptic.* In fact, the two primes containing F, F' meet in a solid whose sections of F, F' are both C.

Mapping F' by $k_3 \equiv B_1 \dots B_5$, we find that C is mapped by one of these curves; hence sections of F' by quadrics through F are represented by this system of cubic curves. There is one involutory net, $k_3 \equiv B_1 \dots B_5 X_1 X_2$, and the corresponding involution is by

Quadrics through F and two points Y_1, Y_2, The free curve common to four quadrics is a prime section of some F', passes through Y_1, Y_2, and meets F in four coplanar points.

This is the analogue of case (2) above.

There is one homaloidal net, $\bar{k}_2 \equiv B_1 \dots B_5 + k_1$. The Cremona transformation is therefore by

Quadrics through F and a line meeting F once. The free curve common to four quadrics is a rational cubic curve meeting F three times and the line twice. *The reverse transformation is therefore cubic.*

11. Take F to be a rational normal quartic ruled surface. f is mapped by a $k_2 \equiv A_1 A_2$ and f' by a $k_4 \equiv A_1 A_2 A_3^2 A_4^2 A_5^2$. Hence *$F'$ is a rational quartic surface, with rational prime sections, meeting F in a sextic curve C.* Taking π to contain a generator and a rational cubic on F, we find that the generator is mapped by $k_1 \equiv A_1 A_2$ and the cubic by a k_1'. Thus *C meets every generator of F twice and every rational cubic four times, and therefore is elliptic.*

Now F' is either a Veronese surface or another rational normal quartic ruled surface. It cannot be the former because it would follow, as in paragraph 9, that through every point of [5] there would pass a chord of a Veronese surface, and this is not true. The alternative holds and we have

the theorem that *through every point of* [5] *there passes one chord of a rational normal ruled quartic surface.*

There are no involutions or Cremona transformations associated with F.

12. Take F to be a Veronese surface. As before, we find that *F' is quartic, has rational prime sections, and meets F in a sextic curve C.* Now we have seen that F' cannot be the ruled surface, such a surface being residual to another of the same kind; *F' is therefore a Veronese surface.* We find that *C meets each conic of F three times and is therefore elliptic.*

F' is mapped on a plane by all the conics of the plane; C is mapped by an elliptic cubic curve; hence sections of F' by quadrics through F are mapped by the lines of the plane, *i.e.* are the conics of F' themselves—which is otherwise obvious.

There is no involution and only one Cremona transformation, namely by *quadrics through F.* The reverse transformation is quadric; it is easily seen to be the same as the direct transformation*.

13. Take F to be a plane ω. We find that *F' is of order* 7, *has prime sections of genus* 3, *and meets F in an elliptic cubic curve C.*

A solid through F meets three quadrics through F each in ω and in another plane; the three planes so obtained meet in one point which is on F'. Hence *F' projects birationally from F on to a plane and therefore is rational.*

Call the three quadrics Ω, Ω', Ω'' and take three arbitrary points U, V, W on F. The primes tangent to Ω', Ω'' at U contain the lines through U common to the two quadrics; the primes meet in a solid which cuts Ω', Ω'' each in a plane, besides F, through U. These planes meet in the line UX, which is then the only one common to Ω', Ω'', passing through U, and not lying in F. We take X to be the second intersection of the line with Ω. In the same way two other lines VY, WZ are obtained.

With $XYZUVW$ as simplex of reference, F will have the equations $x=y=z=0$, and we can take the equations of Ω, Ω', Ω'' respectively to be

$$(yz+zx+xy)+u(x+y+z)+v(z+x)+w(x+y)=0,$$

$$(yz+zx+xy)+u(y+z)+v(x+y+z)+w(x+y)=0,$$

$$(yz+zx+xy)+u(y+z)+v(z+x)+w(x+y+z)=0,$$

where the coefficients of all the terms are omitted.

* Semple, *loc. cit.*, 353, gives this transformation.

These equations lead to

$$u\Delta = \theta, \quad v\Delta = \phi, \quad w\Delta = \psi,$$

where Δ is the determinant of the coefficients of u, v, w, and is therefore of degree 3 in x, y, z; and θ is obtained by replacing the first column of Δ by the left-hand members (with the signs changed) of the above equations, and so is of degree 4 in x, y, z; ϕ and ψ are obtained similarly.

A general prime is

$$Ax+By+Cz+Du+Ev+Fw = 0;$$

it meets F' in a curve which projects from F by means of the cone

$$(Ax+By+Cz)\Delta+D\theta+E\phi+F\psi = 0,$$

which is quartic. The cone contains nine fixed solids whose equations are obtained by equating to zero all the two-rowed minors of Δ. Hence *F' is represented on a plane by quartic curves with nine base points.*

Denoting this system by $k_4 \equiv A_1 \dots A_9$, we find that C is mapped by a $k_3 \equiv A_1 \dots A_9$; the sections of F' by quadrics through F are therefore mapped by $k_5 \equiv A_1 \dots A_9$.

The homaloidal nets are:

(1) $$\bar{k}_2 \equiv A_1 \dots A_5 + k_3 \equiv A_6 \dots A_9 X_1^2,$$

(2) $$\bar{k}_3 \equiv A_1 \dots A_6 + k_2 \equiv A_7 A_8 A_9,$$

(3) $$\bar{k}_3 \equiv A_1 \dots A_7 + k_2 \equiv A_8 A_9 X_1,$$

(4) $$\bar{k}_3 \equiv A_1 \dots A_8 + k_2 \equiv A_9 X_1 X_2,$$

(5) $$\bar{k}_3 \equiv A_1 \dots A_9 + k_2 \equiv X_1 X_2 X_3,$$

(6) $$\bar{k}_4 \equiv A_1 \dots A_9 + k_1.$$

We have then the following homaloidal systems of quadrics. We shall again use ϕ to denote the free curve common to four general quadrics

(1) *Quadrics through F and a rational c_3 meeting F once and which touch a fixed plane σ at a fixed point Y_1.* ϕ is a rational octavic curve meeting F five times, c_3 six times, and having a node at Y_1, each branch touching σ there. The reverse transformation is of order 8.

(2) *Quadrics through F and an elliptic c_6 meeting F three times.* ϕ is a rational quintic curve meeting F three times and c_6 six times. The reverse transformation is of order 5.

(3) *Quadrics through F, an elliptic c_5 which meets F twice, and a fixed point Y_1.* ϕ is a rational sextic curve through Y_1, meeting F four times and c_5 six times. The reverse transformation is sextic.

(4) *Quadrics through F, an elliptic c_4 meeting F once, and two fixed points Y_1, Y_2.* ϕ is a rational septimic curve through Y_1, Y_2, meeting F five times and c_4 six times. The reverse transformation is of order 7.

(5) The curve $\bar{k}_3$ is that which maps C. *All the surfaces F' therefore meet F in a* FIXED *curve C and the quadrics pass through three fixed points Y_1, Y_2, Y_3.* ϕ is a rational octavic curve through Y_1, Y_2, Y_3 and meets C in six points. The reverse transformation is of order 8.

(6) *Quadrics through F and a c_7 of genus 3 which meets F three times.* ϕ is a rational quartic curve meeting F three times and c_7 four times. The reverse transformation is quartic.

The degenerations of the base curves may be discussed in the usual way.

The involutory nets are:

(1) $\bar{k}_2 \equiv A_1 A_2 \quad + k_3 \equiv A_3 \dots A_9,$

(2) $\bar{k}_2 \equiv A_1 A_2 A_3 + k_3 \equiv A_4 \dots A_9 X_1,$

(3) $\bar{k}_2 \equiv A_1 \dots A_4 + k_3 \equiv A_5 \dots A_9 X_1 X_2,$

(4) $\bar{k}_2 \equiv A_1 \dots A_5 + k_3 \equiv A_6 \dots A_9 X_1 X_2 X_3,$

(5) $\bar{k}_3 \equiv A_1 \dots A_7 + k_2 \equiv A_8 A_9 L_1,$

(6) $\bar{k}_3 \equiv A_1 \dots A_8 + k_2 \equiv A_9 X_1 L_1,$

(7) $\bar{k}_3 \equiv A_1 \dots A_9 + k_2 \equiv X_1 X_2 L_1.$

The involutions are as follows:

(1) *Quadrics through F and a rational c_6 meeting F four times.* ϕ is an elliptic quintic curve meeting F twice and c_6 six times.

(2) *Quadrics through F and a rational c_5 meeting F three times and a fixed point Y_1.* ϕ is an elliptic sextic curve meeting F three times, c_5 six times, and passes through Y_1.

(3) *Quadrics through F and a rational c_4 meeting F twice and two fixed points Y_1, Y_2.* ϕ is an elliptic septimic curve through Y_1, Y_2, meeting F four times and c_4 six times.

(4) *Quadrics through F and a rational c_3 meeting F once, and three fixed points Y_1, Y_2, Y_3.* ϕ is an elliptic octavic curve through Y_1, Y_2, Y_3, meeting F five times and c_4 six times.

(5), (6), (7) are similar to the homaloidal cases (3), (4), (5), where we replace a fixed base point by a linear condition not affecting the grade.

14. Take F to be a quadric surface. We find that *F' is sextic, has prime sections of genus 2, and meets F in an elliptic quartic curve.* F' cannot be ruled because the normal space for a sextic ruled surface with prime sections of genus 2 is of not more than 4 dimensions*. F' therefore belongs to a class of surfaces discussed by Castelnuovo†, namely the F^n whose prime sections are of genus $p>1$ and hyper-elliptic. These are rational and the normal space is a $[n-p+1]$. When $n<4p+4$, F^n is the projection of an F^{4p+4} in a $[3p+5]$ from $4p+4-n$ points of itself. It follows that *F' is rational and normal.* Its plane representation is by $k_4 \equiv A_1^2 A_2 \dots A_7$. C is mapped by a $k_3 \equiv A_1 \dots A_7$; hence sections of F' by quadrics through F are mapped by $k_5 \equiv A_1^3 A_2 \dots A_7$.

The homaloidal nets are:

(1) $\qquad k_5 \equiv A_1^3 A_2 \dots A_7 X_1^3,$

(2) $\qquad \bar{k}_1 \qquad +k_4 \equiv A_1^3 A_2 \dots A_7,$

(3) $\qquad \bar{k}_1 \equiv A_7 \qquad +k_4 \equiv A_1^3 A_2 \dots A_6 X_1,$

(4) $\qquad \bar{k}_1 \equiv A_6 A_7 \qquad +k_4 \equiv A_1^3 A_2 \dots A_5 X_1 X_2,$

(5) $\qquad \bar{k}_2 \equiv A_1 A_6 A_7 \qquad +k_3 \equiv A_1^2 A_2 \dots A_5,$

(6) $\qquad \bar{k}_2 \equiv A_1 A_5 A_6 A_7 \; +k_3 \equiv A_1^2 A_2 \dots A_4 X_1,$

(7) $\qquad \bar{k}_2 \equiv A_1 A_4 \dots A_7 \; +k_3 \equiv A_1^2 A_2 A_3 X_1 X_2,$

(8) $\qquad \bar{k}_3 \equiv A_1^2 A_4 \dots A_7 + k_2 \equiv A_1 A_2 A_3,$

(9) $\qquad \bar{k}_3 \equiv A_1^2 A_3 \dots A_7 + k_2 \equiv A_1 A_2 X_1,$

(10) $\qquad \bar{k}_3 \equiv A_1^2 A_2 \dots A_7 + k_2 \equiv A_1 X_1 X_2,$

(11) $\qquad \bar{k}_4 \equiv A_1^3 A_2 \dots A_7 + k_1.$

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* Edge, *Theory of ruled surfaces* (Cambridge, 1931), 310.

† *Rend. di Palermo*, 4 (1890), 73. See also Segre, *Encyk. d. Math. Wiss.*, III C7, 921.

The involutory nets are:

(1) $k_5 \equiv A_1^3 A_2 \dots A_7 X_1^2 X_2^2$,

(2) $\bar{k}_1 \equiv A_7 \quad +k_4 \equiv A_1^3 A_2 \dots A_6 L_1$,

(3) $\bar{k}_1 \equiv A_6 A_7 \quad +k_4 \equiv A_1^3 A_2 \dots A_5 X_1 L_1$,

(4) $\bar{k}_1 \equiv A_1 \quad +k_4 \equiv A_1^2 A_2 \dots A_7 X_1^2$,

(5) $\bar{k}_1 \equiv A_1 A_7 \quad +k_4 \equiv A_1^2 A_2 \dots A_6 X_1^2 X_2$,

(6) $\bar{k}_2 \equiv A_1 A_5 A_6 A_7 \quad +k_3 \equiv A_1^2 A_2 A_3 A_4 L_1$,

(7) $\bar{k}_2 \equiv A_1 A_4 \dots A_7 \quad +k_3 \equiv A_1^2 A_2 A_3 X_1 L_1$,

(8) $\bar{k}_1 \equiv A_1 + \bar{k}_1' \equiv A_1 \quad +k_3 \equiv A_1 \dots A_7$,

(9) $\bar{k}_1 \equiv A_1 A_7 + \bar{k}_1' \equiv A_1 \quad +k_3 \equiv A_1 \dots A_6 X_1$,

(10) $\bar{k}_1 \equiv A_1 A_7 + \bar{k}_1' \equiv A_1 A_6 + k_3 \equiv A_1 \dots A_5 X_1 X_2$,

(11) $\bar{k}_3 \equiv A_1^2 A_3 \dots A_7 \quad +k_2 \equiv A_1 A_2 L_1$,

(12) $\bar{k}_3 \equiv A_1^2 A_2 \dots A_7 \quad +k_2 \equiv A_1 X_1 L_1$.

The properties of the corresponding Cremona transformations and involutions are exhibited in Table II. As in Table I, $\phi \,.\, c_m$ denotes the number of variable points common to ϕ and a base curve c_m; $\phi \,.\, F$ the number of variable points common to ϕ and the base surface F; and $c_m \,.\, F$ the number of intersections of c_m and F. In the case of the Cremona transformations, the order of a reverse transformation is the order of the corresponding curve ϕ. The degenerate base curves are not discussed.

The plane representation of F' shows that ∞^3 quadrics pass through the surface. We can therefore put three independent quadrics through F' and any point P and these will meet in an F through P. Two generators of F pass through P and each meets C (and so F') twice. There remains a residual pencil of quadrics; that which goes through P will automatically contain the two generators of F. Hence *the rational normal sextic surface with prime sections of genus 2 has two apparent double points. All quadrics through the surface form a linear system compounded from the quadratic congruence of its chords.*

15. It will be clear that we can obtain Cremona involutions and transformations in [4] by projection by a method similar to that of paragraph 7. Properties of the "Geiser" involutions in [5] will be easily deduced from those of the analogous involutions in three or four dimensions.

CREMONA TRANSFORMATIONS.

No.	Additional Base Curves		Base Points	No. of Linear Conditions	Contact Conditions	Curve ϕ	$\phi.c_m$	$\phi.c_n$	$\phi.F$	$c_m.c_n$	$c_m.F$	$c_n.F$	Remarks
	c_m	c_n											
1					Fixed point of osculation Y_1	Rational, Octavic			6				ϕ has a triple point at Y_1
2, 5, 8, 11	Rational c_4					Rational, Quartic	4		3		3		
3, 6, 9	Rational c_3		Y_1			Rational, Quintic	4		4		2		$\phi \equiv Y_1$
4, 7, 11	Conic c_2		$Y_1 Y_2$			Rational, Sextic	4		5		1		$\phi \equiv Y_1 Y_2$

INVOLUTIONS.

No.	Additional Base Curves		Base Points	No. of Linear Conditions	Contact Conditions	Curve ϕ	$\phi.c_m$	$\phi.c_n$	$\phi.F$	$c_m.c_n$	$c_m.F$	$c_n.F$	Remarks
	c_m	c_n											
1					Touch planes σ_1 at Y_1 and σ_2 at Y_2	Elliptic, Octavic			6				ϕ has nodes at $Y_1 Y_2$ and touches σ_1, σ_2 there
2, 6, 11	Rational c_4			1		Rational, Quintic	4		4		2		
3, 7, 12	Conic c_2		Y_1	1		Rational, Quintic	4		5		1		$\phi \equiv Y_1$
4	Conic c_2				Touch a plane σ at Y_1	Elliptic, Sextic	2		4		2		ϕ has a node at Y_1 and touches σ there
5	Line c_1		Y_2		Touch a plane σ at Y_1	Elliptic, Septemic	2		5		3		ϕ has a node at Y_1 and touches σ there. $\phi \equiv Y_2$
8	Conic c_2	Conic c_2'				Elliptic, Quartic	2	2	2	0	2	2	
9	Line c_1	Conic c_2	Y_1			Elliptic, Quintic	2	2	3	0	1	2	$\phi \equiv Y_1$
10	Line c_1	Line c_1'	$Y_1 Y_2$			Elliptic, Sextic	2	2	4	0	1	1	$\phi \equiv Y_1 Y_2$

ALGEBRAIC NUMBER-FIELDS WITH TWO INDEPENDENT UNITS

By W. E. H. BERWICK.

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1. A rational irreducible algebraic equation of degree n, with r real roots and s pairs of conjugate complex roots, defines n conjugate fields of algebraic numbers, though these are not necessarily all distinct, in each of which the most general unit is

$$\alpha\epsilon_1^{\,l}\,\epsilon_2^{\,m}\ldots\epsilon_{r+s-1}^{\,t},$$

where $\alpha^q = 1$ and $\epsilon_1, \epsilon_2, \ldots, \epsilon_{r+s-1}$ are a primitive set of independent units of the field. Dedekind's formula for the number of classes of ideals in a number-field has been transformed in various ways, but in each of them it takes the form

$$hR = C \times \text{an analytic function},$$

and fails to separate the class-number from the regulator. When $r+s=2$ all the units are $\alpha\epsilon^t$, where $\alpha^q = 1$, and in each case ϵ can be found by the finite processes of arithmetic. The present purpose is to establish a finite method of finding two primitive units when there are only two, *i.e.* when

$$n=3,\ r=3;\quad n=4,\ r=2,\ s=1;\quad n=5,\ r=1,\ s=2;\quad n=6,\ s=3.$$

Various numerical examples are given, showing that the method is powerful enough to obtain the two independent units in many cases.

A method of obtaining a primitive pair of units in a cubic field of positive discriminant has been previously indicated by Woronoi*, but the subject does not seem to have been developed during the last thirty-five years.

* G. Woronoi, *A generalization of the algorithm of continued fractions* (Warsaw, 1896) (Russian), 1–203. See also Uspensky, *Trans. American Math. Soc.*, 33 (1930), 1–22 (1); and *Jahrbuch über die Fortschritte der Math.*, 27 (1896), 170–174, for a summary of Woronoi's work.

2. We first investigate the units in a cubic field of positive discriminant. Taking θ, θ', θ'' to be the conjugate irrationalities, and η_1, η_2 to be a primitive pair of units in (θ), the general unit of this field is $\pm\eta_1^x\eta_2^y$, where x, y are rational integers.

I. There exists a unit ϵ such that

$$|\epsilon - k|$$

is as small as we please, k being arbitrary. It is sufficient to suppose that k is positive, since $-\epsilon$ is a unit when ϵ is a unit. Putting $\epsilon = \eta_1^x\eta_2^y$, we have

$$\log\epsilon = x\log\eta_1 + y\log\eta_2,$$

and the latter can be made to approach arbitrarily near to $\log k$. Hence, by suitable choice of x and y, $|\epsilon - k|$ can be made arbitrarily small.

II. There exists a unit ϵ such that $\pm\epsilon/\epsilon'$ is as nearly equal to k as we please. Putting $\epsilon = \eta_1^x\eta_2^y$, we have

$$\log\left|\frac{\epsilon}{\epsilon'}\right| = x\log\left|\frac{\eta_1}{\eta_1'}\right| + y\log\left|\frac{\eta_2}{\eta_2'}\right|,$$

and this can be made arbitrarily near to $\log k$ by suitable choice of x, y.

III. There exists a unit ϵ such that

$$\epsilon > 1, \quad |\epsilon'| < 1, \quad |\epsilon''| < 1.$$

It is sufficient to suppose that $\eta_1 > 1$; otherwise η_1 can be changed into $-\eta_1$ or $\pm 1/\eta_1$; similarly $\eta_2 > 1$. If $\epsilon = \eta_1^x\eta_2^y$,

$$\log|\epsilon'| = x\log|\eta_1'| + y\log|\eta_2'|, \quad \log|\epsilon''| = x\log|\eta_1''| + y\log|\eta_2''|.$$

Choosing (x, y) to be an integer point on the negative side of each of the crossing lines

$$x\log|\eta_1'| + y\log|\eta_2'| = 0, \quad x\log|\eta_1''| + y\log|\eta_2''| = 0,$$

we have $|\epsilon'| < 1$, $|\epsilon''| < 1$, whence $\epsilon > 1$ as required.

IV. Among the several units satisfying the inequalities in III we take ϵ_1 to be that for which ϵ is least. Such a unit exists, since the number of solutions of the inequalities $|\epsilon| < K$, $|\epsilon'| < 1$, $|\epsilon''| < 1$ is finite. Similarly ϵ_2 is taken to be the one for which $|\epsilon| < 1$, $|\epsilon''| < 1$, ϵ' is positive and least, and ϵ_3 to be that for which $|\epsilon| < 1$, $|\epsilon'| < 1$, and ϵ'' positive and least.

Since there are only two independent units in the field, there must be a relation

$$\epsilon_1^{\,l}\,\epsilon_2^{\,m}\,\epsilon_3^{\,n} = \pm 1,$$

where l, m, n are rational integers.

(i) It is sufficient to suppose that l, m, n have no common factor, for, if all are divisible by p, $\epsilon_1^{\,l/p}\,\epsilon_2^{\,m/p}\,\epsilon_3^{\,n/p}$ is real and is a cyclotomic unit, so that the above relation can be replaced by one with smaller exponents.

(ii) The indices l, m, n can be assumed to be all positive. For, if

$$l < 0, \quad m > 0, \quad n > 0,$$

then $\quad |\epsilon_1^{\,l}| < 1, \ |\epsilon_2^{\,m}| < 1, \ |\epsilon_3^{\,n}| < 1, \quad \text{and} \quad \epsilon_1^{\,l}\,\epsilon_2^{\,m}\,\epsilon_3^{\,n} < 1,$

contrary to supposition.

(iii) If m, n have a common factor p,

$$\epsilon_1^{\,l/p}(\epsilon_2^{\,m/p}\,\epsilon_3^{\,n/p}) = \pm 1.$$

The second factor being in the field (θ), $\epsilon_1^{\,l'/p} \equiv \eta$, where $l' \equiv l \pmod{p}$ and $l' < p$, is a unit of (θ). Hence

$$1 < \eta < \epsilon_1, \quad |\eta'| < 1, \quad |\eta''| < 1,$$

contrary to the assumption that ϵ_1 is the smallest unit in the θ-approximation. It follows that $dv(m, n) = 1$; similarly

$$dv(n, l) = 1, \quad dv(l, m) = 1.$$

V. We now have $\epsilon_1^{\,l}\,\epsilon_2^{\,m}\,\epsilon_3^{\,n} = 1$, where l, m, n are all positive integers and no two have any common factor. It will next be proved that

$$l = m = n = 1.$$

Assuming that $\qquad m > l \geqslant n,$

since $\qquad (\epsilon_1\,\epsilon_2)^l\,\epsilon_2^{\,m-l}\,\epsilon_3^{\,n} = 1, \ \text{we have} \ \epsilon_1\,\epsilon_2 > 1;$

since $\qquad (\epsilon_1'\,\epsilon_2')^l(\epsilon_2')^{m-l}(\epsilon_3')^n = 1, \ \text{we have} \ \epsilon_1'\,\epsilon_2' \leqslant 1;$

and since $\qquad \epsilon_1'' < 1, \ \epsilon_2'' < 1, \ \text{we have} \ \epsilon_1''\,\epsilon_2'' < 1.$

If $\epsilon_1'\,\epsilon_2' < 1$, the unit $\epsilon_1\,\epsilon_2$ comes before ϵ_1 in the θ-approximation, contrary to the supposition that ϵ_1 comes first.

If $\epsilon_1' \epsilon_2' > 1$, put $\epsilon_1 \epsilon_2 \rho = 1$, then

$$\rho < 1, \quad \rho' < 1, \quad \rho'' > 1;$$

also

$$(\rho'')^l = (\epsilon_2'')^{m-l}(\epsilon_3'')^n$$

$$< (\epsilon_3'')^l, \quad \text{since } m > l.$$

Hence $\rho'' < \epsilon_3''$, contrary to the assumption that ϵ_3'' comes first in the θ''-approximation. It follows as stated that

$$l = m = n = 1.$$

VI. It can also be proved that any two of the units ϵ_1, ϵ_2, ϵ_3 form a primitive pair of units in the field (θ), *i.e.* that any unit ϵ of the field is of the form $\pm \epsilon_1^f \epsilon_2^g$ for integral values of f, g. Since there are only two independent units in the field,

$$\epsilon^s = \pm \epsilon_1^f \epsilon_2^g,$$

and it is sufficient to suppose that $0 < f \leqslant g < s$.

If $\epsilon_1^f \epsilon_2^g > 1$, $\epsilon_1'^f \epsilon_2'^g < 1$, the unit ϵ comes before ϵ_1 in the θ-approximation, and if $\epsilon_1^f \epsilon_2^g < 1$, $\epsilon_1'^f \epsilon_2'^g > 1$, ϵ' comes before ϵ_2' in the θ'-approximation, whereas the inequalities $\epsilon_1^f \epsilon_2^g < 1$, $\epsilon_1'^f \epsilon_2'^g < 1$ are impossible together. There remains the possibility

$$\epsilon_1^f \epsilon_2^g > 1, \quad \epsilon_1'^f \epsilon_2'^g > 1;$$

we then put

$$\epsilon^{-s} = \epsilon_1^{-f} \epsilon_2^{-g}, \quad (\epsilon')^{-s} = (\epsilon_1')^{-f} (\epsilon_2')^{-g}.$$

This involves $|\epsilon^{-1}| < 1$, $|(\epsilon')^{-1}| < 1$, and

$$1 < (\epsilon'')^{-s} = (\epsilon_1'')^{-f} (\epsilon_2'')^{-g} < (\epsilon_3'')^s, \quad |(\epsilon'')^{-1}| < \epsilon_3'',$$

so that ϵ_3'' cannot be the first unit in the θ''-approximation, contrary to supposition.

It follows that any two of the three units ϵ_1, ϵ_2, ϵ_3 form a primitive pair of units of (θ). These three units will be called a primitive set.

To calculate ϵ_1 in a numerical case it is sufficient to find the solution of

$$(x + y\omega_1 + z\omega_2)(x + y\omega_1' + z\omega_2')(x + y\omega_1'' + z\omega_2'') = \pm 1,$$

where $(1, \omega_1, \omega_2)$ is the integral basis of (θ), for which

$$|x + y\omega_1' + z\omega_2'| < 1, \quad |x + y\omega_1'' + z\omega_2''| < 1,$$

and $x+y\omega_1+z\omega_2$ is positive and least. The units ϵ_2 and ϵ_3 are similarly obtained and the relation $\epsilon_1\epsilon_2\epsilon_3 = \pm 1$ provides a check on the work.

When the cubic field is normal, so that $\theta' = R_1(\theta)$, $\theta'' = R_2(\theta)$, we have

$$\epsilon_1 = \epsilon_2' = \epsilon_3'', \quad \epsilon_1' = \epsilon_2'' = \epsilon_3, \quad \epsilon_1'' = \epsilon_2 = \epsilon_3'.$$

In this field the calculation of ϵ_1 determines all the units.

3. As a numerical example we take the field chosen by Woronoi*,

$$\theta^3-7\theta-2=0, \quad D(\theta) = 2^4 . 79,$$

for which $$\mathfrak{O} = \{1, \theta, \tfrac{1}{2}(\theta^2+\theta)\}, \quad D = 2^2 . 79.$$

Taking $$f = x+\theta y+\tfrac{1}{2}(\theta^2+\theta)z,$$

we have $$f_1 = x+2{\cdot}778y+5{\cdot}249z,$$

$$f_2 = x-0{\cdot}289y-0{\cdot}103z,$$

$$f_3 = x-2{\cdot}489y+1{\cdot}854z.$$

If $-1<f_2<1$, $-1<f_3<1$,

$$x : y : z = -0{\cdot}7 : -2{\cdot}0 : -2{\cdot}2 \text{ nearly},$$

and $x=1$, $y=2$, $z=2$ gives

$$f_1 = 17{\cdot}054, \quad f_2 = 0{\cdot}216, \quad f_3 = -0{\cdot}270, \quad f_1f_2f_3 = -1,$$

$$\epsilon_1 = \theta^2+3\theta+1.$$

Next, when $-1<f_1<1$, $-1<f_3<1$,

$$x : y : z = 17 : 3 : -5, \text{ nearly},$$

and $x=10$, $y=2$, $z=-3$ gives

$$f_1 = -0{\cdot}19, \quad f_2 = 9{\cdot}73, \quad f_3 = -0{\cdot}54, \quad f_1f_2f_3 = 1,$$

$$\epsilon_2 = \tfrac{1}{2}(-3\theta^2+\theta+20).$$

Similarly $$\epsilon_3 = \tfrac{1}{2}(\theta^2-3\theta),$$

and, by multiplication, $$\epsilon_1\epsilon_2\epsilon_3 = +1.$$

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* *Loc. cit.*, 163–171, 199–203.

Woronoi* gives the units ϵ_1, ϵ_2 as a fundamental pair. The fact that his longer method leads to the same primitive pair of units as the present method appears to be accidental in fields of smaller discriminant.

To prove that the class-number of this field is unity we note that $\sqrt{D} < 18$, so it is only necessary to show that prime ideals whose norms are powers of 2, 3, 5, 7, 11, 13, 17 are principal.

Since $N(\theta) = 2$, $N(\theta-1) = 8$,

$$N\{\tfrac{1}{2}(\theta^2-\theta)\} = 2,$$

and
$$(2) = \{\tfrac{1}{2}(\theta^2-\theta)\}^2\theta = \mathfrak{p}_1{}^2\mathfrak{p}_2,$$

a product of principal ideals.

The cubic z^3-7z-2 is irreducible (mod 3), (mod 5), (mod 7), and (mod 13), so (3), (5), (7), (13) are prime ideals in (θ), whereas, since

$$z^3-7z-2 \equiv (z-4)(z^2+4z+9) \pmod{17},$$

and
$$\equiv (z-5)(z^2+5z-4) \pmod{11},$$

we have $N(4-\theta) = 2.17$ and $N(5-\theta) = 8.11$.

Hence

$$(17,\ 4-\theta) = (4-\theta)/\mathfrak{p}_2, \quad (11,\ 5-\theta) = (5-\theta)/\mathfrak{p}_1{}^3,$$

and all the ideals are principal.

4. In the numerical examples which follow the class-number of each field is added.

(1)–(4). Symmetric (H_6) cubic fields.

(1) $\theta^3-5\theta-3 = 0$, $D(\theta) = D = 257$,

$\theta = 2{\cdot}49$, $\epsilon_1 = \theta+1$, $h = 1$,

$\theta' = -0{\cdot}66$, $\epsilon_2 = -3\theta^2+2\theta+14$, $\epsilon_1\epsilon_2\epsilon_3 = -1$,

$\theta'' = -1{\cdot}83$, $\epsilon_3 = \theta^2-2\theta-2$.

(2) $\theta^3-7\theta-3 = 0$, $D(\theta) = D = 1129$,

$\theta = 2{\cdot}84$, $\epsilon_1 = \theta^2+3\theta+1$, $h = 1$,

$\theta' = -0{\cdot}44$, $\epsilon_2 = -2\theta^2+\theta+14$, $\epsilon_1\epsilon_2\epsilon_3 = -1$,

$\theta'' = -2{\cdot}40$, $\epsilon_3 = 2\theta^2-5\theta-2$.

* *Loc. cit.*, 196–198.

(3) $\theta^3-12\theta-1=0$, $D(\theta)=D=3^4.5.17$,

$\theta=3{\cdot}50$, $\epsilon_1=4\theta^2+14\theta+1$, $h=3$,

$\theta'=-0{\cdot}08$, $\epsilon_2=-\theta^2+12$, $\epsilon_1\epsilon_2\epsilon_3=-1$,

$\theta''=-3{\cdot}42$, $\epsilon_3=2\theta^2-7\theta$.

(4) $\theta^3-20\theta-3=0$, $D(\theta)=D=11.2887$,

$\theta=4{\cdot}545$, $\epsilon_1=2206\theta^2+10027\theta+1456$, $h=1$,

$\theta'=-0{\cdot}150$, $\epsilon_2=-1152\theta^2+173\theta+23014$, $\epsilon_1\epsilon_2\epsilon_3=1$,

$\theta''=-4{\cdot}395$, $\epsilon_3=38\theta^2-167\theta-26$.

(5)–(6). Cyclical (C_3) cubic fields.

(5) $\theta^3+\theta^2-26\theta+41=0$, $D(\theta)=D=79^2$,

$\theta=3{\cdot}12$, $\epsilon_1=-\theta''+3=\theta^2+4\theta-13$, $h=1$,

$\theta'=2{\cdot}11=\theta^2+3\theta-17$, $\epsilon_1\epsilon_2\epsilon_3=-1$,

$\epsilon_2=-\theta'+3=-\theta^2-3\theta+20$,

$\theta''=-6{\cdot}23=-\theta^2-4\theta+16$, $\epsilon_3=-\theta+3$.

(6) $\theta^3+\theta^2-32\theta-79=0$, $D(\theta)=D=97^2$,

$\theta=6{\cdot}21$, $\epsilon_1=\theta+4$, $h=1$,

$\theta'=-3{\cdot}10=\theta^2-3\theta-23$, $\epsilon_1\epsilon_2\epsilon_3=-1$,

$\epsilon_2=\theta''+4=-\theta^2+2\theta+26$,

$\theta''=-4{\cdot}11=-\theta^2+2\theta+22$, $\epsilon_3=\theta^2-3\theta-19=\theta'+4$.

5. A quartic equation of negative discriminant has two real roots, θ and θ', and two complex roots θ'', $\bar{\theta}''$, the corresponding group being G_{24} or G_8, and each conjugate quartic field has two independent units.

A G_{24} quartic field has no sub-fields and the four conjugate fields (θ) are all distinct. The argument of § 2, in which θ'' is replaced by $\theta''\bar{\theta}''$, applies to this case. The two independent units are any two of the three

defined by

$$\epsilon > 1 \text{ and least}, \quad |\epsilon'| < 1, \quad \epsilon''\bar{\epsilon}'' < 1;$$

$$|\epsilon| < 1, \quad \epsilon' > 1 \text{ and least}, \quad \epsilon''\bar{\epsilon}'' < 1;$$

$$|\epsilon| < 1, \quad |\epsilon'| < 1, \quad |\epsilon''| = |\bar{\epsilon}''| > 1 \text{ and least.}$$

If $\epsilon_1\epsilon_2\epsilon_3 = -1$, ϵ_3 can be replaced by $-\epsilon_3$; thus it is sufficient to take $\epsilon_1\epsilon_2\epsilon_3 = 1$ in all cases.

The same argument applies to a quintic field when four of the conjugate fields are unreal. If θ is the real root and $(\theta', \bar{\theta}')$, $(\theta'', \bar{\theta}'')$ the conjugate pairs of complex ones, $|\epsilon'|$ in the above is replaced by $\epsilon'\bar{\epsilon}'$. The group of the equation is H_{120}, Γ_{60}, or G_{20}, and again $\epsilon_1\epsilon_2\epsilon_3 = 1$.

A sextic field whose conjugates are all unreal, and having no cubic sub-field, *i.e.* when in the notation of a former paper* the group is H_{720}, H_{120}, G_{72}, G_{36} or G_{18}, admits a similar determination of the units. The conditions to be satisfied are

$$|\epsilon_1| = |\bar{\epsilon}_1| > 1 \text{ and least}, \quad |\epsilon_1'| = |\bar{\epsilon}_1'| < 1, \quad |\epsilon_1''| = |\bar{\epsilon}_1''| < 1, \text{ etc.},$$

and, after replacing ϵ_r by $\epsilon_r i$ or $\epsilon_r\omega$ if necessary, when the field contains these cyclotomic units, $\epsilon_1\epsilon_2\epsilon_3 = 1$ as before.

6. The following examples give the sets of primitive units in some quartic, metacyclic quintic, and sextic fields. In each case ϵ_3 has been so adjusted that $\epsilon_1\epsilon_2\epsilon_3 = 1$. When the class-number h has been determined, it is added.

(7)–(9). Three symmetric quartics.

(7) $\theta^4 - 4\theta - 2 = 0$, $\quad D(\theta) = D = -2^8 . 5 . 7,$

$\theta = 1{\cdot}7$, $\quad \epsilon_1 = \theta^3 + 2\theta^2 + 3\theta + 1,$

$\theta' = -0{\cdot}5$, $\quad \epsilon_2 = -8\theta^3 + 4\theta^2 - 2\theta + 33,$

$\theta'' = -0{\cdot}6 + 1{\cdot}4i$, $\quad \epsilon_3 = \theta^2 - \theta - 1.$

(8) $\theta^4 - 4\theta - 3 = 0$, $\quad D(\theta) = D = -2^9 . 3^3,$

$\theta = 1{\cdot}8$, $\quad \epsilon_1 = 4\theta^3 + 7\theta^2 + 13\theta + 7, \quad h = 1,$

$\theta' = -0{\cdot}7$, $\quad \epsilon_2 = -42\theta^3 + 29\theta^2 - 20\theta + 182,$

$\theta'' = -0{\cdot}5 + 1{\cdot}5i$, $\quad \epsilon_3 = \theta^2 - \theta - 1.$

* W. E. H. Berwick, *Proc. London Math. Soc.* (2), 29 (1929), 1–28.

(9) $\theta^4-\theta-7=0$, $D(\theta)=D\doteq -5\,.\,11\,.\,1597$,

$\theta=1{\cdot}7$, $\epsilon_1=25\theta^3+43\theta^2+74\theta+102$, $h=1$,

$\theta'=-1{\cdot}5$, $\epsilon_2=-4\theta^3+6\theta^2-9\theta+18$,

θ'' complex, $\epsilon_3=\theta^2-3$.

(10)–(13) Four metacyclic quintics, in which θ is real and $\alpha=\exp(2\pi i/5)$.

(10) $\theta^5-2=0$, $D(\theta)=D=5^5\,.\,2^4$,

$\theta=\sqrt[5]{2}$, $\epsilon_1=\theta^4+\theta^3+\theta^2+\theta+1$,

$\theta'=\alpha\theta$, $\epsilon_2=\theta^4-\theta^2-\theta+1$,

$\theta''=\alpha^2\theta$, $\epsilon_3=\theta^4-\theta^3+\theta^2-1$.

(11) $\theta^5-3=0$, $D(\theta)=D=5^5\,.\,3^4$,

$\theta=\sqrt[5]{3}$, $\epsilon_1=13\theta^4+16\theta^3+20\theta^2+25\theta+31$,

$\theta'=\alpha\theta$, $\epsilon_2=\theta^3+\theta^2-\theta-2$,

$\theta''=\alpha^2\theta$, $\epsilon_3=\theta^4-\theta^3+\theta^2-2$.

(12) $\theta^5-5=0$, $D(\theta)=D=5^9$,

$\theta=\sqrt[5]{5}$, $h=1$,

$\epsilon_1=21\theta^4+29\theta^3+40\theta^2+55\theta+76$,

$\theta'=\alpha\theta$, $\epsilon_2=4\theta^4+3\theta^3-5\theta^2-10\theta+1$,

$\theta''=\alpha^2\theta$, $\epsilon_3=5\theta^4-2\theta^3-5\theta^2+15\theta-24$.

(13) $\theta^5-7=0$, $D(\theta)=5^5\,.\,7^4$,

$\mathfrak{O}=[1,\ \theta,\ \theta^2,\ \theta^3,$ $D=5^3\,.\,7^4$,

$\frac{1}{5}(\theta^4+2\theta^3+4\theta^2+3\theta+1)]$, $h=1$,

$\theta=\sqrt[5]{7}$,

$\epsilon_1=\frac{1}{5}(19\theta^4+28\theta^3+41\theta^2+62\theta+89)$,

$\theta'=\alpha\theta$, $\epsilon_2=\frac{1}{5}(-\theta^4-7\theta^3-4\theta^2+12\theta+19)$,

$\theta''=\alpha^2\theta$, $\epsilon_3=\frac{1}{5}(-3\theta^4+4\theta^3-2\theta^2-4\theta+12)$.

(14)–(15) Two cubic super-fields of (i).

(14) G_{36} $\theta^3 = 5+2i$, $D_i(\theta) = -27(5+2i)^2$,

Amp $\theta = 7°\,16'$, $\epsilon_1 = (5+4i)\theta^2+(8+8i)\theta+12+16i$,

Amp $\theta' = 127°\,16'$, $\epsilon_2 = (1+5i)\theta^2-(8+4i)\theta+14-7i$,

Amp $\theta'' = 247°\,16'$, $\epsilon_3 = (-22-8i)\theta^2+(2+41i)\theta+65-31i$.

This field was chosen on account of the fact that $\theta^3 = 29$ gives a large primitive unit, exceeding $3 \cdot 10^{17}$.

(15) G_{18} $\theta^3-3i\theta+2-i = 0$, $D_i(\theta) = -3^4$,

$\epsilon_1 = -(\theta+1+i)$,

$\theta' = \theta^2-(1+i)\theta-2i$, $\epsilon_2 = -(\theta'+1+i)$,

$\theta'' = -\theta^2+i\theta+2i$, $\epsilon_3 = -(\theta''+1+i)$.

The three fields (θ), (θ'), (θ'') are identical, but not the same as the fields defined by $\theta^3+3i\theta+2+i = 0$, which are also identical.

7. The other type of quartic field, belonging to the group G_8, is $[\sqrt{(a+b\sqrt{m})}]$, which has two independent units when

$$m > 0, \quad a+b\sqrt{m} > 0, \quad a-b\sqrt{m} < 0.$$

In this case $(\theta) = (\theta')$, both real, and $(\theta'') = (\bar{\theta}'')$, both complex. In the argument of § 2, III, a possibility is $|\epsilon''| = |\bar{\epsilon}''| = 1$, so the restrictions are taken to be

$$\epsilon_1 > 1 \text{ and least}, \quad |\epsilon_1'| < 1, \quad |\epsilon_1''| = |\bar{\epsilon}_1''| \leqslant 1,$$

we have $$|\epsilon_2| < 1, \quad \epsilon_2' > 1 \text{ and least}, \quad |\epsilon_2''| = |\bar{\epsilon}''| \leqslant 1.$$

Since $(\theta) = (\theta')$, $\epsilon_1 = \epsilon_2'$, involving $\epsilon_1' = \epsilon_2$, $\epsilon_1'' = \bar{\epsilon}_2''$, $\bar{\epsilon}_1'' = \epsilon_2''$, whence $\epsilon_1\epsilon_2$ is a unit of the sub-field $(\sqrt{m})$, *i.e.*,

$$\epsilon_1\epsilon_2 = \iota^{2s} \text{ or } \iota^{2s+1}, \quad \epsilon_1''\epsilon_2'' = \iota^{-2s} \text{ or } \iota^{-2s-1},$$

where ι, greater than 1 when $\sqrt{m}$ is taken positively, is the primitive unit of $(\sqrt{m})$. Evidently $s = 0$, otherwise $\epsilon_1\iota^{-s}$ comes before ϵ_1 in the θ-approximation. So the only cases which need be considered are

$$\text{(A)} \quad \epsilon_1\epsilon_2 = \iota, \quad \epsilon_1''\epsilon_2'' = \iota^{-1}; \qquad \text{(B)} \quad \epsilon_1\epsilon_2 = 1, \quad \epsilon_1''\epsilon_2'' = 1.$$

(A) We examine whether the units ϵ_1, ϵ_2 form a primitive pair. Since ϵ_1, ϵ_2' are the first units in the θ, θ' approximations

$$\epsilon_1 > 1, \quad |\epsilon_1'| < 1, \quad \iota^{-1} < 1; \quad |\epsilon_2| < 1, \quad \epsilon_2' > 1, \quad \iota^{-1} < 1.$$

Further $$|\epsilon_1''| < 1, \quad |\epsilon_2''| < 1, \quad \iota > 1,$$

and this is the first unit in the θ''-approximation unless $\sqrt{\iota}$ belongs to the field, in which case $|\epsilon_1''| = |\epsilon_2''| < 1$, $\sqrt{\iota} > 1$. The field then, however, falls under (B), since

$$\frac{\epsilon_1}{\sqrt{\iota}} > 1, \quad \left|\frac{\epsilon_1'}{\sqrt{\iota}}\right| < 1, \quad |\epsilon_1''\sqrt{\iota}| = 1.$$

Hence, in all cases under (A), ϵ_1, ϵ_2, ι^{-1} are a primitive set of units.

(B) $$\epsilon_1\epsilon_2 = \epsilon_1\epsilon_1' = 1; \quad \epsilon_1''\epsilon_2'' = \epsilon_1''\bar{\epsilon}_1'' = 1.$$

In this case ϵ_3, greater than 1 when $\sqrt{m}$ is taken positively, is the primitive unit of $(\sqrt{m})$ except for the field $(\sqrt{\iota})$, when $\epsilon_3 = \sqrt{\iota}$.

The units ϵ_1, ϵ_3 being an independent pair of units, any unit v of the field is such that

$$v^f = \epsilon_1^g \epsilon_3^h, \quad (v')^f = \epsilon_2^g \epsilon_3^h.$$

If f, g have a common factor, this must divide h and can be removed. From the last,

$$\left(\frac{v}{v'}\right)^f = \left(\frac{\epsilon_1}{\epsilon_2}\right)^g = \epsilon_1^{2g},$$

hence $f = 1$ or 2, and, if $f = 1$, v belongs to the system $\epsilon_1^g \epsilon_3^h$ as required.

Under $f = 2$ it is necessary to examine

(i) $g = 0$, $h = 1$, $\epsilon_3 = \iota$, (ii) $g = 0$, $h = 1$, $\epsilon_3 = \sqrt{\iota}$,

(iii) $g = 1$, $h = 0$, $\epsilon_3 = \iota$, (iv) $g = 1$, $h = 0$, $\epsilon_3 = \sqrt{\iota}$,

(v) $g = 1$, $h = 1$, $\epsilon_3 = \iota$, (vi) $g = 1$, $h = 1$, $\epsilon_3 = \sqrt{\iota}$.

Of these (i) must be excluded since $\sqrt{\iota}$ does not belong to the field, whereas, if (ii) holds, $\sqrt[4]{\iota}$ belongs to the field, which is not possible for a field of this degree. If (iii) holds, $\sqrt{\epsilon_1}$ belongs to the field and ϵ_1 is not the first unit in the θ-approximation, and (iv) is excluded for a similar reason. There remain (v) and (vi) when

$$v^2 = \epsilon_1\epsilon_3, \text{ whence } (v')^2 = \epsilon_2\epsilon_3 \text{ and } \left(\frac{v}{v'}\right)^2 = \frac{\epsilon_1}{\epsilon_2} = \epsilon_1^2, \quad vv' = \epsilon_3.$$

Taking ϵ to be any unit such that $\epsilon\epsilon' = \iota$, $\epsilon''\bar{\epsilon}'' = \iota^{-1}$, we can put $\Omega = \epsilon^2/\iota$, then $\Omega\Omega' = 1$.

If $\Omega/\epsilon_1{}^t = \tau$, $\Omega'/\epsilon_2{}^t = \tau'$, $\tau\tau' = 1$, t can be so chosen that

$$1 \leqslant \tau < \epsilon_1, \quad |\tau'| < 1,$$

then either $\tau = 1$ or comes before ϵ_1 in the θ-approximation. If $\tau = 1$, $\epsilon = \sqrt{\iota}$. Hence, in every case under (B), ϵ_1, ϵ_3 are a primitive pair of units.

The fields (θ'') are the simplest class of number-fields possessing an infinite number of units whose absolute values are 1. In (A) $|\epsilon_1''/\epsilon_2''| = 1$, and an alternative pair of fundamental units is $\epsilon_1''/\epsilon_2''$, ϵ_2''. In (B) $|\epsilon_1''| = 1$. These fields with an infinite number of units on the unit circle are not cyclotomic, since for no finite value of the positive integer n is $(\epsilon'')^n = 1$.

The biquadratic fields $\sqrt{(a+b\sqrt{m})}$, $\sqrt{(a-b\sqrt{m})}$ are also the simplest conjugate fields of which one is real and the other complex, and with a one-one correspondence between units and equivalent ideals. These fields are generally distinct from the conjugate pair $\sqrt{(-a+b\sqrt{m})}$, $\sqrt{(-a-b\sqrt{m})}$. In one set of cases, however, when

$$\sqrt{(a+b\sqrt{m})}/\sqrt{(-a+b\sqrt{m})}$$

is in $(\sqrt{m})$, the fields

$$\sqrt{(a+b\sqrt{m})}, \quad \sqrt{(-a+b\sqrt{m})}$$

are identical. The condition involves

$$m = f^2+g^2, \quad \text{and} \quad N(\iota) = -1.$$

The general field of this type is $\sqrt{(c\iota)}$, where c is any rational integer. In such fields $\epsilon_3 = \iota$ or $\sqrt{\iota}$, according as $c \neq 1$ or $= 1$. When $c = 1$ it is easily seen that $\epsilon_1\epsilon_2 = +1$. For $\epsilon_1\epsilon_1'$ and $\epsilon_1''\bar{\epsilon}_1''$ are conjugate in $(\sqrt{m})$ and the latter of these, which is necessarily positive, equals $+1$. Hence also $\epsilon_1\epsilon_1' = +1$. In fields of this type which fall under (A) the primitive set of units is such that $\epsilon_1\epsilon_2\iota = -1$. For $\epsilon_1''\epsilon_2''$ is essentially positive, and ι' negative. Hence $\epsilon_1''\epsilon_2''\iota'$ and its conjugate $\epsilon_1\epsilon_2\iota$ are both equal to -1.

8. In the following examples of G_8 quartic fields it is sufficient to record ϵ_1 and the type, (A) or (B), to which each field belongs.

(16)–(26). Eleven fields $\theta^4 = n$.

(16) (A) $\theta^4 = 2$, $\epsilon_1 = (1+\theta)(1+\theta^2)$;

(17) (B) $\theta^4 = 3$, $\epsilon_1 = 2+\theta+\theta^2+\theta^3$;

(18) (B) $\theta^4 = 5$, $\epsilon_1 = \frac{1}{2}(3+\theta+\theta^2+\theta^3)$;

(19) (B) $\theta^4 = 6$, $\epsilon_1 = 11425+7300\theta+4664\theta^2+2980\theta^3$;

(20) (B) $\theta^4 = 7$, $\epsilon_1 = 43+26\theta+16\theta^2+10\theta^3$;

(21) (B) $\theta^4 = 10$ $\epsilon_1 = 259+146\theta+82\theta^2+46\theta^3$;

(22) (B) $\theta^4 = 12$, $\epsilon_1 = \frac{1}{4}(2+2\theta+\theta^2)$;

(23) (B) $\theta^4 = 13$, $\epsilon_1 = \frac{1}{2}(63+33\theta+17\theta^2+9\theta^3)$;

(24) (B) $\theta^4 = 14$, $\epsilon_1 = 15+8\theta+4\theta^2+2\theta^3$;

(25) (A) $\theta^4 = 15$, $\epsilon_1 = 8+4\theta+2\theta^2+\theta^3$;

(26) (B) $\theta^4 = 20$, $\epsilon_1 = 9+4\theta+2\theta^2+\theta^3$.

(27)–(33). Seven fields $\sqrt{\iota}$. $\epsilon_1\epsilon_2 = 1$, $\epsilon_3 = \theta$.

(27) $\theta^2 = 1+\sqrt{2}$, $\epsilon_1 = 1+\sqrt{2}+\theta\sqrt{2}$;

(28) $\theta^2 = \frac{1}{2}(1+\sqrt{5})$, $\epsilon_1 = \frac{1}{2}(1+\sqrt{5})+\theta$;

(29) $\theta^2 = \frac{1}{2}(3+\sqrt{13})$, $\epsilon_1 = \frac{1}{2}(5+\sqrt{13})+\frac{1}{2}(1+\sqrt{13})\theta$;

(30) $\theta^2 = 4+\sqrt{17}$, $\epsilon_1 = \frac{1}{2}(5+\sqrt{17})+\frac{1}{2}(-1+\sqrt{17})\theta$;

(31) $\theta^2 = \frac{1}{2}(5+\sqrt{29})$, $\epsilon_1 = \frac{1}{2}(25+5\sqrt{29})+(6+\sqrt{29})\theta$;

(32) $\theta^2 = 6+\sqrt{37}$, $\epsilon_1 = \theta^6+42\theta^3$;

(33) $\theta^2 = \frac{1}{2}(7+\sqrt{53})$, $\epsilon_1 = \frac{1}{2}(65+9\sqrt{53})+\frac{1}{2}(27+3\sqrt{53})\theta$.

(34)–(37). Four fields $\sqrt{(c\iota)}$.

(34) (B) $\theta^2 = 3(1+\sqrt{2})$, $\epsilon_1 = 5+3\sqrt{2}+(2+\sqrt{2})\theta$;

(35) (A) $\theta^2 = 5(1+\sqrt{2})$, $\epsilon_1 = 4+3\sqrt{2}+(1+\sqrt{2})\theta$;

(36) (B) $\theta^2 = 7(1+\sqrt{2})$, $\epsilon_1 = 7+5\sqrt{2}+(2+\sqrt{2})\theta$;

(37) (A) $\theta^2 = 1+\sqrt{5}$, $\epsilon_1 = \frac{1}{2}(3+\sqrt{5})+\frac{1}{2}(1+\sqrt{5})\theta$.

9. If α is the real root of a cubic equation of negative discriminant, and $f(\alpha)$ is negative, the sextic field of $\theta^2 = f(\alpha)$ has two independent units. Putting

$$z^2 - f(\alpha) = (z-\theta)(z-\bar{\theta}),$$

$$z^2 - f(\alpha') = (z-\theta')(z-\bar{\theta}'),$$

$$z^2 - f(\alpha'') = (z-\theta'')(z-\bar{\theta}''),$$

the conjugate pairs of values of θ, in the complex sense, are

$$(\theta, \bar{\theta}), \quad (\theta', \bar{\theta}''), \quad (\bar{\theta}', \theta'').$$

Any member of (θ) is $g(\alpha)+\theta h(\alpha)$, and the conditions to be satisfied by ϵ_1, ϵ_2 are

$$\epsilon_1\bar{\epsilon}_1 < 1, \quad \epsilon_1'\bar{\epsilon}_1'' > 1 \text{ and least}, \quad \bar{\epsilon}_1'\epsilon_1'' < 1;$$

$$\epsilon_2\bar{\epsilon}_2 < 1, \quad \epsilon_2'\bar{\epsilon}_2'' < 1, \quad \bar{\epsilon}_2'\epsilon_2'' > 1 \text{ and least}.$$

As in § 7, ϵ_1 determines ϵ_2. The third unit is ϵ, the primitive unit of (α), or $\sqrt{-\epsilon}$, according as the field is not or is $(\sqrt{-\epsilon})$.

There are again the two possibilities (A) and (B),

$$\text{(A)} \quad \epsilon_1\epsilon_2\epsilon^{-1} = 1; \qquad \text{(B)} \quad \epsilon_1\epsilon_2 = 1, \ \epsilon_3 = \epsilon \text{ or } \sqrt{-\epsilon}.$$

A special field coming under this head is $(\alpha, \sqrt{-n})$, including as a more particular case $\sqrt[6]{-n}$.

Another special case, when $-n$ is the discriminant of the cubic field (α), is the normal field $(\alpha, \alpha', \alpha'')$. The primitive units ϵ, ϵ', ϵ'' of (α), (α'), (α'') are all units of the normal field, but two of them are not necessarily a primitive pair. If ϵ_1, ϵ_2 are a primitive pair of units of the normal field and

$$\epsilon = \epsilon_1^p\epsilon_2^q, \quad \epsilon' = \epsilon_1^r\epsilon_2^s,$$

the constant $ps-qr$ will be called the index I of the normal field. It can be shown that the index is either 1 or 3. The distinction between these values of I generally depends on the existence or otherwise of an integer β in (α) such that

$$m\epsilon = \beta^3.$$

First, when (α) contains such an integer β,

$$m\epsilon' = \beta'^3 \quad \text{and} \quad \epsilon'/\epsilon = (\beta'/\beta)^3,$$

so that β'/β is a unit of the sextic field. If (α) contains another integer γ such that

$$m'\epsilon^s = \gamma^3,$$

then $$\left(\frac{\gamma'}{\gamma}\right)^3 = \left(\frac{\beta'}{\beta}\right)^{3s}, \quad i.e. \quad \left(\frac{\gamma}{\beta^s}\right)^3 = \left(\frac{\gamma'}{\beta'^s}\right)^3,$$

and this involves $\gamma'/\gamma = (\beta'/\beta)^s$ unless $(\alpha) = (\sqrt[3]{l})$. In the latter event $\omega(\gamma'/\gamma) = (\beta'/\beta)^s$, where ω is a cyclotomic unit. Hence in all cases the only units of this type are $\omega^t(\beta/\beta')^s$.

It will now be proved that ϵ, β/β' are a primitive pair of units in the sextic field. If E is any unit therein and

$$E^k = \epsilon^a(\beta'/\beta)^b,$$

it is sufficient to suppose that $b = 1$. This involves the existence of another unit E_0 such that

$$E_0{}^k = \epsilon^a(\beta''/\beta),$$

whence $$\frac{\beta'}{\beta''} = \left(\frac{E}{E_0}\right)^k = f(\beta', \beta'')^k,$$

requiring that $$\beta'/\beta = f(\beta', \beta)^k$$

and $$\frac{\beta'^2}{\beta\beta''} = \epsilon' = \{f(\beta', \beta)f(\beta', \beta'')\}^k,$$

which can only be satisfied when $k = \pm 1$. Accordingly the three units $\epsilon, \beta'/\beta, \beta''/\beta$ form a primitive set and $I = 3$ as stated.

Next, when, as in $\sqrt[3]{7}$, there is no solution of $m\epsilon = \beta^3$, so that ϵ'/ϵ is not a cube in the normal field, it will be shown that any two of the units $\epsilon, \epsilon', \epsilon''$ form a primitive pair. As in the last case it is enough to suppose that

$$E^k = \epsilon^a\epsilon', \quad E_0{}^k = \epsilon^a\epsilon'',$$

whence $$\frac{\epsilon'}{\epsilon''} = \left(\frac{E}{E_0}\right)^k = f(\alpha', \alpha'')^k,$$

involving

$$\epsilon'/\epsilon = f(\alpha', \alpha)^k \quad \text{and} \quad (\epsilon')^3 = \{f(\alpha', \alpha'')f(\alpha', \alpha)\}^k,$$

so that $k = 1$ or 3. If $k = 3$, $a = 0$, then $E^3 = \epsilon'$, so that ϵ' is not the fundamental unit of (α'). Similarly, if $k = 3$, $a = 1$, then $E^3 = \epsilon\epsilon' = 1/\epsilon''$. If, however, $k = 3$, $a = 2$, then $E^3 = \epsilon'/\epsilon''$, and this has already been excluded.

Hence the only possibility is $k=1$ and, as stated, ϵ, ϵ', ϵ'' is a primitive set of units and $I=1$. [The field $(\sqrt[3]{3}, \sqrt{-3})$ is exceptional, *cf.* Ex. 50 below.]

10. As in § 8, it is sufficient to record ϵ_1 and the type (A) or (B) of each field.

(38)–(47). Ten super-fields of $\beta^3=2$ for which $\epsilon=\beta^2+\beta+1$.

(38) H_6 $\omega^2+\omega+1=0$,

$$\epsilon_1=\tfrac{1}{3}(1+\beta)^2(1+\omega^2\beta),$$

$$\epsilon_2=\tfrac{1}{3}(1+\omega^2\beta)^2(1+\omega\beta),$$

$$\epsilon_3=\tfrac{1}{3}(1+\omega\beta)^2(1+\beta).$$

In $(\omega^l\beta)$, $\epsilon=1+\omega^l\beta+\omega^{2l}\beta^2=\tfrac{1}{3}(1+\omega^l\beta)^3$ and

$$\left.\begin{aligned} 1+\beta+\beta^2&=\epsilon_2^{-2}\epsilon_3^{-1}=\epsilon_3\epsilon_1^{\,2}=\epsilon_1\epsilon_2^{-1},\\ 1+\omega\beta+\omega^2\beta^2&=\epsilon_2\epsilon_3^{\,2}=\epsilon_3\epsilon_1^{-1}=\epsilon_1^{-2}\epsilon_2^{-1},\\ 1+\omega^2\beta+\omega\beta^2&=\epsilon_2\epsilon_3^{-1}=\epsilon_3^{-2}\epsilon_1^{-1}=\epsilon_1\epsilon_2^{\,2},\end{aligned}\right\}\quad I=3.$$

(39) G_{12} (A) $i^2+1=0$, $\epsilon_1=1+\tfrac{1}{2}(1+i)\beta^2$;

(40) G_{48} (B) $\theta^2+\epsilon=0$, $\epsilon_1=\beta^2-1+(-\beta^2+2)\theta$;

(41) G_{12} (A) $\alpha^2+2=0$, $\epsilon_1=\beta^2+\beta-1+\tfrac{1}{2}(\beta^2-2\beta)\alpha$

$\lambda=\alpha/\beta$, $\lambda^6+2=0$, $\quad=\lambda^4-\lambda^2-1+(\lambda^4-1)\lambda$;

(42) G_{48} (A) $\theta^2+\epsilon\beta=0$, $\epsilon_1=-\beta^2+2\beta+1+(-\beta^2-\beta+3)\theta$;

(43) G_{48} (A) $\theta^2+1+\beta=0$, $\epsilon_1=\beta+\theta$;

(44) G_{48} (A) $\theta^2+\beta+\beta^2=0$, $\epsilon_1=1+\theta$;

(45) G_{12} (A) $\theta^2+6=0$, $\epsilon_1=\beta-1+\tfrac{1}{2}\beta^2\theta$;

(46) G_{12} (B) $\alpha^2+5=0$, $\epsilon_1=11\beta^2-17\beta+3+(4\beta^2+3\beta-10)\alpha$;

(47) G_{12} (A) $\alpha^2-\alpha+2=0$, $\epsilon_1=-3\beta^2-\beta+7+3(\beta^2-\beta)\alpha$.

(48) $\theta^3=5$, $\epsilon=41+24\theta+14\theta^2$, $\epsilon^{-1}=1-4\theta+2\theta^2$;

G_{12} (B) $\alpha^2=-5$, $\epsilon_1=\theta^2-7\theta+9+\tfrac{1}{5}(9\theta^2-5\theta-20)\alpha$.

11. A quadratic super-field of a cubic field (α) of positive discriminant is $\sqrt{f(\alpha)}$, and this has two independent units when

$$f(\alpha), \quad f(\alpha'), \quad f(\alpha'')$$

are all negative. If ϵ_1, ϵ_2, ϵ_3 is a primitive set of units in (α), any two of these are also independent units of $\sqrt{f(\alpha)}$. Further, if E_1, E_2 are a primitive pair of units of the latter field and

$$\epsilon_1 = E_1^{\,p} E_2^{\,q}, \quad \epsilon_2 = E_1^{\,r} E_2^{\,s}, \quad ps - qr = l,$$

then $E_1^{\,l} = \epsilon_1^{\,s} \epsilon_2^{-q}$, and the right-hand member is real. Since the only real members of $\sqrt{f(\alpha)}$ also belong to the sub-field (α), if E_1 is real it must also belong to (α), and $l = 1$, $E_1 = \epsilon_1^{\,s} \epsilon_2^{-q}$.

On the other hand, when E_1 is unreal the only possibility is

$$E_1 = \epsilon_b \sqrt{-\epsilon_a},$$

since anything else would involve E_1 being an algebraic number of higher degree than the sixth. In this case, ϵ_1, ϵ_1', ϵ_1'', say, are all positive, $l = 2$, and if

$$\epsilon_2 \sqrt{\epsilon_1} < 1, \quad \epsilon_3 \sqrt{\epsilon_1} > 1,$$

a primitive set of units of $\sqrt{-\epsilon_1}$ is

$$\sqrt{-\epsilon_1}, \quad \epsilon_2 \sqrt{-\epsilon_1}, \quad \epsilon_3.$$

An assigned cubic field of positive discriminant may have no super-field of this type or it may have one or three. The only possible fields are

$$\sqrt{-\epsilon_1}, \quad \sqrt{-\epsilon_2}, \quad \sqrt{-\epsilon_3},$$

and usually ϵ_a, ϵ_a', ϵ_a'' are not all positive, so that the field $\sqrt{-\epsilon_a}$ is not of the type with two independent units.

When $$\theta^3 - 8\theta - 5 = 0, \quad D = 1373,$$

$$\epsilon_1 = 2\theta^2 + 6\theta + 3, \quad \epsilon_2 = -9\theta^2 + 6\theta + 68, \quad \epsilon_3 = 2\theta^2 - 5\theta - 4,$$

ϵ, ϵ', ϵ'' have both signs for each unit and no field $(\sqrt{-\epsilon})$ enjoys the present property.

But when

$$\theta^3 - 5\theta - 3 = 0, \quad D = 257, \quad \epsilon = -3\theta^2 + 2\theta + 14,$$

the three conjugates ϵ, ϵ', ϵ'' are all positive, and one super-field $(\sqrt{-\epsilon})$ has two independent units.

Thirdly, when ϵ_1, ϵ_2, ϵ_3 and their conjugates are all positive there are three of these two-unit super-fields. An example is given by

$$\theta^3-32\theta^2+30\theta-1=0, \quad D=D(\theta)=37.18913, \tag{49}$$

in which

$$(\theta-1)=\mathfrak{p}, \quad (\theta+1)=\mathfrak{p}^6, \quad (31-\theta)=\mathfrak{p}^5,$$

$$\theta=31{\cdot}033, \quad \epsilon_1=\theta,$$

$$\theta'=0{\cdot}931, \quad \epsilon_2=\frac{\theta^4(\theta+1)}{(\theta-1)^6}=-4\,99176\theta^2+155\,08888\theta-5\,36159,$$

$$\theta''=0{\cdot}035, \quad \epsilon_3=\frac{(\theta-1)^6}{\theta^5(\theta+1)}=5\,67009\theta^2-181\,24664\theta+163\,82982.$$

It is noticeable that the fields $(\alpha, \sqrt{-n})$ have the same primitive set of units for every positive value of the integer n.

Our last numerical example is the exception noted at the end of § 9.

$$\lambda^6+3=0, \quad D(\lambda)=-6^6.3^5, \quad D=-3^{11}. \tag{50}$$

$$(\lambda)=(\theta, \omega), \quad \theta^3-3=0, \quad \omega^2+\omega+1=0, \quad D(\theta)=-3^5.$$

$$\epsilon_1=f(\lambda)=\tfrac{1}{2}(\lambda^5+\lambda^4-\lambda^2-\lambda-2),$$

$$\epsilon_2=f\{\tfrac{1}{2}(-\lambda+\lambda^4)\}=\tfrac{1}{2}(-\lambda^4+2\lambda^2-\lambda-2),$$

$$\epsilon_3=f\{\tfrac{1}{2}(-\lambda-\lambda^4)\}=\tfrac{1}{2}(-\lambda^5-\lambda^2+2\lambda-2),$$

$\epsilon_1\epsilon_2\epsilon_3=\tfrac{1}{2}(1-\lambda^3)$, a cyclotomic unit. The primitive units of the sub-fields (θ), $(\omega\theta)$, $(\omega^2\theta)$ are respectively

$$2\theta^2+3\theta+4=\epsilon_2/\epsilon_3, \quad 2\omega^2\theta^2+3\omega\theta+4=\epsilon_1/\epsilon_2, \quad 2\omega\theta^2+3\omega^2\theta+4=\epsilon_3/\epsilon_1.$$

Although in (θ) there is no solution of $m\epsilon=\beta^3$, the unit

$$(2\theta^2+3\theta+4)/(2\omega^2\theta^2+3\omega\theta+4)=-\epsilon_2{}^3\omega^2,$$

and the index $I=3$.

12. It remains to mention the cyclotomic units in the types of number-field here discussed. Cubic, quartic, and quintic fields have none. A sextic field which is a super-field of (i) or (ω) has the cyclotomic units i or ω, ω^2.

There are also two special sextic fields with cyclotomic units, viz.:

$$\alpha^6+\alpha^5+\alpha^4+\alpha^3+\alpha^2+\alpha+1=0, \quad \alpha^7=1, \tag{1}$$

for which

$$\epsilon_1 = -(\alpha+\alpha^6)(\alpha^3+\alpha^4), \quad \epsilon_2 = -(\alpha^2+\alpha^5)(\alpha^6+\alpha), \quad \epsilon_3 = -(\alpha^4+\alpha^3)(\alpha^5+\alpha^2),$$

and

$$\gamma^6+\gamma^3+1 = 0, \quad \gamma^9 = 1, \tag{2}$$

when

$$\epsilon_1 = -(\gamma+\gamma^8)(\gamma^4+\gamma^5), \quad \epsilon_2 = -(\gamma^2+\gamma^7)(\gamma+\gamma^8), \quad \epsilon_3 = -(\gamma^4+\gamma^5)(\gamma^2+\gamma^7).$$

[*Addendum*, September 1932. While this paper has been in the press I have seen a recent paper by Daus* containing a method of obtaining units in a cubic field by three-dimensional continued fractions. In Daus' paper the discriminants of the cubic fields are not given and the integral bases are not always chosen aright. His method does not always lead to a fundamental set of units; thus in the one-unit field $\theta^3 = 4$ the author gives

$$2\theta^2+3\theta+5 = (\tfrac{1}{2}\theta^2+\theta+1)^2$$

as the fundamental unit. Again, in Woronoi's example $\theta^3-7\theta-2 = 0$ he gives

$$e_1 = 18\theta^2+50\theta+13 = (\theta^2+3\theta+1)^2,$$

$$e_2 = -14\theta^2+4\theta+97 = \{\tfrac{1}{2}(-3\theta^2+\theta+20)\}^2,$$

$$e_3 = 4\theta^2-10\theta-3 = \{\tfrac{1}{2}(\theta^2-3\theta)\}^2,$$

no two of which form a primitive pair of units in (θ). In other cubic fields of positive discriminant Daus obtains three units connected by the relations

$$e_1 e_2{}^9 e_3{}^2 = 1, \quad e_1 e_2{}^2 e_3{}^3 = 1, \quad e_1 e_2{}^5 e_3{}^2 = 1,$$

which are not so convenient as the relation $\epsilon_1 \epsilon_2 \epsilon_3 = \pm 1$ of the present paper.]

* P. H. Daus, "Normal ternary continued fraction expansions for cubic irrationalities", *American J. of Math.*, 51 (1929), 67–98.

REMARKS ON THE THEORY OF CONJUGATE FUNCTIONS

By J. D. TAMARKIN.

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1. Let

$$w(z) \equiv u(z)+iv(z), \quad z = re^{i\theta} \quad (0 \leqslant r < 1), \tag{1.1}$$

be a function analytic in the unit circle $|z| < 1$. It will be assumed throughout this paper that

$$\Im w(0) = v(0) = 0, \quad \Re w(0) = u(0). \tag{1.2}$$

There are several theorems showing relationships between the averages over the circles $|z| = r$ of the real and imaginary parts of $w(z)$. To state these results in a compact form the following notation will be used. If $f(\theta)$ is any periodic function, of period 2π, then

$$M_p(f) \equiv \frac{1}{2\pi}\int_{-\pi}^{\pi} |f(\theta)|^p \, d\theta, \tag{1.3}$$

and similarly

$$M_p(r, \omega) \equiv \frac{1}{2\pi}\int_{-\pi}^{\pi} |\omega(z)|^p \, d\theta, \tag{1.4}$$

where $\omega(z)$ is any function of the complex variable z, not necessarily analytic.

THEOREM $\mathbf{A}_1$ (M. Riesz)*. *If p is any number greater than* 1, *then*

$$M_p(r, v) \leqslant B_p{}^p M_p(r, u), \tag{1.5}$$

where B_p is a positive number which depends only on p and $w(z)$, and is such that

$$B_p = B_{p'}, \quad 1/p+1/p' = 1,$$

* M. Riesz, **1**, 220–224.

while

$$B_p = O\left(\frac{1}{p-1}\right) \quad \text{as} \quad p \to 1,$$

$$B_p = O(p) \qquad \text{as} \quad p \to \infty.$$

THEOREM $\mathbf{A}_2$*. *If q is any number such that $0 < q < 1$, then*

$$M_q(r, v) \leqslant A_q [M_1(r, u)]^q, \tag{1.6}$$

where A_q is used as a generic notation for a number which depends only on q and is such that

$$A_q = O\left(\frac{1}{1-q}\right) \quad \text{as} \quad q \to 1.$$

In fact, in the case of formula (1.6),

$$A_q = \frac{2}{\cos \frac{1}{2} q\pi}.$$

The object of the present note is to prove

THEOREM $\mathbf{A}_3$. *With the notation of Theorem $\mathbf{A}_2$,*

$$M_1(r,v) \leqslant A_q[M_1(r,u)]^q + \frac{c}{q} \int_{-\pi}^{\pi} |u(z)| \log\left[\frac{|u(z)|}{q} + 1\right] d\theta \equiv \mathfrak{Z}_q(r,u), \tag{1.7}$$

where c is a numerical constant, independent of q and $w(z)$.

The proof of this theorem will be given in Section 2. It is well known that Theorems $\mathbf{A}_1$ and $\mathbf{A}_2$ lead to important applications in the theory of conjugate functions and in the theory of the convergence in the mean of Fourier series. The following results should be mentioned in this connection.

THEOREM $\mathbf{B}_1$ (M. Riesz)†. *If $f(\theta)$ is any function of the class L_p, $p > 1$, then its conjugate function*

$$\tilde{f}(\theta) \equiv -\frac{1}{2\pi} P.V. \int_{-\pi}^{\pi} f(\theta + t) \cot \tfrac{1}{2} t \, dt$$

also $\subset L_p$, and

$$M_p(\tilde{f}) \leqslant B_p{}^p M_p(f). \tag{1.8}$$

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* A proof of this theorem is contained in the papers of Hardy, **1**, and Smirnoff, **1**.

† M. Riesz, **1**, 225.

THEOREM $\mathbf{C}_1$ (M. Riesz)*. *Under the assumptions of Theorem* $\mathbf{B}_1$, *if*

$$\tfrac{1}{2}a_0+\sum_{\nu=1}^{\infty}(a_\nu\cos\nu\theta+b_\nu\sin\nu\theta)\sim f(\theta)$$

is the Fourier series of $f(\theta)$, *and*

$$s_n(\theta;\,f)\equiv s_n(f)\equiv s_n\equiv\tfrac{1}{2}a_0+\sum_{\nu=1}^{n}(a_\nu\cos\nu\theta+b_\nu\sin\nu\theta)$$

is its n-th partial sum, then $s_n(f)$ *converges to* $f(\theta)$ *in the mean of order* p, *that is*

$$M_p(s_n-f)\to 0 \quad \text{as} \quad n\to\infty \quad (p>1). \tag{1.9}$$

THEOREM $\mathbf{B}_2$ (Kolmogoroff)†. *If* $f(\theta)$ *is an arbitrary function* $\subset L_1$, *its conjugate* $\tilde{f}(\theta)\subset L_q$, *for any* q *such that* $0<q<1$, *and*

$$M_q(\tilde{f})\leqslant\frac{2}{\cos\frac{1}{2}q\pi}\,[M_1(f)]^q. \tag{1.10}$$

THEOREM $\mathbf{C}_2$ (Kolmogoroff)†. *Under the assumptions of Theorem* $\mathbf{B}_2$, $s_n(f)$ *converges to* $f(\theta)$ *in the mean of any order* q,

$$M_q(s_n-f)\to 0 \quad \text{as} \quad n\to\infty \quad (0<q<1). \tag{1.11}$$

Two analogous theorems are due to Zygmund‡.

THEOREM $\mathbf{B}_3$. *If* $f(\theta)\subset L_1$, *and also* $f\log^+|f|\subset L_1$, *then the conjugate function* $\tilde{f}(\theta)\subset L_1$, *and, with the notation of Theorem* $\mathbf{A}_3$,

$$M_1(\tilde{f})\leqslant A_q\,[M_1(f)]^q+\frac{c}{q}\int_{-\pi}^{\pi}|f(\theta)|\log\left[\frac{|f(\theta)|}{q}+1\right]d\theta\equiv\mathfrak{B}_q(f). \tag{1.12}$$

THEOREM $\mathbf{C}_3$. *Under the assumptions of Theorem* $\mathbf{B}_3$, $s_n(f)\,[s_n(\tilde{f})]$ *converges to* $f(\theta)\,[\tilde{f}(\theta)]$ *in the mean of order* 1,

$$M_1[s_n(f)-f]\to 0, \quad M_1[s_n(\tilde{f})-\tilde{f}]\to 0 \quad \text{as} \quad n\to\infty. \tag{1.13}$$

A proof of the first part of Theorem $\mathbf{B}_3$ [without the inequality (1.12)] was given by Zygmund§, and, subsequently, a simpler proof was given by

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* M. Riesz, 1, 230.

† Kolmogoroff, 1, Littlewood, 1.

‡ Zygmund, 1.

§ Zygmund, 1, 285.

Littlewood*. In Section 3 we give a complete proof of Theorem $\mathbf{B}_3$. In Section 4 we give a proof of Theorem $\mathbf{C}_3$, alternative to Zygmund's proof†. It should be observed that Zygmund's proof is rather condensed, and that it is difficult to follow it in all its details‡. Finally, in Section 5 it is shown that Theorem $\mathbf{A}_3$ gives the " best possible " result, in a certain sense. Theorems $\mathbf{B}_3$ and $\mathbf{C}_3$ are easily derived from the main Theorem $\mathbf{A}_3$. In proving this theorem we use a method which is essentially an extension of the method used by Littlewood§.

2. In the proof of Theorem $\mathbf{A}_3$ we shall need some simple lemmas which we state for the convenience of the reader.

LEMMA 2.1. (Young's inequality)‖. *The inequality*

$$ab \leqslant (a+1)\log(a+1)-a+e^b-b-1 \equiv \xi(a)+\eta(b) \tag{2.1}$$

holds whenever $a \geqslant 0$ *and* $b \geqslant 0$.

LEMMA 2.2. *Let* $\Omega(x)$ *be convex, non-negative, and non-decreasing for* $x \geqslant 0$. *Let* $f(\theta)$ *be such that*

$$f(\theta) \subset L_1, \quad \Omega\{|f(\theta)|\} \subset L_1.$$

Then, putting

$$\omega(z) \equiv P(z, f) \equiv \int_{-\pi}^{\pi} f(t)\, K_r(t, \theta)\, dt,$$

$$K_r(t, \theta) \equiv \frac{1}{2\pi} \frac{1-r^2}{1+r^2-2r\cos(t-\theta)} \quad (z = re^{i\theta}),$$

we have

$$\int_{-\pi}^{\pi} \Omega\{|\omega(z)|\}\, d\theta \leqslant \int_{-\pi}^{\pi} \Omega\{|f(\theta)|\}\, d\theta. \tag{2.2}$$

This lemma is due to Littlewood¶, except for non-essential modifications.

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* Littlewood, 2.
† Zygmund, 1, 302–303.
‡ [Added in proof]. *Cf.* Zygmund's note, 2.
§ Littlewood, 2.
‖ Young, 1; Zygmund, 1, 291–293.
¶ Littlewood, 2.

LEMMA 2.3. *Let*

$$w(z) \equiv u(z)+iv(z), \quad v(0)=0,$$

be analytic in the unit circle $|z|<1$. *Let* $\mathfrak{F}(r, u)$ *be an increasing functional of* $u(z)$ *on the circle* $|z|=r<1$, *in the sense that*

$$\mathfrak{F}(r, u_1) \leqslant \mathfrak{F}(r, u_2), \quad \text{whenever} \quad |u_1(z)| \leqslant |u_2(z)| \quad (|z|=r). \tag{2.3}$$

Suppose that the inequality

$$M_k(r, v) \leqslant \mathfrak{F}(r, u) \quad (k>0) \tag{2.4}$$

holds when $u(z) \equiv \Re w(z) > 0$ *in the circle* $|z| \leqslant r$. *Then, in the general case of any analytic function* $w(z)$,

$$M_k(r, v) \leqslant c\,\mathfrak{F}(r, u), \tag{2.5}$$

where $c=2$ *or* 2^k, *according as* $k \leqslant 1$ *or* $k \geqslant 1$.

To prove this lemma we have only to observe that

$$w(z) = \frac{1}{2\pi}\int_{-\pi}^{\pi} u(\zeta)\,\frac{\zeta+z}{\zeta-z}\,dt, \quad \zeta=\rho e^{it} \quad (r<\rho<1).$$

Using the notation

$$\begin{cases} a^+ = a \quad \text{or} \quad 0, \quad \text{according as} \quad a \geqslant 0 \quad \text{or} \quad a \leqslant 0, \\ a^- = 0 \quad \text{or} \quad a, \quad \text{according as} \quad a \leqslant 0 \quad \text{or} \quad a \geqslant 0, \end{cases} \tag{2.6}$$

we put

$$u(\zeta) = u^+(\zeta)+u^-(\zeta), \quad |u^{\pm}(\zeta)| \leqslant |u(\zeta)|. \tag{2.7}$$

The function $w(z)$ can then be written as

$$w(z) = w_1(z)+w_2(z),$$

where

$$u_1(z) = \Re w_1(z) > 0, \quad -u_2(z) = -\Re w_2(z) > 0 \quad \text{in} \quad |z| \leqslant r,$$

except in the trivial case when one of these functions vanishes identically. Since, by hypothesis,

$$M_k(r, v_j) \leqslant \mathfrak{F}(r, u_j) \quad (j=1, 2),$$

(2.5) follows immediately from (2.3), (2.7), and the inequality

$$(a+b)^k \leqslant \tfrac{1}{2}c(a^k+b^k).$$

We now proceed to the proof of Theorem $\mathbf{A}_3$. Let

$$w(z) \equiv u(z)+iv(z), \quad v(0)=0, \quad w(0)=u(0),$$

be analytic in the unit circle $|z|<1$. Assume in addition that

$$u(z)>0 \quad \text{in} \quad |z| \leqslant r<1.$$

Following Littlewood, we put

$$w(z) \equiv Re^{i\Phi}, \quad R \equiv (u^2+v^2)^{\frac{1}{2}},$$

$$\log w(z) = \log R+i\Phi \equiv U(z)+iV(z),$$

$$W(z) \equiv w(z)\log w(z) = (u+iv)(U+iV),$$

and observe that a branch of the many-valued function $\log w(z)$ can be selected in such a way that, in $|z| \leqslant r$,

$$-\tfrac{1}{2}\pi < V(z) < \tfrac{1}{2}\pi, \quad \operatorname{sgn} V(z) = \operatorname{sgn} v(z),$$

while $$|V(z)| \geqslant \tfrac{1}{4}\pi > \tfrac{1}{2} \quad \text{when} \quad |v(z)| \geqslant u(z).$$

It follows that the product $v(z)\,V(z) \geqslant 0$ always, and

$$|v(z)| \leqslant \max\,[u(z),\ 2v(z)\,V(z)] \leqslant u(z)+2v(z)\,V(z).$$

Hence, on integrating, we have

$$\int_{-\pi}^{\pi} |v(z)|\,d\theta \leqslant \int_{-\pi}^{\pi} u(z)\,d\theta+2\int_{-\pi}^{\pi} v(z)\,V(z)\,d\theta \tag{2.8}$$

$$= 2\pi u(0)+2\int_{-\pi}^{\pi} v(z)\,V(z)\,d\theta.$$

On the other hand, by Cauchy's theorem,

$$W(0) = w(0)\log w(0) = u(0)\log u(0) = \frac{1}{2\pi i}\int_{-\pi}^{\pi} W(z)\,\frac{dz}{z}$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} W(z)\,d\theta = \frac{1}{2\pi}\int_{-\pi}^{\pi} (u+iv)(U+iV)\,d\theta,$$

whence, on taking the real parts, we get

$$\int_{-\pi}^{\pi} v(z)\, V(z)\, d\theta = \int_{-\pi}^{\pi} u(z)\, U(z)\, d\theta - 2\pi u(0) \log u(0),$$

and, on substituting into (2.8),

$$(2.9) \qquad \int_{-\pi}^{\pi} |v(z)|\, d\theta \leqslant 2\pi u(0)\,[1-2\log u(0)] + 2\int_{-\pi}^{\pi} u(z)\, U(z)\, d\theta.$$

From this point on our argument will be different from that of Littlewood. We put, for brevity,

$$X \equiv \int_{-\pi}^{\pi} u(z)\, d\theta = 2\pi u(0), \quad u(0) = \frac{1}{2\pi} X.$$

Since, by hypothesis, $u(z) > 0$ in $|z| \leqslant r$, it readily follows from (2.9) that

$$(2.10) \qquad \int_{-\pi}^{\pi} |v(z)|\, d\theta \leqslant X\,[1+2\log 2\pi - 2\log X] + 2\int_{-\pi}^{\pi} u(z)\, U(z)\, d\theta$$
$$\equiv J_1 + J_2.$$

We begin by estimating the second term, J_2. Let E_r be the set of points of $(-\pi, \pi)$ for which

$$U(z) = U^+(z) > 0.$$

Then, by (2.1),

$$J_2 = 2\int_{E_r} \frac{u(z)}{q}\, qU(z)\, d\theta \leqslant 2\int_{E_r} \xi\left(\frac{u}{q}\right) d\theta + 2\int_{E_r} \eta(qU)\, d\theta \quad (0 < q < 1).$$

Here

$$\int_{E_r} \eta(qU)\, d\theta = \int_{E_r} [e^{qU} - qU - 1]\, d\theta$$
$$\leqslant \int_{E_r} (u^2+v^2)^{\frac{1}{2}q}\, d\theta \leqslant \int_{-\pi}^{\pi} u^q\, d\theta + \int_{-\pi}^{\pi} |v|^q\, d\theta,$$

whence, by Hölder's inequality and Theorem $\mathbf{A}_2$,

$$2\int_{E_r} \eta(qU)\, d\theta \leqslant 2(2\pi)^{1-q} X^q + \frac{4}{\cos \frac{1}{2}q\pi}\, (2\pi)^{1-q} X^q$$
$$\leqslant 2(2\pi)^{1-q}\left(1 + \frac{2}{\cos \frac{1}{2}q\pi}\right) X^q = A_q X^q.$$

On the other hand,

$$2\int_{E_r}\xi\left(\frac{u}{q}\right)d\theta = 2\int_{E_r}\left[\left(\frac{u}{q}+1\right)\log\left(\frac{u}{q}+1\right)-\frac{u}{q}\right]d\theta$$

$$\leqslant \frac{2}{q}\int_{-\pi}^{\pi} u\log\left(\frac{u}{q}+1\right)d\theta.$$

Hence $$J_2 \leqslant A_q X^q + \frac{2}{q}\int_{-\pi}^{\pi} u(z)\log\left[\frac{u(z)}{q}+1\right]d\theta.$$

The term J_1 is readily disposed of. We have

$$J_1 = X(1+2\log 2\pi - 2\log X) \leqslant X(1+2\log 2\pi + 2|\log X|) \equiv \Psi(X).$$

Now, if $0 \leqslant X \leqslant 1$, we have

$$\Psi(X) = (1+2\log 2\pi)X + 2X|\log X| \leqslant 5X + 2X|\log X|$$

$$\leqslant 5X^q + \frac{2}{e(1-q)}X^q = A_q X^q.$$

If, however, $X > 1$, then

$$\Psi(X) = (1+2\log 2\pi)X < 5X.$$

If D_r denotes the set of points of $(-\pi, \pi)$ for which $u(z) > \frac{1}{2}$, and CD_r denotes its complementary set, we have

$$X = \int_{D_r} u(z)\,d\theta + \int_{CD_r} u(z)\,d\theta.$$

Since $\log[u(z)+1] > \log\frac{3}{2} = c^{-1}$ in D_r, we have

$$\int_{D_r} u(z)\,d\theta < c\int_{D_r} u\log(u+1)\,d\theta < \frac{c}{q}\int_{-\pi}^{\pi} u(z)\log\left[\frac{u(z)}{q}+1\right]d\theta,$$

while $$\int_{CD_r} u(z)\,d\theta \leqslant \int_{CD_r} u^q\,d\theta \leqslant \int_{-\pi}^{\pi} u^q\,d\theta \leqslant (2\pi)^{1-q}X^q.$$

On combining these results we obtain the final estimate,

$$\int_{-\pi}^{\pi}|v(z)|\,d\theta \leqslant A_q\left[\int_{-\pi}^{\pi} u(z)\,d\theta\right]^q + \frac{c}{q}\int_{-\pi}^{\pi} u(z)\log\left[\frac{u(z)}{q}+1\right]d\theta.$$

Thus Theorem $\mathbf{A}_3$ is established in the case when $u(z) > 0$ in $|z| \leqslant r$, and, of course, also when $u(z) < 0$. If we replace $u(z)$ by $|u(z)|$ in the right-hand member of the preceding formula, we get an increasing functional of $u(z)$. Hence Lemma 2.3 can be applied and furnishes the complete proof of Theorem $\mathbf{A}_3$.

3. Theorem $\mathbf{B}_3$ follows almost at once from Theorem $\mathbf{A}_3$. The hypotheses of Theorem B_3 imply the existence of the integral

$$\int_{-\pi}^{\pi} |f(\theta)| \log\left[\frac{|f(\theta)|}{q}+1\right] d\theta \quad (0<q<1).$$

Let $$w(z) \equiv \frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)\,\frac{\zeta+z}{\zeta-z}\,dt \equiv u(z)+iv(z) \quad (z=re^{i\theta},\ \zeta=e^{it}).$$

By Theorem $\mathbf{A}_3$ and Lemma 2.2,

$$M_1(r, v) \leqslant A_q[M_1(f)]^q + \frac{c}{q}\int_{-\pi}^{\pi} |f(\theta)| \log\left[\frac{|f(\theta)|}{q}+1\right] d\theta,$$

whence (1.12) follows, by an easy application of Fatou's theorem.

4. We pass on to the proof of Theorem C_3. Using a familiar device*, we write

$$\begin{aligned} S_n(f) &\equiv s_n(f) - \tfrac{1}{2}(a_n \cos n\theta + b_n \sin n\theta) \\ &= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)\left[\frac{\sin \frac{1}{2}(2n+1)(t-\theta)}{\sin \frac{1}{2}(t-\theta)} - \cos n(t-\theta)\right] dt \\ &= \sin n\theta\left[-\frac{1}{2\pi}\,P.V.\int_{-\pi}^{\pi} f(t)\cos nt \cot \tfrac{1}{2}(t-\theta)\,dt\right] \\ &\qquad -\cos n\theta\left[-\frac{1}{2\pi}\,P.V.\int_{-\pi}^{\pi} f(t)\sin nt \cot \tfrac{1}{2}(t-\theta)\right] dt. \end{aligned}$$

To prove Theorem $\mathbf{C}_3$, it suffices to show that

$$\int_{-\pi}^{\pi} |f(\theta) - S_n(f)|\, d\theta \to 0 \quad \text{as} \quad n \to \infty.$$

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* Kolmogoroff, 1; M. Riesz, 1, 230.

On putting $f_n'(\theta) \equiv f(\theta) \cos n\theta, \quad f_n''(\theta) \equiv f(\theta) \sin n\theta,$

and denoting the corresponding conjugate functions by $\tilde{f}_n'(\theta)$, $\tilde{f}_n''(\theta)$, we have

$$S_n(f) = \tilde{f}_n'(\theta) \sin n\theta - \tilde{f}_n''(\theta) \cos n\theta. \tag{4.1}$$

Hence, by Theorem $\mathbf{B}_3$,

$$\int_{-\pi}^{\pi} |S_n(f)|\, d\theta \leqslant \int_{-\pi}^{\pi} |\tilde{f}_n'(\theta)|\, d\theta + \int_{-\pi}^{\pi} |\tilde{f}_n''(\theta)|\, d\theta \leqslant \mathfrak{Z}_q(f). \tag{4.2}$$

It is plain that, under the hypotheses of Theorem $\mathbf{C}_3$, the function $f(\theta)$ can be approximated by a bounded function $f_\epsilon(\theta)$ in such a way that

$$\int_{-\pi}^{\pi} |f - f_\epsilon|\, d\theta < \epsilon, \quad \mathfrak{Z}_q(f - f_\epsilon) < \epsilon. \tag{4.3}$$

Indeed, each of the integrals which constitute the functional $\mathfrak{Z}_q(f)$, when taken over the set E_N for which $|f(\theta)| > N$, can be made arbitrarily small, for N sufficiently large, so that we may put

$$f_\epsilon(\theta) = f(\theta) \quad \text{or} \quad 0, \quad \text{according as} \quad |f(\theta)| \leqslant N \quad \text{or} \quad > N.$$

The function $f_\epsilon(\theta)$ being thus fixed, we have

$$\int_{-\pi}^{\pi} |f - S_n(f)|\, d\theta \leqslant \int_{-\pi}^{\pi} |f_\epsilon - S_n(f_\epsilon)|\, d\theta + \int_{-\pi}^{\pi} |f - f_\epsilon|\, d\theta + \int_{-\pi}^{\pi} |S_n(f - f_\epsilon)|\, d\theta.$$

It is well known that the first integral of the right-hand member tends to 0 as $n \to \infty$. Hence, by (4.2), (4.3),

$$\overline{\lim_{n\to\infty}} \int_{-\pi}^{\pi} |f - S_n(f)|\, d\theta \leqslant \int_{-\pi}^{\pi} |f - f_\epsilon|\, d\theta + \mathfrak{Z}_q(f - f_\epsilon) \leqslant 2\epsilon,$$

and, on making $\epsilon \to 0$, we have

$$\lim_{n\to\infty} \int_{-\pi}^{\pi} |f - S_n(f)|\, d\theta = \lim_{n\to\infty} \int_{-\pi}^{\pi} |f - s_n(f)|\, d\theta = 0,$$

which is the desired result. The second formula (1.13) is proved in an analogous way.

The proof of the following theorem is implicitly contained in the above arguments.

THEOREM $\mathbf{D}_3$. *Under the assumptions of Theorem* $\mathbf{B}_3$, *we have*

$$M_1\{s_n(f)\} \leqslant C_f \mathfrak{Z}_q(f), \tag{4.4}$$

$$M_1\{s_n(\tilde{f})\} \leqslant C_f \mathfrak{Z}_q(f), \tag{4.5}$$

where C_f *is a positive number which depends on* f *and* q.

The proof of (4.4) follows immediately from the definition of $S_n(f)$ and (4.2), while that of (4.5) can be derived in an analogous way from

$$\begin{aligned} S_n(\tilde{f}) &\equiv s_n(\tilde{f}) - \tfrac{1}{2}(-b_n \cos n\theta + a_n \sin n\theta) \\ &= -\frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)\left[\cot \tfrac{1}{2}(t-\theta) - \frac{\cos \tfrac{1}{2}(2n+1)(t-\theta)}{\sin \tfrac{1}{2}(t-\theta)}\right] dt \\ &= \tilde{f}(\theta) - \tilde{f}_n'(\theta)\cos n\theta - \tilde{f}_n''(\theta)\sin n\theta. \end{aligned}$$

For the sake of completeness we state the known analogues of Theorem $\mathbf{D}_3$.

THEOREM $\mathbf{D}_1$ (M. Riesz)*. *If* $f(\theta) \subset L_p$, $p > 1$, *then*

$$M_p[s_n(f)] \leqslant C_f B_p{}^p M_p(f), \tag{4.6}$$

$$M_p[s_n(\tilde{f})] \leqslant C_f B_p{}^p M_p(f). \tag{4.7}$$

THEOREM $\mathbf{D}_2$ (Kolmogoroff)†. *If* $f(\theta) \subset L_1$, *then, for any* q, $0 < q < 1$,

$$M_q[s_n(f)] \leqslant C_f A_q [M_1(f)]^q, \tag{4.8}$$

$$M_q[s_n(\tilde{f})] \leqslant C_f A_q [M_1(f)]^q. \tag{4.9}$$

5. We recall the following theorem of Zygmund‡.

THEOREM $\mathbf{B}_4$. *If* $\epsilon(x)$ *is an arbitrary function, bounded, non-negative, and satisfying the condition*

$$\epsilon(x) = o(1) \quad as \quad x \to \infty,$$

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* M. Riesz, **1**, 230.
† Kolmogoroff, **1**.
‡ Zygmund, **1**, 285.

there exists a function $f(\theta)$ such that

$$f(\theta) \subset L_1, \quad f(\theta) \log^+ |f(\theta)| \epsilon \{|f(\theta)|\} \subset L_1,$$

while $\tilde{f}(\theta)$ does not $\subset L_1$.

On the basis of this theorem we can prove

THEOREM $\mathbf{A}_4$. *The result of Theorem $\mathbf{A}_3$ is the "best possible", in the sense that, corresponding to every function $\delta(x)$, convex, non-decreasing, and non-negative for $x \geqslant 0$, and such that*

$$\delta(x) = o(x \log^+ x) \quad as \quad x \to \infty,$$

there always exists a function

$$w(z) \equiv u(z) + iv(z), \quad v(0) = 0,$$

analytic in the unit circle $|z| < 1$, for which no inequality of the type

$$M_1(r, v) \leqslant A_q [M_1(r, u)]^q + \int_{-\pi}^{\pi} \delta(|u(z)|)\, d\theta \tag{5.1}$$

can be satisfied as $r \to 1$.

Indeed, the function

$$\epsilon(x) \equiv \frac{\delta(x)}{x \log^+ x} \quad (x > 1); \quad \epsilon(x) \equiv 0 \quad (1 \geqslant x \geqslant 0)$$

satisfies the conditions of Zygmund's Theorem $\mathbf{B}_4$. Let $f(\theta)$ be the function mentioned in this theorem. Let $w(z)$ be the corresponding analytic function,

$$w(z) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(t) \frac{\zeta + z}{\zeta - z} dt \quad (z = re^{i\theta}, \ \zeta = e^{it}).$$

If the inequality (5.1) was satisfied for all $r < 1$, it would follow from Lemma 2.2 that

$$M_1(r, v) \leqslant A_q [M_1(f)]^q + \int_{-\pi}^{\pi} \delta\{|f(\theta)|\}\, d\theta.$$

The boundedness of $M_1(r, v)$, hence the integrability of $\tilde{f}(\theta)$, would follow, in contradiction to the definition of $f(\theta)$.

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A REMARK ON CONJUGATE SERIES

By A. ZYGMUND.

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1. It has been shown that, if $|f(x)|\overset{+}{\log}|f(x)|$ $(0 \leqslant x \leqslant 2\pi)$ is integrable, the function $\bar{f}$, conjugate to f, is also integrable and

$$\int_0^{2\pi} |\bar{f}|\,dx \leqslant A\int_0^{2\pi} |f|\overset{+}{\log}|f|\,dx + B, \tag{1.1}$$

A and B denoting absolute constants†.

Let a_n, b_n be the Fourier coefficients of f, $A_n(x) = a_n\cos nx + b_n \sin nx$, and let $s_n = \frac{1}{2}A_0 + A_1 + \ldots + A_n$. Starting from the identity‡

$$\begin{aligned} s_n - \tfrac{1}{2}A_n &= \frac{\cos nx}{2\pi}\int_0^{2\pi} f(t)\cos nt\cot\tfrac{1}{2}(x-t)\,dt \\ &\qquad - \frac{\sin nx}{2\pi}\int_0^{2\pi} f(t)\sin nt\cot\tfrac{1}{2}(x-t)\,dt \\ &= \cos nx\,.f_n^*(x) - \sin nx\,.f_n^{**}(x), \end{aligned}$$

where f_n^* and f_n^{**} are respectively conjugate to $f(x)\cos nx$ and $f(x)\sin nx$, we deduce from (1.1) that

$$\overline{\lim_{n\to\infty}}\int_0^{2\pi} |s_n|\,dx \leqslant 2A\int_0^{2\pi} |f|\overset{+}{\log}|f|\,dx + 2B.$$

Applying this inequality to the function kf, k denoting a positive constant, we have

$$\overline{\lim_{n\to\infty}}\int_0^{2\pi} |s_n|\,dx \leqslant 2A\int_0^{2\pi} |f|\,.\overset{+}{\log}|kf|\,dx + \frac{2B}{k}. \tag{1.2}$$

† See A. Zygmund, *Fundamenta Math.*, 13 (1929), 284–303; J. E. Littlewood, *Journal London Math. Soc.*, 4 (1929), 305–307. In the first of these papers a less strong result (Theorem I) is stated, but the method used easily gives the inequality (1.1).

‡ This identity has already been used for the same purpose by Kolmogoroff, *Fundamenta Math.*, 7 (1925), 24–29, and M. Riesz, *Math. Zeitschrift*, 27 (1928), 218–244.

Let k be very large but fixed. We put $f=f_1+f_2$ and, correspondingly, $s_n=s_n^{(1)}+s_n^{(2)}$, where f_1 is bounded and the integrals of $|f_2|$ and $|f_2|\overset{+}{\log}|kf_2|$ over $(0, 2\pi)$ are very small. Then

$$\int_0^{2\pi}|f-s_n|\,dx\leqslant\int_0^{2\pi}|s_n^{(1)}-f_1|\,dx+\int_0^{2\pi}|s_n^{(2)}|\,dx+\int_0^{2\pi}|f_2|\,dx,$$

where, on the right-hand side, the first term tends to zero as $n\to\infty$, and, as follows from (1.2), the two other terms are very small. Hence we get the following theorem†:

THEOREM A. *If $|f|\overset{+}{\log}|f|$ is integrable and s_n denotes the partial sum of the Fourier series of f, then*

$$\int_0^{2\pi}|f-s_n|\,dx\to 0. \tag{1.3}$$

2. Now we may ask under what conditions the relation

$$\int_0^{2\pi}|\bar{f}-\bar{s}_n|\,dx\to 0 \tag{2.1}$$

is true, $\bar{s}_n$ being the partial sums of the conjugate series. That (1.3) and (2.1) are both true if $f\subset L^p$ $(p>1)$ has already been shown by M. Riesz (*loc. cit.*). For $p=1$ the theorem is false, even if we *assume* that $|\bar{f}|$ is integrable; there exists a power series in $e^{i\vartheta}$, whose real and imaginary parts are Fourier series and whose partial sums $t_n(e^{i\theta})$ satisfy the relation‡

$$\overline{\lim_{n\to\infty}}\int_0^{2\pi}|t_n(e^{i\theta})|\,d\theta=+\infty.$$

However, the following theorem is true:

THEOREM B. *If the conjugate series to the Fourier series of an integrable function f is itself a Fourier series, then, as $n\to\infty$, the integrals* (1.3) *and* (2.1) *behave similarly; if one of them*

(i) *converges to zero,* (ii) *is bounded,* (iii) *is unbounded,*

the same may be said of the other.

† *Cf.* A. Zygmund, *loc. cit.*, and *Fundamenta Math.*, 18 (1932), Errata, where a different proof is given. See also J. Tamarkin, "Remarks on the theory of conjugate functions", *Proc. London Math. Soc.* (2), 34 (1932), 379–391.

‡ The only example that I know of such a power series was communicated to me by Prof. F. Riesz; see the addition below, pp. 397–400.

3. The proof of the theorem will be based on the following

LEMMA. *Let $s(\theta)$ be an arbitrary trigonometric polynomial of order n, and let $\phi(u)$ $(u \geqslant 0)$ be a convex function, increasing with u. If*

$$\int_0^{2\pi} \phi\{|s(\theta)|\}\,d\theta = M, \tag{3.1}$$

then

$$\int_0^{2\pi} \phi\left(\left|\frac{s'(\theta)}{n}\right|\right)d\theta \leqslant M. \tag{3.2}$$

This is a slight generalization of a well-known theorem of S. Bernstein. The proof which we give follows very easily from the argument by which F. Riesz proves Bernstein's theorem†. In the identity

$$s'(\theta) = \frac{1}{\pi}\int_0^{2\pi} s(\theta+u)\,[\sin u + \sin 2u + \ldots + n\sin nu]\,du,$$

add the polynomial

$$(n-1)\sin(n+1)u + (n-2)\sin(n+2)u + \ldots + \sin(2n-1)u,$$

to the expression in the square brackets and take together $k\sin ku$ and $k\sin(2n-k)u$; we get

$$s'(\theta) = \frac{2n}{\pi}\int_0^{2\pi} s(\theta+u)\sin nu\,K_{n-1}(u)\,du,$$

where K_n is Fejér's kernel. Let $s(\theta+u)\,K_{n-1}(u) = \lambda(u)$. An integration by parts gives

$$s'(\theta) = \frac{2}{\pi}\int_0^{2\pi} \lambda'(u)\cos nu\,du.$$

In this integral $\lambda'(u)$ is a trigonometric polynomial of order $2n-1$ and so $\cos nu$ may be replaced by any function whose Fourier series is of the form $\cos nu + a_{2n}\cos 2nu + a_{2n+1}\cos(2n+1)u + \ldots$. Let $\psi(u)$ denote the even periodic function with period 2π, which is equal to 1 for $0 \leqslant u < \frac{1}{2}\pi$, to -1 for $\frac{1}{2}\pi < u \leqslant \pi$, and to 0 for $u = \frac{1}{2}\pi$. We have

$$\psi(u) = \frac{2}{\pi}\left[\cos u - \tfrac{1}{3}\cos 3u + \ldots\right],$$

and so, in the last integral, the factor $2\cos nu/\pi$ may be replaced by $\psi(nu)$.

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† F. Riesz, *Comptes rendus*, 158 (1914), 1657–1661.

An integration by parts gives†

$$(3.3) \qquad s'(\theta) = -\int_0^{2\pi} s(\theta+u)\, K_{n-1}(u)\, d\psi(nu).$$

Putting $\theta = 0$, $s(u) = \sin nu$, we get

$$1 = \int_0^{2\pi} K_{n-1}(u) \left| \frac{d\psi(nu)}{n} \right|,$$

where the integral reduces in reality to a sum $\alpha_1+\alpha_2+\ldots+\alpha_{2n}$ $(\alpha_i > 0)$. From (3.3) it follows that

$$(3.4) \qquad \left| \frac{s_n'(\theta)}{n} \right| \leqslant \int_0^{2\pi} |s(\theta+u)|\, K_{n-1}(u) \left| \frac{d\psi(nu)}{n} \right| = \sum_{i=1}^{2n} \alpha_i |s(\theta+u_i)|,$$

$u_1, u_2, \ldots, u_{2n}$ denoting the consecutive points of discontinuity of $\psi(nu)$ in $(0, 2\pi)$. Since $\alpha_1+\ldots+\alpha_{2n} = 1$, by applying Jensen's inequality, we get

$$(3.5) \qquad \phi\left(\left| \frac{s_n'(\theta)}{n} \right| \right) \leqslant \phi\left(\sum_{i=1}^{2n} \alpha_i |s(\theta+u_i)| \right) \leqslant \sum_{i=1}^{2n} \alpha_i \phi\{|s(\theta+u_i)|\},$$

$$(3.6) \qquad \int_0^{2\pi} \phi\left(\left| \frac{s_n'(\theta)}{n} \right| \right) d\theta \leqslant \sum_{i=1}^{2n} \alpha_i \int_0^{2\pi} \phi\{|s(\theta+u_i)|\}\, d\theta = M \sum_{i=1}^{2n} \alpha_i = M.$$

If we assume that $\phi(u)$ is strictly increasing, it is not difficult to find when we have the sign of equality in (3.2). When equality occurs we deduce from (3.6) and (3.5) that the integrals, over $(0, 2\pi)$, of the first two members in (3.5) are equal; hence, from (3.4), it follows that the sign $\leqslant$ may be replaced there by $=$, for any value of θ. Thence and from (3.3) we see that any two consecutive terms of the sequence

$$s(\theta+u_1),\ s(\theta+u_2),\ \ldots,\ s(\theta+u_n)$$

are of opposite signs, whatever θ may be. Consequently the distance between any two consecutive zeros of $s(u)$ cannot exceed π/n; since the number of zeros of $s(u)$ in $(0, 2\pi)$ is at most $2n$, this means that the zeros of $s(u)$ are exactly of the form $\alpha+\pi k/n$ $(k = 0, 1, \ldots, 2n-1)$, *i.e.* $s(u) = A \sin n(u-\alpha)$, where α, A are two constants‡.

† F. Riesz, *loc cit.*; M. Riesz, *Jahresbericht d. Deutschen Mathematiker Vereinigung*, 23 (1915), 354–368.

‡ In the same way we prove that (3.1) involves the inequality

$$\int_0^{2\pi} \phi\left(\left| \frac{\bar{s}'}{n} \right| \right) d\theta \leqslant M,$$

$\bar{s}$ denoting the polynomial conjugate to s. See also G. Szegö, *Schriften der Königsberger Gelehrten Gesellschaft*, 5 (1928), 59–70.

COROLLARY. *If $p \geqslant 1$, the inequality*

$$\mathfrak{M}_p[s(u)] = M \tag{3.7}$$

involves

$$\mathfrak{M}_p[s'(u)] \leqslant nM\dagger. \tag{3.8}$$

4. Now we can easily prove Theorem B. We begin by proving that, if the given series and its conjugate are both Fourier series (s_n, $\bar{s}_n$, σ_n, $\bar{\sigma}_n$ denoting respectively their partial sums and first arithmetic means), and if one of the integrals $\mathfrak{M}_1[s_n]$, $\mathfrak{M}_1[\bar{s}_n]$ is bounded as $n \to \infty$, so is the other. In fact, from the inequality $\mathfrak{M}_1[s_n] \leqslant M$, noticing that

$$s_n'/(n+1) = \bar{\sigma}_n - \bar{s}_n,$$

and the lemma, we find that

$$\mathfrak{M}_1[\bar{\sigma}_n - \bar{s}_n] = \mathfrak{M}_1[s_n'/n] \leqslant M. \tag{4.1}$$

It is well known that $\mathfrak{M}_1[f-\sigma_n] \to 0$, $\mathfrak{M}_1[\bar{f}-\bar{\sigma}_n] \to 0$. Hence, noticing that $\mathfrak{M}_1[\sigma_n]$ is less than a constant N, we have from (4.1)

$$\mathfrak{M}_1[\bar{s}_n] \leqslant \mathfrak{M}_1[\bar{s}_n - \bar{\sigma}_n] + \mathfrak{M}_1[\bar{\sigma}_n] \leqslant M+N.$$

In order to prove that (1.3) involves (2.1), we choose n_0 so large that $\mathfrak{M}_1[f-\sigma_{n_0}] < \epsilon$. From (1.3) we see that, for all values of n sufficiently large, we have $\mathfrak{M}_1[s_n - \sigma_{n_0}] < \epsilon$. It follows that $\mathfrak{M}_1[(s_n' - \sigma_{n_0}')/n] < \epsilon$; consequently, for $n > n_1$,

$$\epsilon > \mathfrak{M}_1[s_n'/n] \geqslant \mathfrak{M}_1[s_n'/(n+1)] = \mathfrak{M}_1[\bar{\sigma}_n - \bar{s}_n]. \tag{4.2}$$

Since $\mathfrak{M}_1[\bar{\sigma}_n - \bar{f}] \to 0$, we have $\overline{\lim}\, \mathfrak{M}_1[\bar{f}-\bar{s}_n] \leqslant \epsilon$ from (4.2); since ϵ is arbitrary, this is equivalent to (2.1). We may add that Theorem B remains true if, instead of all the positive integers $n = 1, 2, \ldots$, we consider only a subsequence $n_1 < n_2 < n_3 < \ldots$.

COROLLARY. *If $|f| . \log^+ |f|$ is integrable, the relation (2.1) holds, in addition to (1.3).*

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† $\mathfrak{M}_p[s]$ denotes the $1/p$-th power of the integral of $|s|^p/2\pi$ taken over $(0, 2\pi)$. Making $p \to \infty$, we get Bernstein's theorem. I have learnt from Prof. Hardy that the inequality (3.8) was previously known to him and to Prof. Littlewood.

If $f \subset L^p (p \geqslant 1)$, let $\rho_n^{(p)} = \operatorname{Min} \mathfrak{M}_p[f-T]$, where T denotes trigonometric polynomials of order less than or equal to n. Using the inequality (3.7)–(3.8) we can easily prove some results concerning the dependance of the numbers $\rho_n^{(p)}$ (best approximation in mean), and the mean modulus of continuity $\omega_r(\delta) = \operatorname{Sup} \mathfrak{M}_p[f(x+h)-f(x)]$ $(0 \leqslant h \leqslant \delta)$. The theorems and the proofs are analogous to those in the limiting case p is infinite. See Ch. J. de la Vallée Poussin, *Leçons sur l'approximation des fonctions* (Paris, 1919), especially Chapter IV.

5. Theorem B may be generalized by considering functions f such that $\phi(|f|)$ is integrable, where ϕ is convex and increasing, and $\phi(0)=0$. It has been shown† that, then

$$\int_0^{2\pi}\phi(|\sigma_n|)\,dx=O(1), \qquad \int_0^{2\pi}\phi(\tfrac{1}{4}|f-\sigma_n|)\,dx\to 0.$$

THEOREM C. *If $\phi(|f|)$ and $\phi(|\bar{f}|)$ are both integrable, the relations*

$$(a)\quad \int_0^{2\pi}\phi(|s_n|)\,dx=O(1), \qquad (b)\quad \int_0^{2\pi}\phi(|f-s_n|)\,dx\to 0,$$

involve respectively

$$(a')\quad \int_0^{2\pi}\phi(\tfrac{1}{2}|\bar{s}_n|)\,dx=O(1), \qquad (b')\quad \int_0^{2\pi}\phi(\tfrac{1}{16}|\bar{f}-\bar{s}_n|)\,dx\to 0.$$

The proof, which involves the use of Jensen's inequality, is very similar to that of Theorem B and may be left to the reader.

[*Added 27 February*, 1932.] 6. With Prof. F. Riesz's kind permission I reproduce here his example referred to on p. 393. Let

$$P_n(z)=\frac{1}{n}(1+z+z^2+\ldots+z^{n-1})^2$$

$$=\frac{1}{n}[1+2x+3z^2+\ldots+nz^{n-1}+(n-1)z^n+\ldots+z^{2n-2}],$$

$$Q_n(z)=\frac{1}{n}(1+2z+3z^2+\ldots+nz^{n-1}).$$

We have

$$\int_0^{2\pi}|P_n(e^{i\theta})|\,d\theta=2\pi, \quad \int_0^{2\pi}|Q_n(e^{i\theta})|\,d\theta$$

$$=\frac{1}{n}\int_0^{2\pi}|1+2e^{i\theta}+\ldots+ne^{i(n-1)\theta}|\,d\theta>C\log n,$$

where $C>0$ is an absolute constant. Let $\{\alpha_\kappa\}$ be an arbitrary sequence of (say) positive numbers with $\Sigma\alpha_\kappa<\infty$, and let

$$(6.1)\qquad f(z)=\sum_{\kappa=1}^{\infty}\alpha_\kappa z^{m_\kappa}P_{n_\kappa}(z).$$

The sequences $\{m_\kappa\}$ and $\{n_\kappa\}$ of natural numbers are subject only to the

† A. Zygmund, *loc cit.*, 288–291.

conditions that

$$(\alpha) \quad m_\kappa + 2n_\kappa \leqslant m_{\kappa+1}, \qquad (\beta) \quad a_\kappa \log n_\kappa \to \infty.$$

From the first of these conditions we see that the polynomials $z^{m_\kappa} P_{n_\kappa}(z)$ do not overlap. From the results

$$|f(e^{i\theta})| \leqslant \Sigma a_\kappa |P_{n_\kappa}(e^{i\theta})|, \qquad \Sigma a_\kappa \int_0^{2\pi} |P_{n_\kappa}(e^{i\theta})|\, d\theta = 2\pi \Sigma a_\kappa,$$

$$|P_n(e^{i\theta})| = \frac{1}{n}\left|\frac{1-e^{in\theta}}{1-e^{i\theta}}\right|^2 = \frac{1}{n}\left(\frac{\sin \frac{1}{2}n\theta}{\sin \frac{1}{2}\theta}\right)^2, \tag{6.2}$$

it follows that the series of polynomials (6.1) converges on the circle $|z| = 1$, except at the point $z = 1$, and that its partial sums are majorized there by an integrable function. Consequently, the series in (6.1), written in the form

$$\sum_0^\infty c_\nu e^{i\nu\theta}, \tag{6.3}$$

is the Fourier series of $f(e^{i\theta})$. Since

$$a_\kappa \int_0^{2\pi} |Q_{n_\kappa}(e^{i\theta})|\, d\theta > C a_\kappa \log n_\kappa \to \infty, \tag{6.4}$$

we get that

$$\overline{\lim_{n\to\infty}} \int_0^{2\pi} |s_n(\theta)|\, d\theta = \infty, \tag{6.5}$$

where s_n denotes the n-th partial sum of the series (6.3).

Using Prof. Riesz's example, we are able to prove the following

THEOREM D. *Given an arbitrary function $\epsilon(u)$ $(0 \leqslant u < \infty)$, positive, bounded, measurable, and tending to 0 with $1/u$, we can find a power series $g(z)$ whose real and imaginary parts, for $z = e^{i\theta}$, are both Fourier series, and whose partial sums s_n satisfy the condition* (6.5), *although*

$$\int_0^{2\pi} |g(e^{i\theta})| \overset{+}{\log} |g(e^{i\theta})|\, \epsilon[\,|g(e^{i\theta})|\,]\, d\theta < \infty. \tag{6.6}$$

Proof. Putting $\epsilon(u)\, u \overset{+}{\log} u = \phi(u)$, $\epsilon(u) \overset{+}{\log} u = \psi(u)$, we may suppose, without loss of generality, that $\psi(u)$ is an increasing function and that, *e.g.*, $\epsilon(u) \geqslant 1/\sqrt{(\overset{+}{\log} u)}$ $(u \geqslant 1)$. We define $g(z)$ as $f(z)/40$, where $f(z)$ is given by the formula (6.1). Supposing for a moment that the numbers n_κ are already defined, we put

$$a_\kappa = \frac{1}{\sqrt{\{\epsilon(n_\kappa)\} \log n_\kappa}}.$$

Then

$$(6.7) \quad (a) \quad a_\kappa \psi(n_\kappa) = \sqrt{\{\epsilon(n_\kappa)\}}, \quad (b) \quad a_\kappa \log n_\kappa = \frac{1}{\sqrt{\{\epsilon(n_\kappa)\}}} \to \infty.$$

Since $\epsilon(u) \to 0$, we may take for n_κ a sequence of integers increasing so rapidly that

$$\text{(i)} \quad |a_\nu| \leqslant 1; \qquad \text{(ii)} \quad \Sigma \sqrt{\{\epsilon(n_\iota)\}} < \infty;$$

$$\text{(iii)} \quad a_\kappa \text{ decreases monotonically.}$$

On the other hand, since we have

$$a_\kappa n_\kappa \geqslant n_\kappa / \log n_\kappa \to \infty,$$

for κ sufficiently large, we can require that

$$\text{(iv)} \quad a_{\kappa+1} n_{\kappa+1} > 2 a_\kappa n_\kappa \quad (\kappa = 1, 2, \ldots).$$

Since $|\sin nu| < n |\sin u|$, we deduce from the equality (2) that $|P_n(e^{i\theta})| \leqslant n$. In the interval $(1/n, \pi)$ we have $|P_n(e^{i\theta})| \leqslant 1/(n \sin^2 \frac{1}{2}\theta) < \pi^2/(n\theta^2)$. Now supposing that $1/n_{\kappa+1} \leqslant \theta \leqslant 1/n_\kappa$, and taking one of the two estimates for every term of (6.1), we get

$$\begin{aligned}
|f(e^{i\theta})| &\leqslant a_1 n_1 + a_2 n_2 + \ldots + a_\kappa n_\kappa + \pi^2 \theta^{-2} [a_{\kappa+1} n_{\kappa+1}^{-1} + a_{\kappa+2} n_{\kappa+2}^{-1} + \ldots] \\
&< a_\kappa n_\kappa (1 + 2^{-1} + 2^{-2} + \ldots) + \pi^2 \theta^{-2} \left[\frac{a_{\kappa+1}^2}{a_{\kappa+1} n_{\kappa+1}} + \frac{a_{\kappa+2}^2}{a_{\kappa+2} n_{\kappa+2}} + \ldots \right] \\
&< 2 a_\kappa n_\kappa + \frac{\pi^2 \theta^{-2} a_{\kappa+1}^2}{a_{\kappa+1} n_{\kappa+1}} (1 + 2^{-1} + 2^{-2} + \ldots) \\
&< 20 \{ a_\kappa n_\kappa + a_{\kappa+1} / (\theta^2 n_{\kappa+1}) \}.
\end{aligned}$$

The function $\phi(u)$ being increasing, we have $\phi(\lambda + \mu) \leqslant \phi(2\lambda) + \phi(2\mu)$ and so

$$\begin{aligned}
\int_{1/n_{\kappa+1}}^{1/n_\kappa} \phi\{|g(e^{i\theta})|\} d\theta &= \int_{1/n_{\kappa+1}}^{1/n_\kappa} \phi\{|f(e^{i\theta})|/40\} d\theta \\
&\leqslant \int_{1/n_{\kappa+1}}^{1/n_\kappa} \phi\left(\tfrac{1}{2} a_\kappa n_\kappa + \frac{a_{\kappa+1}}{2\theta^2 n_{\kappa+1}} \right) d\theta \\
&\leqslant \int_{1/n_{\kappa+1}}^{1/n_\kappa} \phi(a_\kappa n_\kappa)\, d\theta + \int_{1/n_{\kappa+1}}^{1/n_\kappa} \phi\left(\frac{a_{\kappa+1}}{\theta^2 n_{\kappa+1}} \right) d\theta = A_\kappa + B_\kappa.
\end{aligned}$$

$$A_\kappa < \phi(a_\kappa n_\kappa)/n_\kappa = a_\kappa n_\kappa \psi(a_\kappa n_\kappa)/n_\kappa \leqslant a_\kappa \psi(n_\kappa),$$

$$\begin{aligned}
B_\kappa &= \frac{a_{\kappa+1}}{n_{\kappa+1}} \int_{1/n_{\kappa+1}}^{1/n_\kappa} \theta^{-2} \psi\left(\frac{a_{\kappa+1}}{\theta^2 n_{\kappa+1}} \right) d\theta \\
&\leqslant \frac{a_{\kappa+1} \psi(n_{\kappa+1})}{n_{\kappa+1}} \int_{1/n_{\kappa+1}}^{1/n_\kappa} \theta^{-2} d\theta = a_{\kappa+1} \psi(n_{\kappa+1}).
\end{aligned}$$

Hence $\Sigma(A_\kappa+B_\kappa)<\infty$ and the condition (6.6) is satisfied. At the same time we deduce from (6.7*b*) that

$$\int_0^{2\pi}|s_{n_\kappa+m_\epsilon}(e^{i\theta})|\,d\theta\to\infty. \tag{6.8}$$

Additional remark. From the relation (6.8) we deduce, in particular, that the sequence $s_{m_\kappa+n_\kappa}$ cannot be majorized by any integrable function. Since, roughly speaking, we may take for n_κ an arbitrary sequence tending to ∞ sufficiently rapidly, so that the conditions (i), (ii), (iii), (iv) are satisfied, we see that we may choose $\{m_\kappa\}$ and $\{n_\kappa\}$ such that $m_\kappa+n_\epsilon$ is a subsequence of an arbitrary sequence of integers $N_1<N_2<\ldots$. Hence, whatever be the sequence $\{N_\kappa\}$ and the function $\epsilon(u)$, we can always find a $g(e^{i\theta})$ satisfying the condition (6) and yet such that $s_{n_\epsilon}(e^{i0})$ cannot be majorized by any integrable function†.

† See J. E. Littlewood and R. E. A. C. Paley, *Journal London Math. Soc.*, 6 (1931), 230-233, Theorem 5.

ON A TRIGONOMETRIC SUM

By S. CHOWLA *and* A. WALFISZ.

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Introduction.

Let $d(n)$ denote the number of divisors of the positive integer n. The trigonometric sum*

$$D(x) = \sum_{n \leqslant x} d(n)\, e^{2n\pi\theta i},$$

where θ is real, has recently been considered by us†. In particular, Walfisz has proved that, for almost all θ (that is, for all θ with the exception of a set of Lebesgue measure zero),

$$D(x) = O(x^{\frac{1}{2}} \log^{2+\epsilon} x),$$

$$D(x) = \Omega(x^{\frac{1}{2}} \log^{\frac{1}{2}} x \log\log x). \tag{1}$$

In proving (1) use was made of the following deep result due to T. Estermann‡:

I. *Let*

$$h \text{ and } k \text{ be integers, } k > 0,\ (h, k) = 1;$$

$$\Re(y) > 0, \quad k|y| < 8\sqrt{\{\Re(y)\}};$$

$$z = e^{2\pi i(h/k)-y}, \quad f(s) = \sum_{r=1}^{\infty} d(r)\, s^r \quad \text{for} \quad |s| < 1;$$

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* When the lower limit of summation is not explicitly stated it is always 1.

† See S. D. Chowla, "Some problems of diophantine approximation (I)", *Math. Zeitschrift*, 33 (1931), 544–563, and A. Walfisz, "Über einige trigonometrische Summen", *Math. Zeitschrift*, 33 (1931), 564–601.

‡ T. Estermann, "On the representations of a number as the sum of three products", *Proc. London Math. Soc.* (2), 29 (1929), 453–478, Lemma 11.

then, $\log y$ *denoting the principal branch, we have**

$$f(z) = \frac{1}{ky}(\gamma - \log y - 2\log k) + \frac{B|y|k\log 2k}{\Re(y)},$$

where γ *is Euler's constant.*

We are now able, with the simplest apparatus, to prove the following result, which is indeed weaker than I, but is sufficient to prove (1).

II. *Let*

(2) $$h \text{ and } k \text{ be integers, } k > 1,\ (h, k) = 1;$$

(3) $$y = \sigma + ti,\ \sigma \text{ and } t \text{ real}, \quad 0 < \sigma \leqslant 1/k^2, \quad |t| \leqslant \sigma;$$

(4) $$z = e^{2\pi i(h/k) - y}, \quad f(s) = \sum_{r=1}^{\infty} d(r)s^r \quad \text{for} \quad |s| < 1.$$

Then we have

(5) $$f(z) = \frac{1}{ky}(\gamma - \log y - 2\log k) + B(\sigma^{-\frac{1}{2}} + k\log k).$$

With the help of II we shall prove the following results which go beyond (1): for almost all θ,

(6) $$\Re\{D(x)\} = \Omega_R\{x^{\frac{1}{4}}\log^{\frac{1}{2}}x(\log\log x)^{\frac{3}{2}}\};$$

(7) $$\Im\{D(x)\} = \Omega_{\substack{R\\L}}\{x^{\frac{1}{4}}\log^{\frac{1}{2}}x(\log\log x)^{\frac{3}{2}}\}.$$

Our method, which leads to II, can be applied to related questions; these applications will be given elsewhere.

1. *Proof of* II.

Let $x \geqslant 3$ throughout. We put†

(8) $$S(x) = \sum_{m \leqslant x} d(m)\, e\left(\frac{mh}{k}\right), \quad T(x) = \sum_{n \leqslant x} S(n).$$

We shall show that, under the conditions (2), (3), and (4),

(9) $$T(x) = \frac{1}{2k}x^2\log x - \frac{1}{k}(\log k - \gamma + \tfrac{3}{4})x^2 + B(x^{\frac{3}{2}} + xk\log k + x^{\frac{1}{2}}k^2).$$

(5) will then be deduced without difficulty from (9).

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* We denote by A positive absolute constants, by B complex numbers for which $|B| < A$, by c positive constants which may depend only on θ, by (u, v) the greatest common divisor of u and v; $a \mid b$ means that a divides b, $a \nmid b$ that a does not divide b.

† We write, for brevity, $e(u)$ in place of $e^{2\pi iu}$.

From (8) we obtain

$$S(x)=\sum_{ab\leqslant x} e\left(\frac{abh}{k}\right)=2\sum_{a\leqslant\sqrt{x}}\sum_{a\leqslant b\leqslant x/a} e\left(\frac{abh}{k}\right)-\sum_{a\leqslant\sqrt{x}} e\left(\frac{a^2h}{k}\right)$$

$$=2\sum_{a\leqslant\sqrt{x}}\sum_{a\leqslant b\leqslant x/a} e\left(\frac{abh}{k}\right)+B\sqrt{x}. \tag{10}$$

Let $[x]$ denote the greatest integer contained in x. For $a\not\equiv 0 \pmod k$ we have

$$\sum_{a\leqslant b\leqslant x/a} e\left(\frac{abh}{k}\right)=\frac{e\left\{\frac{ah}{k}\left(\left[\frac{x}{a}\right]+1\right)\right\}-e\left(\frac{a^2h}{k}\right)}{e\left(\frac{ah}{k}\right)-1}$$

$$=\frac{1}{2i}\frac{2i}{e\left(\frac{ah}{2k}\right)-e\left(\frac{-ah}{2k}\right)}\left\{e\left(\frac{ah}{k}\left[\frac{x}{a}\right]\right)e\left(\frac{ah}{2k}\right)-e\left(\frac{a^2h}{k}\right)e\left(\frac{-ah}{2k}\right)\right\}$$

$$=\frac{1}{2i}\cot\frac{ah\pi}{k}\,e\left(\frac{ah}{k}\left[\frac{x}{a}\right]\right)+\tfrac{1}{2}e\left(\frac{ah}{k}\left[\frac{x}{a}\right]\right)$$

$$-\frac{1}{2i}\cot\frac{ah\pi}{k}\,e\left(\frac{a^2h}{k}\right)+\tfrac{1}{2}e\left(\frac{a^2h}{k}\right). \tag{11}$$

From (10) and (11) it follows that

$$S(x)=-i\sum_{\substack{a\leqslant\sqrt{x}\\ k\nmid a}}\cot\frac{ah\pi}{k}\,e\left(\frac{ah}{k}\left[\frac{x}{a}\right]\right)+i\sum_{\substack{a\leqslant\sqrt{x}\\ k\nmid a}}\cot\frac{ah\pi}{k}\,e\left(\frac{a^2h}{k}\right)$$

$$+2\sum_{\substack{a\leqslant\sqrt{x}\\ k\mid a}}\sum_{a\leqslant b\leqslant x/a}1+B\sqrt{x} \tag{12}$$

$$=-iS_1(x)+iS_2(x)+2S_3(x)+B\sqrt{x}. \tag{13}$$

We split $S_1(x)$ into the two sums $S_{11}(x)$ and $S_{12}(x)$, where

$$ah\equiv r \pmod k,\quad 1\leqslant r\leqslant\tfrac{1}{2}k \quad \text{in } S_{11}(x), \tag{14}$$

$$ah\equiv r \pmod k,\quad \tfrac{1}{2}k<r\leqslant k-1 \quad \text{in } S_{12}(x). \tag{15}$$

From the equations (12) to (15) it follows that

$$S_1(x)=S_{11}(x)+S_{12}(x), \tag{16}$$

$$S_{11}(x)=\sum_{r\leqslant\frac{1}{2}k}\cot\frac{r\pi}{k}\sum_{\substack{a\leqslant\sqrt{x}\\ ah\equiv r\,(\mathrm{mod}\,k)}} e\left(\frac{r}{k}\left[\frac{x}{a}\right]\right), \tag{17}$$

$$S_{12}(x)=\sum_{\frac{1}{2}k<r\leqslant k-1}\cot\frac{r\pi}{k}\sum_{\substack{a\leqslant\sqrt{x}\\ ah\equiv r\,(\mathrm{mod}\,k)}} e\left(\frac{r}{k}\left[\frac{x}{a}\right]\right). \tag{18}$$

For real u_1 and u_2 we have

$$|e(u_1)-e(u_2)| = \left|2\pi i\int_{u_2}^{u_1} e(u)\,du\right| \leqslant 2\pi|u_1-u_2|. \tag{19}$$

From (19) we obtain

$$e\left(\frac{r}{k}\left[\frac{x}{a}\right]\right)-e\left(\frac{r}{k}\,\frac{x}{a}\right) = B\,\frac{r}{k}.$$

Applied to (17) this gives

$$S_{11}(x) = \sum_{r\leqslant \frac{1}{2}k} \cot\frac{r\pi}{k} \sum_{\substack{a\leqslant\sqrt{x}\\ ah\equiv r\,(\mathrm{mod}\,k)}} e\left(\frac{r}{k}\,\frac{x}{a}\right)+B\sum_{r\leqslant\frac{1}{2}k}\cot\frac{r\pi}{k}\sum_{\substack{a\leqslant\sqrt{x}\\ ah\equiv r\,(\mathrm{mod}\,k)}}\frac{r}{k}. \tag{20}$$

In the latter sum

$$\sum_{r\leqslant\frac{1}{2}k}\cot\frac{r\pi}{k}\sum_{\substack{a\leqslant\sqrt{x}\\ ah\equiv r\,(\mathrm{mod}\,k)}}\frac{r}{k} = B\sum_{r\leqslant\frac{1}{2}k}\frac{r}{k}\operatorname{cosec}\frac{r\pi}{k}\sum_{\substack{a\leqslant\sqrt{x}\\ ah\equiv r\,(\mathrm{mod}\,k)}} 1$$

$$= B\sum_{r\leqslant\frac{1}{2}k}\sum_{\substack{a\leqslant\sqrt{x}\\ ah\equiv r\,(\mathrm{mod}\,k)}} 1 = B\sum_{a\leqslant\sqrt{x}} 1 = B\sqrt{x}. \tag{21}$$

From (20) and (21) we obtain

$$S_{11}(x) = \sum_{r\leqslant\frac{1}{2}k}\cot\frac{r\pi}{k}\sum_{\substack{a\leqslant\sqrt{x}\\ ah\equiv r\,(\mathrm{mod}\,k)}} e\left(\frac{r}{k}\,\frac{x}{a}\right)+B\sqrt{x}. \tag{22}$$

Writing $k-r$ for r in the sum (18) we obtain

$$S_{12}(x) = -\sum_{r<\frac{1}{2}k}\cot\frac{r\pi}{k}\sum_{\substack{a\leqslant\sqrt{x}\\ ah\equiv -r\,(\mathrm{mod}\,k)}} e\left(-\frac{r}{k}\left[\frac{x}{a}\right]\right).$$

From this, with the help of (19) and an equation similar to (20), we obtain

$$S_{12}(x) = -\sum_{r<\frac{1}{2}k}\cot\frac{r\pi}{k}\sum_{\substack{a\leqslant\sqrt{x}\\ ah\equiv -r\,(\mathrm{mod}\,k)}} e\left(-\frac{r}{k}\,\frac{x}{a}\right)+B\sqrt{x}. \tag{23}$$

We now make an evaluation of the sum

$$T_{11}(x) = \sum_{n\leqslant x} S_{11}(n). \tag{24}$$

From (22) and (24) it follows that

$$T_{11}(x) = \sum_{r\leqslant\frac{1}{2}k}\cot\frac{r\pi}{k}\sum_{n\leqslant x}\sum_{\substack{a\leqslant\sqrt{n}\\ ah\equiv r\,(\mathrm{mod}\,k)}} e\left(\frac{r}{k}\,\frac{n}{a}\right)+Bx^{\frac{3}{2}}, \tag{25}$$

and here we have

$$\sum_{n\leqslant x}\sum_{\substack{a\leqslant\sqrt{n}\\ ah\equiv r\,(\mathrm{mod}\,k)}} e\left(\frac{r}{k}\frac{n}{a}\right) = \sum_{\substack{a\leqslant\sqrt{x}\\ ah\equiv r\,(\mathrm{mod}\,k)}}\sum_{a^2\leqslant n\leqslant x} e\left(\frac{r}{k}\frac{n}{a}\right)$$

$$= B\sum_{\substack{a\leqslant\sqrt{x}\\ ah\equiv r\,(\mathrm{mod}\,k)}} \mathrm{cosec}\,\frac{r\pi}{ka} = B\sum_{\substack{a\leqslant\sqrt{x}\\ ah\equiv r\,(\mathrm{mod}\,k)}}\frac{ka}{r}$$

$$= \frac{Bk}{r}\sum_{\substack{a\leqslant\sqrt{x}\\ ah\equiv r\,(\mathrm{mod}\,k)}} a = B\sqrt{x}\,\frac{k}{r}\sum_{\substack{a\leqslant\sqrt{x}\\ ah\equiv r\,(\mathrm{mod}\,k)}} 1. \tag{26}$$

From (25) and (26) we obtain

$$T_{11}(x) = B\sqrt{x}\sum_{r\leqslant\frac{1}{2}k}\frac{k}{r}\cot\frac{r\pi}{k}\sum_{\substack{a\leqslant\sqrt{x}\\ ah\equiv r\,(\mathrm{mod}\,k)}} 1 + Bx^{\frac{3}{2}}$$

$$= B\sqrt{x}\,k^2\sum_{r\leqslant\frac{1}{2}k}\frac{1}{r^2}\sum_{\substack{a\leqslant\sqrt{x}\\ ah\equiv r\,(\mathrm{mod}\,k)}} 1 + Bx^{\frac{3}{2}}. \tag{27}$$

In the inner sum in (27) we have

$$a\equiv\rho\ (\mathrm{mod}\,k),\quad 1\leqslant\rho\leqslant k-1.$$

Hence

$$\sum_{\substack{a\leqslant\sqrt{x}\\ ah\equiv r\,(\mathrm{mod}\,k)}} 1 = \sum_{\substack{a\leqslant\sqrt{x}\\ a\equiv\rho\,(\mathrm{mod}\,k)}} 1 = \sum_{mk+\rho\leqslant\sqrt{x}} 1 = \sum_{0\leqslant m\leqslant(\sqrt{x}-\rho)/k} 1$$

$$= \left[\frac{\sqrt{x}-\rho}{k}\right]+1 \leqslant \frac{\sqrt{x}}{k}+1. \tag{28}$$

(27) and (28) give

$$T_{11}(x) = B\sqrt{x}k^2\sum_{r\leqslant\frac{1}{2}k}\frac{1}{r^2}\left(\frac{\sqrt{x}}{k}+1\right)+Bx^{\frac{3}{2}} \tag{29}$$

$$= B(xk+\sqrt{x}k^2+x^{\frac{3}{2}}).$$

Similarly, starting from (23), we find exactly as above that, if

$$T_{12}(x) = \sum_{n\leqslant x} S_{12}(n), \tag{30}$$

then

$$T_{12}(x) = B(xk+\sqrt{x}k^2+x^{\frac{3}{2}}). \tag{31}$$

We now put

$$T_1(x) = \sum_{n\leqslant x} S_1(n). \tag{32}$$

From (32), (16), (24), (30), (29), and (31) it follows that

$$T_1(x) = B(x^{\frac{3}{2}}+xk+x^{\frac{1}{2}}k^2). \tag{33}$$

We consider now the sum

$$S_2(x) = \sum_{\substack{a \leqslant \sqrt{x} \\ k \nmid a}} \cot \frac{ah\pi}{k}\, e\left(\frac{a^2 h}{k}\right). \tag{34}$$

If a takes the values $1, 2, \ldots, k-1 \pmod k$, we have

$$\sum_a \cot \frac{ah\pi}{k}\, e\left(\frac{a^2 h}{k}\right) = \sum_{a=1}^{k-1} \cot \frac{ah\pi}{k}\, e\left(\frac{a^2 h}{k}\right)$$

$$= \sum_{a < \frac{1}{2}k} \left(\cot \frac{ah\pi}{k} + \cot \frac{(k-a)h\pi}{k}\right) e\left(\frac{a^2 h}{k}\right) = 0.$$

It now follows from (34) that there is a positive integer $m \leqslant k-1$ such that

$$S_2(x) = \sum_{a \leqslant m} \cot \frac{ah\pi}{k}\, e\left(\frac{a^2 h}{k}\right),$$

whence

$$S_2(x) = B \sum_{a \leqslant k-1} \left| \operatorname{cosec} \frac{a\pi}{k} \right| = B \sum_{a \leqslant \frac{1}{2}k} \operatorname{cosec} \frac{a\pi}{k}$$

$$= Bk \sum_{a \leqslant \frac{1}{2}k} \frac{1}{a} = Bk \log k$$

and so

$$T_2(x) = \sum_{n \leqslant x} S_2(n) = Bx\, k \log k. \tag{35}$$

It remains to evaluate the sum $S_3(x)$ [given by (12) and (13)] and the iterated sum

$$T_3(x) = \sum_{n \leqslant x} S_3(n). \tag{36}$$

We have

$$S_3(x) = \sum_{\substack{a \leqslant \sqrt{x} \\ k \nmid a}} \sum_{a \leqslant b \leqslant x/a} 1 = \sum_{a \leqslant \sqrt{x/k}} \sum_{ka \leqslant b \leqslant x/ka} 1$$

$$= \sum_{a \leqslant \sqrt{x/k}} \left(\frac{x}{ka} - ka\right) + B\sqrt{x} = \frac{x}{k} \sum_{a \leqslant \sqrt{x/k}} \frac{1}{a} - k \sum_{a \leqslant \sqrt{x/k}} a + B\sqrt{x}$$

$$= \frac{x}{k}\left(\log \frac{\sqrt{x}}{k} + \gamma + \frac{Bk}{\sqrt{x}}\right) - \tfrac{1}{2}k\left[\frac{\sqrt{x}}{k}\right]\left[\frac{\sqrt{x}}{k} + 1\right] + B\sqrt{x}$$

$$= \frac{x}{k}\left(\log \frac{\sqrt{x}}{k} + \gamma + \frac{Bk}{\sqrt{x}}\right) - \tfrac{1}{2}k\,\frac{\sqrt{x}}{k}\,\frac{\sqrt{x}}{k} + B\sqrt{x} + Bk$$

$$= \frac{1}{2k}\, x \log x - \frac{1}{k}\,(\log k - \gamma + \tfrac{1}{2})\, x + B(\sqrt{x} + k). \tag{37}$$

From (36) and (37) it follows that

$$2T_3(x) = \frac{1}{k} \sum_{n\leqslant x}{}' n\log n - \frac{2}{k}(\log k-\gamma+\tfrac{1}{2}) \sum_{n\leqslant x} n + B(x^{\frac{3}{2}}+xk)$$

$$= \frac{1}{k} \sum_{n\leqslant x} n\left(\sum_{m=1}^{n}\frac{1}{m}-\gamma+\frac{B}{n}\right)$$

$$-\frac{2}{k}(\log k-\gamma+\tfrac{1}{2}) \sum_{n\leqslant x} n + B(x^{\frac{3}{2}}+xk)$$

$$= \frac{1}{k} \sum_{n\leqslant x} n \sum_{m=1}^{n}\frac{1}{m} - \frac{1}{k}(2\log k-\gamma+1) \sum_{n\leqslant x} n + B(x^{\frac{3}{2}}+xk)$$

$$= \frac{1}{k} \sum_{m\leqslant x}\frac{1}{m} \sum_{m\leqslant n\leqslant x} n - \frac{1}{2k}(2\log k-\gamma+1)[x][x+1] + B(x^{\frac{3}{2}}+xk)$$

$$= \frac{1}{2k}[x][x+1] \sum_{m\leqslant x}\frac{1}{m} - \frac{1}{2k} \sum_{m\leqslant x}(m-1)$$

$$-\frac{1}{2k}(2\log k-\gamma+1)x^2 + B(x^{\frac{3}{2}}+xk)$$

$$= \frac{1}{2k}x^2 \sum_{m\leqslant x}\frac{1}{m} - \frac{1}{2k} \sum_{m\leqslant x} m - \frac{1}{2k}(2\log k-\gamma+1)x^2 + B(x^{\frac{3}{2}}+xk)$$

$$= \frac{1}{2k}x^2(\log x+\gamma) - \frac{1}{2k}\frac{x^2}{2} - \frac{1}{2k}(2\log k-\gamma+1)x^2 + B(x^{\frac{3}{2}}+xk)$$

$$= \frac{1}{2k}x^2\log x - \frac{1}{k}(\log k-\gamma+\tfrac{3}{4})x^2 + B(x^{\frac{3}{2}}+xk). \tag{38}$$

In order to prove (9) we have only to combine (8), (13), (32), (33), (35), (36) and (38).

We shall now obtain (5) from (9), making frequent use of (2), (3), and (4). For brevity we write

$$p = 2\log k - 2\gamma + \tfrac{3}{2}, \quad q = 2\log k - \gamma + \tfrac{3}{2}, \quad r = \sigma^{-\frac{1}{2}} + k\log k. \tag{39}$$

From (8), (9), and (39) we obtain

$$f(z) = (1-e^{-y}) \sum_{n=1}^{\infty} S(n)e^{-ny} = (1-e^{-y})^2 \sum_{n=1}^{\infty} T(n)e^{-ny}$$

$$= \frac{1}{2k}(1-e^{-y})^2 \sum_{n=1}^{\infty} n^2(\log n-p)e^{-ny}$$

$$+B\sigma^2 \sum_{n=1}^{\infty}(n^{\frac{3}{2}}+nk\log k+n^{\frac{1}{2}}k^2)e^{-n\sigma}. \tag{40}$$

The error term here is, by (39),

$$B\sigma^2(\sigma^{-\frac{5}{2}}+\sigma^{-2}k\log k+\sigma^{-\frac{3}{2}}k^2)=B(\sigma^{-\frac{1}{2}}+k\log k+\sigma^{\frac{1}{2}}k^2)=Br.$$

If we substitute $\sum_{m=1}^{n}\frac{1}{m}-\gamma$ for $\log n$ in (40), we obtain an error

$$B\sigma^2\sum_{n=1}^{\infty}n^2\frac{1}{n}e^{-n\sigma}=B.$$

Hence

$$f(z)=\frac{1}{2k}(1-e^{-y})^2\sum_{n=1}^{\infty}n^2\Big(\sum_{m=1}^{n}\frac{1}{m}-q\Big)e^{-ny}+Br$$

$$=\frac{1}{2k}(1-e^{-y})^2\Big(\sum_{m=1}^{\infty}\frac{1}{m}\sum_{n=m}^{\infty}n^2e^{-ny}-q\sum_{n=1}^{\infty}n^2e^{-ny}\Big)+Br. \tag{41}$$

We write for brevity

$$e^{-y}=u. \tag{42}$$

Since

$$\sum_{n=m}^{\infty}n^2u^n=\frac{m^2u^m}{1-u}+\frac{(2m+1)u^{m+1}}{(1-u)^2}+\frac{2u^{m+2}}{(1-u)^3},$$

we obtain from (41), noting (42) and (39),

$$f(z)=\frac{1}{2k}(1-u)^2\Big\{\sum_{m=1}^{\infty}\frac{mu^m}{1-u}+2\sum_{m=1}^{\infty}\frac{u^{m+1}}{(1-u)^2}+\sum_{m=1}^{\infty}\frac{u^{m+1}}{m(1-u)^2}$$

$$+2\sum_{m=1}^{\infty}\frac{u^{m+2}}{m(1-u)^3}-q\Big(\frac{u}{1-u}+\frac{3u^2}{(1-u)^2}+\frac{2u^3}{(1-u)^3}\Big)\Big\}+Br$$

$$=\frac{1}{2k}(1-u)^2\Big(\frac{u}{(1-u)^2}+\frac{u^2}{(1-u)^3}+\frac{2u^2}{(1-u)^3}+\frac{u}{(1-u)^2}\log\frac{1}{1-u}$$

$$+\frac{2u^2}{(1-u)^3}\log\frac{1}{1-u}\Big)-\frac{1}{k}q\frac{u^3}{1-u}+Br$$

$$=\frac{1}{2k}\Big(\frac{3u^2}{1-u}+u\log\frac{1}{1-u}+\frac{2u^2}{1-u}\log\frac{1}{1-u}\Big)-\frac{1}{k}q\frac{u^3}{1-u}+Br$$

$$=\frac{1}{2k}\Big(\frac{3}{1-u}-3\log\frac{1}{1-u}+\frac{2}{1-u}\log\frac{1}{1-u}\Big)-\frac{1}{k}q\frac{1}{1-u}+Br. \tag{43}$$

In (43) we have, from (42),

$$\frac{1}{1-u}=\frac{1}{y}+B,\quad \log\frac{1}{1-u}=-\log y+B\sigma=B\log\frac{1}{\sigma}.$$

Making these substitutions and noting (39), we obtain

$$f(z) = \frac{1}{2k}\left(\frac{3}{y} + B\log\frac{1}{\sigma} - \frac{2}{y}\log y + B\log\frac{1}{\sigma}\right) - \frac{1}{k}\,q\,\frac{1}{y} + Br$$

$$= \frac{1}{ky}\left(\tfrac{3}{2} - \log y - q\right) + B(\sigma^{-\frac{1}{2}} + k\log k)$$

$$= \frac{1}{ky}(\gamma - \log y - 2\log k) + B(\sigma^{-\frac{1}{2}} + k\log k). \tag{5}$$

Thus II is proved.

2. *Proof of* (6) *and* (7).

Under the hypotheses of II we have $\sigma^{-\frac{1}{2}} \leqslant 1/(k\sigma)$. Hence it follows from (5) that

$$f(z) = \frac{1}{ky}(\gamma - \log y - 2\log k) + B\left(\frac{1}{k\sigma} + k\log k\right)$$

$$= \frac{\sigma - ti}{k|y|^2}\{\gamma - \log|y| - i\Im(\log y) - 2\log k\} + B\left(\frac{1}{k\sigma} + k\log k\right)$$

$$= \frac{\sigma - ti}{k|y|^2}\log\frac{1}{k^2|y|} + B\left(\frac{1}{k\sigma} + k\log k\right),$$

$$\Re\{f(z)\} = \frac{\sigma}{k|y|^2}\log\frac{1}{k^2|y|} + B\left(\frac{1}{k\sigma} + k\log k\right), \tag{44}$$

$$\Im\{f(z)\} = -\frac{t}{k|y|^2}\log\frac{1}{k^2|y|} + B\left(\frac{1}{k\sigma} + k\log k\right). \tag{45}$$

We shall now make use of the following theorem, due to Khintchine*:

Let $F(u)$ be positive and continuous for $u \geqslant 1$ and let $uF(u)$ be steadily decreasing. When the integral

$$\int_1^\infty F(u)\,du \tag{46}$$

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* A. Khintchine, "Einige Sätze über Kettenbrüche, mit Anwendungen auf die Theorie der Diophantischen Approximationen", *Math. Annalen*, 92 (1924), 115–125. Khintchine only considers the inequality $|\theta - (h/k)| < F(k)/k$. His proof leads, however, without further trouble to the theorem attributed to him in the text when we remember that (after Bernstein) each of the inequalities

$$a_{2n} > \frac{1}{G(2n-1)}, \quad a_{2n+1} > \frac{1}{G(2n)}$$

(for the elements in the simple continued fraction for θ) has infinitely many solutions, for almost all θ, whenever $G(n)$ is a positive, steadily decreasing function and $\sum_{n=1}^{\infty} G(n)$ diverges.

converges, each of the inequalities

$$0 \leqslant \pm\left(\theta - \frac{h}{k}\right) < \frac{F(k)}{k} \tag{47}$$

has only a finite number of integral solutions for almost all θ. *If, however, the integral* (46) *is divergent, then, for almost all* θ, *each of the inequalities* (47) *has infinitely many solutions with* $k > 0$, $(h, k) = 1$.

With the help of this result we shall deduce the following relations* from (44) and (45), for almost all θ:

$$\Re\{f(e^{-\rho+2\pi\theta i})\} = \Omega_R\left\{\left(\frac{1}{\rho}\,\mathrm{l}\,\frac{1}{\rho}\right)^{\frac{1}{2}}\left(\mathrm{ll}\,\frac{1}{\rho}\right)^{\frac{3}{2}}\left(\mathrm{lll}\,\frac{1}{\rho}\right)^{\frac{1}{8}}\right\} \quad (\rho > 0,\ \rho \to 0). \tag{48}$$

$$\Im\{f(e^{-\rho+2\pi\theta i})\} = \Omega_{\substack{R\\L}}\left\{\left(\frac{1}{\rho}\,\mathrm{l}\,\frac{1}{\rho}\right)^{\frac{1}{2}}\left(\mathrm{ll}\,\frac{1}{\rho}\right)^{\frac{3}{2}}\left(\mathrm{lll}\,\frac{1}{\rho}\right)^{\frac{1}{8}}\right\} \quad (\rho > 0,\ \rho \to 0). \tag{49}$$

To obtain (6) from (48) and (7) from (49) we make use of an artifice which has been used by Hardy and Littlewood† in a similar connection.

We apply Khintchine's theorem with

$$F(u) = (2\pi u\,\mathrm{l}\,2u\,\mathrm{ll}\,3u\,\mathrm{lll}\,27u)^{-1} \quad (u \geqslant 1),$$

so that the integral (46) is divergent. Hence, corresponding to almost every θ, there is an infinite sequence of positive integers $k_1, k_2, k_3, \ldots$ such that

$$3^{27} \leqslant k_1 < k_2 < k_3 < \ldots \tag{50}$$

and an infinite sequence of integers $h_1, h_2, h_3, \ldots$ such that, for all positive integral values of n,

$$(h_n, k_n) = 1, \tag{51}$$

$$(-1)^n\left(\theta - \frac{h_n}{k_n}\right) = \left|\theta - \frac{h_n}{k_n}\right| < (2\pi k_n{}^2\,\mathrm{l}\,k_n\,\mathrm{ll}\,k_n\,\mathrm{lll}\,k_n)^{-1}. \tag{52}$$

We put

$$\left\{\begin{array}{l} \sigma_n = (k_n{}^2\,\mathrm{l}\,k_n\,\mathrm{ll}\,k_n\,\mathrm{lll}\,k_n)^{-1}, \quad t_n = -2\pi\left(\theta - \dfrac{h_n}{k_n}\right), \quad y_n = \sigma_n + t_n i, \\ \qquad\qquad z_n = e^{(2h_n\pi i/k_n) - y_n}, \end{array}\right. \tag{53}$$

* For the sake of brevity we now write l, ll, lll for log, log log, log log log.

† G. H. Hardy and J. E. Littlewood, "Some problems of Diophantine approximation, II: The trigonometrical series associated with the elliptic ϑ-functions", *Acta Math.*, 37 (1914), 193–239 (228).

and we can apply (44) and (45) with

$$h = h_n, \quad k = k_n, \quad \sigma = \sigma_n, \quad t = t_n, \quad y = y_n, \quad z = z_n, \tag{54}$$

since it follows from the equations (50) to (54) that the conditions (2), (3), (4) are satisfied. We notice, further, that, in consequence of (50),

$$\mathrm{l}\, k_n > 27, \quad \mathrm{ll}\, k_n > 3, \quad \mathrm{lll}\, k_n > 1. \tag{55}$$

It follows now that

$$\Re\{f(z_n)\} = \frac{\sigma_n}{k_n |y_n|^2}\, \mathrm{l}\, \frac{1}{k_n^2 |y_n|} + B\left(\frac{1}{k_n \sigma_n} + k_n\, \mathrm{l}\, k_n\right), \tag{56}$$

$$\Im\{f(z_n)\} = -\frac{t_n}{k_n |y_n|^2}\, \mathrm{l}\, \frac{1}{k_n^2 |y_n|} + B\left(\frac{1}{k_n \sigma_n} + k_n\, \mathrm{l}\, k_n\right). \tag{57}$$

We have here, on account of (53) and (55),

$$\frac{1}{|y_n|^2} = \frac{1}{\sigma_n^2 + t_n^2} \geqslant \frac{1}{2\sigma_n^2} = \tfrac{1}{2}(k_n^2\, \mathrm{l}\, k_n\, \mathrm{ll}\, k_n\, \mathrm{lll}\, k_n)^2,$$

$$\frac{1}{k_n |y_n|^2} \geqslant \tfrac{1}{2} k_n^3 (\mathrm{l}\, k_n\, \mathrm{ll}\, k_n\, \mathrm{lll}\, k_n)^2, \tag{58}$$

$$\frac{1}{k_n^2 |y_n|} \geqslant \tfrac{1}{2}\, \mathrm{l}\, k_n\, \mathrm{ll}\, k_n\, \mathrm{lll}\, k_n \geqslant \mathrm{l}\, k_n, \tag{59}$$

$$\frac{1}{k_n \sigma_n} = k_n\, \mathrm{l}\, k_n\, \mathrm{ll}\, k_n\, \mathrm{lll}\, k_n. \tag{60}$$

From (56), (58), (53), (59), and (60) we obtain*

$$\Re\{f(z_n)\} \geqslant \tfrac{1}{2} k_n^3 (\mathrm{l}\, k_n\, \mathrm{ll}\, k_n\, \mathrm{lll}\, k_n)^2 (k_n^2\, \mathrm{l}\, k_n\, \mathrm{ll}\, k_n\, \mathrm{lll}\, k_n)^{-1}\, \mathrm{ll}\, k_n - A k_n\, \mathrm{l}\, k_n\, \mathrm{ll}\, k_n\, \mathrm{lll}\, k_n$$

$$= \tfrac{1}{2} k_n\, \mathrm{l}\, k_n (\mathrm{ll}\, k_n)^2\, \mathrm{lll}\, k_n - A k_n\, \mathrm{l}\, k_n\, \mathrm{ll}\, k_n\, \mathrm{lll}\, k_n,$$

whence

$$\Re\{f(z_n)\} \geqslant \tfrac{1}{3} k_n\, \mathrm{l}\, k_n (\mathrm{ll}\, k_n)^2\, \mathrm{lll}\, k_n \quad \text{for} \quad n > A. \tag{61}$$

We now need a lower limit for $|t_n|$ and to this end we apply Khintchine's theorem with

$$F(u) = \{u\, \mathrm{l}\, 2u\, \mathrm{ll}\, 3u\, (\mathrm{lll}\, 27u)^{\frac{1}{4}}\}^{-1} \quad (u \geqslant 1).$$

The integral (46) converges and hence, for almost all θ, (47) can have only a finite number of integral solutions. We may suppose here that the set of values of θ specified is exactly the same as that for which (57) was

* See foot-note *, p. 402.

proved (for the common part of two sets of " almost all " θ is a set of almost all θ, as is seen immediately by considering the complementary sets). Hence, on account of (52) and (53), we have*

$$(62) \qquad (-1)^{n+1} t_n = |t_n| \geqq c\{k_n^2 \, \mathrm{l}\, k_n \, \mathrm{ll}\, k_n \, (\mathrm{lll}\, k_n)^{\frac{1}{2}}\}^{-1}.$$

From (57), (58), (62), (59), and (60) it follows that

$$(-1)^n \Im\{f(z_n)\} \geqq c\, k_n^3 (\mathrm{l}\, k_n \, \mathrm{ll}\, k_n \, \mathrm{lll}\, k_n)^2 \{k_n^2 \, \mathrm{l}\, k_n \, \mathrm{ll}\, k_n (\mathrm{lll}\, k_n)^{\frac{1}{2}}\}^{-1} \mathrm{ll}\, k_n - A k_n \, \mathrm{l}\, k_n \, \mathrm{ll}\, k_n \, \mathrm{lll}\, k_n,$$

so that

$$(63) \qquad (-1)^n \Im\{f(z_n)\} \geqq c\, k_n \, \mathrm{l}\, k_n (\mathrm{ll}\, k_n)^2 (\mathrm{lll}\, k_n)^{\frac{3}{2}} \text{ for } n > c.$$

We now put

$$(64) \qquad k_n^2 \, \mathrm{l}\, k_n \, \mathrm{ll}\, k_n \, \mathrm{lll}\, k_n = \frac{1}{\rho_n}.$$

We then have the following asymptotic relations as $n \to \infty$:

$$\mathrm{l}\, k_n \sim \tfrac{1}{2} \mathrm{l} \frac{1}{\rho_n}, \quad \mathrm{ll}\, k_n \sim \mathrm{ll} \frac{1}{\rho_n}, \quad \mathrm{lll}\, k_n \sim \mathrm{lll} \frac{1}{\rho_n},$$

$$k_n^2 \sim 2 \left(\rho_n \, \mathrm{l} \frac{1}{\rho_n} \, \mathrm{ll} \frac{1}{\rho_n} \, \mathrm{lll} \frac{1}{\rho_n}\right)^{-1}, \quad k_n \sim \sqrt{2} \left(\rho_n \, \mathrm{l} \frac{1}{\rho_n} \, \mathrm{ll} \frac{1}{\rho_n} \, \mathrm{lll} \frac{1}{\rho_n}\right)^{-\frac{1}{2}},$$

whence

$$(65) \qquad k_n \, \mathrm{l}\, k_n (\mathrm{ll}\, k_n)^2 \, \mathrm{lll}\, k_n \sim \frac{1}{\sqrt{2}} \left(\frac{1}{\rho_n} \, \mathrm{l} \frac{1}{\rho_n}\right)^{\frac{1}{2}} \left(\mathrm{ll} \frac{1}{\rho_n}\right)^{\frac{3}{2}} \left(\mathrm{lll} \frac{1}{\rho_n}\right)^{\frac{1}{2}},$$

$$(66) \qquad k_n \, \mathrm{l}\, k_n (\mathrm{ll}\, k_n)^2 (\mathrm{lll}\, k_n)^{\frac{3}{4}} \sim \frac{1}{\sqrt{2}} \left(\frac{1}{\rho_n} \, \mathrm{l} \frac{1}{\rho_n}\right)^{\frac{1}{2}} \left(\mathrm{ll} \frac{1}{\rho_n}\right)^{\frac{3}{2}} \left(\mathrm{lll} \frac{1}{\rho_n}\right)^{\frac{1}{4}}.$$

From (53) and (64) it follows that

$$z_n = e^{-\rho_n + 2\pi\theta i}.$$

Hence, from (61), (65), and (63), (66), we obtain

$$\Re\{f(e^{-\rho_n + 2\pi\theta i})\} \geqq \left(\frac{1}{\rho_n} \, \mathrm{l} \frac{1}{\rho_n}\right)^{\frac{1}{2}} \left(\mathrm{ll} \frac{1}{\rho_n}\right)^{\frac{3}{2}} \left(\mathrm{lll} \frac{1}{\rho_n}\right)^{1} \quad (n > A),$$

$$(-1)^n \Im\{f(e^{-\rho_n + 2\pi\theta i})\} \geqq \left(\frac{1}{\rho_n} \, \mathrm{l} \frac{1}{\rho_n}\right)^{\frac{1}{2}} \left(\mathrm{ll} \frac{1}{\rho_n}\right)^{\frac{3}{2}} \left(\mathrm{lll} \frac{1}{\rho_n}\right)^{\frac{1}{2}} \quad (n > c).$$

Since $\rho > 0$, $\rho \to 0$, these two inequalities show the truth of (48) and (49).

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* See foot-note *, p. 402.

Let us suppose that

$$\Re\{D(n)\} \leqslant c(n \,\mathrm{l}\, n)^{\frac{1}{4}}(\mathrm{ll}\, n)^{\frac{3}{2}} \quad (n \geqslant 27). \tag{67}$$

It would then follow that, for $\rho > 0$, $\rho \to 0$,

$$\begin{aligned}\Re\{f(e^{-\rho+2\pi\theta i})\} &= \sum_{n=1}^{\infty} d(n) \cos 2n\pi\theta \, e^{-n\rho} \\ &= (1-e^{-\rho}) \sum_{n=1}^{\infty} \Re\{D(n)\} e^{-n\rho} \leqslant c + c\rho \sum_{n=27}^{\infty} (n \,\mathrm{l}\, n)^{\frac{1}{4}}(\mathrm{ll}\, n)^{\frac{3}{2}} e^{-n\rho} \\ &\leqslant c + c\rho \frac{1}{\rho}\left(\frac{1}{\rho}\,\mathrm{l}\,\frac{1}{\rho}\right)^{\frac{1}{4}}\left(\mathrm{ll}\,\frac{1}{\rho}\right)^{\frac{3}{2}} \leqslant c\left(\frac{1}{\rho}\,\mathrm{l}\,\frac{1}{\rho}\right)^{\frac{1}{4}}\left(\mathrm{ll}\,\frac{1}{\rho}\right)^{\frac{3}{2}}.\end{aligned}$$

Since this contradicts (48), (67) cannot be true for any value of c. This proves (6).

If we now suppose that either of the inequalities

$$\pm\Im\{D(n)\} \leqslant c(n \,\mathrm{l}\, n)^{\frac{1}{4}}(\mathrm{ll}\, n)^{\frac{3}{2}} \quad (n \geqslant 27)$$

is true, we are led, by exactly similar reasoning, to a contradiction of (49), and thus (7) is proved.

ON THE CONVERGENCE AND SUMMABILITY OF DIRICHLET'S SERIES*

By K. Ananda Rau.

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I. *Introduction.*

1. In a paper† published several years ago in these *Proceedings* Hardy and Littlewood proved a series of theorems of great generality, whose nature they have called Tauberian; and in the same paper they applied these theorems to the study of a variety of problems. The theorems were concerned, to put the matter roughly, with the relations connecting the behaviour of a function and of its derivatives of integral orders. Hardy and Littlewood foresaw that theorems similar to theirs can be proved involving Riemann-Liouville derivatives of non-integral orders. M. Riesz has since proved some of these extensions by making use of an important mean value theorem‡.

2. Among the applications which Hardy and Littlewood made of their Tauberian theorems, there is one class with which the present paper is connected, namely the applications to questions of convergence and summability of Dirichlet's series. Hardy and Littlewood concerned themselves only with ordinary Dirichlet's series, that is, series of the type $\Sigma a_n n^{-s}$; the methods of summation which they employed were those associated with the name of Cesàro, and the orders of summation were integral. But they foresaw not only that their theorems can be extended to non-integral orders of Cesàro summation, but also that the more general methods of summation by typical means introduced by Riesz

* The contents of this paper form a slightly modified version of part of an essay for which a Smith's Prize was awarded in 1918 in the University of Cambridge.

† G. H. Hardy and J. E. Littlewood, "Contributions to the arithmetic theory of series", *Proc. London Math. Soc.* (2), 11 (1913), 411–478. This paper will be referred to as the *Contributions*.

‡ Marcel Riesz, "Sur un théorème de la moyenne et ses applications", *Acta Lit. ac Sc. Reg. Univ. Hungaricae* (*Sectio Sci. Math.*), 1 (1923), 114–126.

can be employed and theorems of a similar character proved for Dirichlet's series of arbitrary type. Riesz, in the paper referred to, has proved some extensions of this nature.

3. The present paper deals with some of the generalizations suggested by Hardy and Littlewood and also some additional matter. The mean value theorem of Riesz and his extended forms of the Tauberian theorems of Hardy and Littlewood play an important part in the present paper*. In Section II I prove an Abelian theorem (Theorem 1) and obtain a few other results which, besides being required in the sequel, have also some interest of their own. Section III is devoted to some Tauberian theorems closely related to those proved in the *Contributions*. In Section IV are proved theorems on the convergence and summability (by typical means) of Dirichlet's series. Some of these theorems are purely arithmetical and form extensions of the corresponding theorems of Hardy and Littlewood; and the others are of a function-theoretic character. The classical theorem of Schnee and Landau is here extended by combining a Tauberian theorem with an important theorem of Riesz on the summability of Dirichlet's series. This method of proof resembles, in general nature, that by which Hardy deduced the convergence of Fourier series of functions of bounded variation by applying a Tauberian theorem to Fejér's classical result on the summability of such series by Cesàro's first mean.

II. *Auxiliary theorems.*

4. Let $l_1, l_2, \ldots$ be a strictly increasing sequence tending to infinity, and let $l_1 \geqslant 1$. Let $c_1+c_2+\ldots$ be an infinite series and let

$$C^\kappa(\omega) = \sum_{l_n \leqslant \omega} c_n(\omega - l_n)^\kappa,$$

where $\omega > 0$ is a variable and $\kappa \geqslant 0$ is a constant. We shall write $C(\omega)$ for $C^0(\omega)$. It is well known that, for $\kappa > 0$,

$$C^\kappa(\omega) = \kappa \int_0^\omega C(\tau)(\omega-\tau)^{\kappa-1}\,d\tau;$$

and we say, as usual, that the series Σc_n is summable (l, κ) to sum s if $\omega^{-\kappa} C^\kappa(\omega) \to s$ when $\omega \to \infty$.

In the present paper we shall have many order equations involving ω as the variable; and to avoid repetition we shall always understand that ω tends to infinity. Besides the series $c_1+c_2+\ldots$ we shall also have to

* I had communicated most of this paper along with some additional matter to Prof. M. Riesz several years ago; and it is this that he refers to on page 126 of his paper cited above.

employ other series denoted by

$$a_1+a_2+\ldots+a_n+\ldots,$$

$$b_1+b_2+\ldots+b_n+\ldots,$$

and, when we do so, we shall understand without further explanation that

$$A(\omega)=\sum_{l_n\leqslant\omega} a_n, \quad A^\kappa(\omega)=\sum_{l_n\leqslant\omega} a_n(\omega-l_n)^\kappa,$$

$$B(\omega)=\sum_{l_n\leqslant\omega} b_n, \quad B^\kappa(\omega)=\sum_{l_n\leqslant\omega} b_n(\omega-l_n)^\kappa.$$

5. THEOREM 1*. *Let* $\sum_1^\infty c_n$ *be an infinite series, and let* $b_n=c_n l_n^\beta$, *where* $\beta>0$ *is a constant. Let* $C^\kappa(\omega)=o(\omega^\gamma)$, *where* $\kappa\geqslant 0$, $\gamma\geqslant\kappa$. *Then* $B^\kappa(\omega)=o(\omega^{\gamma+\beta})$.

Let $l_n\leqslant\omega<l_{n+1}$. We shall have to consider various cases according to the value of κ.

(i) Let $\kappa=0$. Then

$$\begin{aligned} B(\omega) &= c_1 l_1^\beta+c_2 l_2^\beta+\ldots+c_n l_n^\beta \\ &= \sum_{\nu=1}^{n} C(l_\nu)(l_\nu^\beta-l_{\nu+1}^\beta)+C(l_n)\,l_n^\beta \\ &= -\int_1^\omega C(\tau)\,\frac{d}{d\tau}(\tau^\beta)\,d\tau+\omega^\beta C(\omega) \\ &= o\left(\int_1^\omega \tau^\gamma.\tau^{\beta-1}\,d\tau\right)+o(\omega^{\gamma+\beta}) \\ &= o(\omega^{\gamma+\beta}). \end{aligned}$$

(ii) Let κ be a positive integer. Then

$$\begin{aligned} B^\kappa(\omega) &= \sum_{\nu=1}^{n} c_\nu l_\nu^\beta(\omega-l_\nu)^\kappa \\ &= \sum_{\nu=1}^{n} C(l_\nu)[l_\nu^\beta(\omega-l_\nu)^\kappa-l_{\nu+1}^\beta(\omega-l_{\nu+1})^\kappa]+C(l_n)\,l_n^\beta(\omega-l_n)^\kappa \\ &= -\int_1^\omega C(\tau)\,\frac{d}{d\tau}\{\tau^\beta(\omega-\tau)^\kappa\}\,d\tau. \end{aligned}$$

Integrate by parts κ times; all the integrated terms vanish except one, which is

$$\omega^\beta C^\kappa(\omega)=o(\omega^{\gamma+\beta}).$$

* *Cf.* Theorem 14 of the *Contributions*, 435.

by hypothesis. We have to consider the integral remaining over, which is, save for a constant factor,

$$\int_1^\omega C^\kappa(\tau)\left(\frac{d}{d\tau}\right)^{\kappa+1}\{\tau^\beta(\omega-\tau)^\kappa\}\,d\tau. \tag{1}$$

Now
$$\left(\frac{d}{d\tau}\right)^\kappa\{\tau^\beta(\omega-\tau)^\kappa\} = \sum_{r=0}^{\kappa} H_r(\omega-\tau)^{\kappa-r}\tau^{\beta-\kappa+r}$$
$$= H_\kappa\tau^\beta + G(\tau),$$

say, where the H's are numbers independent of τ and ω. The integral (1) is therefore equal to

$$\beta H_\kappa\int_1^\omega C^\kappa(\tau)\tau^{\beta-1}\,d\tau + \int_1^\omega C^\kappa(\tau)\frac{dG(\tau)}{d\tau}\,d\tau.$$

The contribution from the first integral to $B^\kappa(\omega)$ is clearly equal to

$$o\left(\int_1^\omega \tau^\gamma.\tau^{\beta-1}\,d\tau\right) = o(\omega^{\gamma+\beta}).$$

The typical term in $dG(\tau)/d\tau$ is

$$H_r[(\beta-\kappa+r)(\omega-\tau)^{\kappa-r}\tau^{\beta-\kappa+r-1} + (\kappa-r)(\omega-\tau)^{\kappa-r-1}\tau^{\beta-\kappa+r}], \tag{2}$$

r taking the values 0, 1, ..., $\kappa-1$. Since $\kappa-r-1\geqslant 0$, $\beta+r-1>-1$ and $\gamma\geqslant\kappa$, the contribution due to the typical term (2) is

$$o\left(\int_1^\omega \tau^\gamma.\tau^{\beta-\kappa+r-1}(\omega-\tau)^{\kappa-r}\,d\tau\right) + o\left(\int_1^\omega \tau^\gamma.\tau^{\beta-\kappa+r}(\omega-\tau)^{\kappa-r-1}\,d\tau\right)$$
$$= o(\omega^{\gamma+\beta}) + o(\omega^{\gamma+\beta})^* = o(\omega^{\gamma+\beta}).$$

Combining these results, we see that the theorem is proved when κ is a positive integer.

(iii) Let $0<\kappa<1$. Then, as before,

$$B^\kappa(\omega) = -\int_1^\omega C(\tau)\frac{d}{d\tau}\{\tau^\beta(\omega-\tau)^\kappa\}\,d\tau. \tag{3}$$

Now
$$\frac{d}{d\tau}\{\tau^\beta(\omega-\tau)^\kappa\} = -\omega^\beta\kappa(\omega-\tau)^{\kappa-1} - \frac{d}{d\tau}\{(\omega^\beta-\tau^\beta)(\omega-\tau)^\kappa\}.$$

---

* Here, and elsewhere, we make use of the simple result that, if $a>-1$, $b>-1$, then

$$\int_1^\omega \tau^a(\omega-\tau)^b\,d\tau < \int_0^\omega \tau^a(\omega-\tau)^b\,d\tau = \frac{\Gamma(a+1)\,\Gamma(b+1)}{\Gamma(a+b+2)}\omega^{a+b+1} = O(\omega^{a+b+1}).$$

Therefore

$$B^\kappa(\omega) = \kappa\omega^\beta \int_1^\omega C(\tau)(\omega-\tau)^{\kappa-1}\,d\tau + \int_1^\omega C(\tau)\frac{d}{d\tau}\{(\omega^\beta-\tau^\beta)(\omega-\tau)^\kappa\}\,d\tau.$$

The first term on the right is

$$\omega^\beta C^\kappa(\omega) = o(\omega^{\gamma+\beta})$$

by hypothesis. The second term, when integrated by parts, becomes

$$\left[C^1(\tau)\frac{d}{d\tau}\{(\omega^\beta-\tau^\beta)(\omega-\tau)^\kappa\}\right]_1^\omega - \int_1^\omega C^1(\tau)\frac{d^2}{d\tau^2}\{(\omega^\beta-\tau^\beta)(\omega-\tau)^\kappa\}\,d\tau.$$

The first term in the above is easily seen to be zero; and the second term can be put as the sum of three integrals J_1, J_2, J_3 which contain under the sign of integration the function $C^1(\tau)$ multiplied respectively by

$$\beta(\beta-1)\tau^{\beta-2}(\omega-\tau)^\kappa,$$

$$-2\kappa\beta\tau^{\beta-1}(\omega-\tau)^{\kappa-1},$$

$$-\kappa(\kappa-1)(\omega-\tau)^{\kappa-2}(\omega^\beta-\tau^\beta).$$

Now, since $0<\kappa<1$, we have*

$$C^1(\omega) = \frac{1}{\Gamma(1+\kappa)\,\Gamma(1-\kappa)}\int_1^\omega C^\kappa(\tau)(\omega-\tau)^{-\kappa}\,d\tau.$$

Using this result along with the hypothesis $C^\kappa(\omega) = o(\omega^\gamma)$, we get

$$C^1(\omega) = O\left(\left|\int_1^\omega C^\kappa(\tau)(\omega-\tau)^{-\kappa}\,d\tau\right|\right)$$

$$= o\left(\int_1^\omega \tau^\gamma(\omega-\tau)^{-\kappa}\,d\tau\right) = o(\omega^{\gamma-\kappa+1}).$$

Hence

$$J_1 = O\left(\left|\int_1^\omega C^1(\tau)\tau^{\beta-2}(\omega-\tau)^\kappa\,d\tau\right|\right)$$

$$= o\left(\int_1^\omega \tau^{\gamma-\kappa+1}.\tau^{\beta-2}(\omega-\tau)^\kappa\,d\tau\right) = o(\omega^{\gamma+\beta}).$$

Similarly

$$J_2 = o\left(\int_1^\omega \tau^{\gamma-\kappa+1}.\tau^{\beta-1}(\omega-\tau)^{\kappa-1}\,d\tau\right) = o(\omega^{\gamma+\beta}).$$

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* For a proof see the Cambridge Tract, G. H. Hardy and M. Riesz, *The general theory of Dirichlet's series* (1915), 27–28 (Lemma 6). This work will be referred to in future as the *Tract*.

Finally, to estimate the order of J_3, we make use of the elementary result that, for $1 \leqslant \tau \leqslant \omega$,

$$\omega^\beta - \tau^\beta \leqslant H\omega^{\beta-1}(\omega-\tau),$$

where H is a positive constant depending only on β. We then get

$$J_3 = o\left(\int_1^\omega \omega^{\beta-1}\tau^{\gamma-\kappa+1}(\omega-\tau)^{\kappa-1}d\tau\right) = o(\omega^{\gamma+\beta}).$$

Combining these results we see that the proof of the theorem for the case $0 < \kappa < 1$ is completed.

The proof of the case when $\kappa > 1$ and is not an integer is obtained by a combination of the arguments used in cases (ii) and (iii) above, and does not present any serious difficulty. I shall indicate the line of proof briefly, without going into details.

We start as before from the formula

$$B^\kappa(\omega) = -\int_1^\omega C(\tau)\frac{d}{d\tau}\{\tau^\beta(\omega-\tau)^\kappa\}d\tau.$$

Let k be the integral part of κ. We integrate the above integral by parts k times and obtain (except for a constant factor)

$$\int_1^\omega C^k(\tau)\left(\frac{d}{d\tau}\right)^{k+1}\{\tau^\beta(\omega-\tau)^\kappa\}d\tau.$$

As in case (ii),

$$\left(\frac{d}{d\tau}\right)^k\{\tau^\beta(\omega-\tau)^\kappa\} = H_k\tau^\beta(\omega-\tau)^{\kappa-k} + G(\tau),$$

where

$$G(\tau) = \sum_{r=0}^{k-1} H_r(\omega-\tau)^{\kappa-r}\tau^{\beta-k+r},$$

the H's* being independent of τ and ω. The contribution to $B^\kappa(\omega)$ from $dG(\tau)/d\tau$ is proved as in case (ii) to be $o(\omega^{\gamma+\beta})$; so that it remains only to show that

$$\int_1^\omega C^k(\tau)\frac{d}{d\tau}\{\tau^\beta(\omega-\tau)^{\kappa-k}\}d\tau = o(\omega^{\gamma+\beta}).$$

The discussion of this integral is exactly similar to that of the integral on

* It need hardly be pointed out that these H's are not the same as the H's in case (ii) above. A similar remark applies to the function $G(\tau)$.

the right-hand side of (3) in case (iii), $C^k(\tau)$ taking the place of $C(\tau)$ and $\kappa-k$ (which is less than 1) taking the place of κ.

6. We now require two lemmas in order to prove Theorems 2 and 3, the latter of which forms an extension of Theorem 13 of the *Contributions*.

LEMMA 1. *Let $\kappa \geqslant 0$, and let p be a positive integer; also let $b_n = c_n l_n{}^p$. Then*

$$B^\kappa(\omega) = \sum_{m=0}^{p} (-1)^m \binom{p}{m} \omega^{p-m} C^{\kappa+m}(\omega).$$

Let $l_n \leqslant \omega < l_{n+1}$. First let $\kappa = 0$, so that

$$B^\kappa(\omega) = B(\omega) = c_1 l_1{}^p + c_2 l_2{}^p + \ldots + c_n l_n{}^p.$$

Now $$l_\nu{}^p = [\omega-(\omega-l_\nu)]^p = \sum_{m=0}^{p} (-1)^m \binom{p}{m} (\omega-l_\nu)^m \omega^{p-m}.$$

Therefore

$$B(\omega) = \sum_{\nu=1}^{n} c_\nu \sum_{m=0}^{p} (-1)^m \binom{p}{m} (\omega-l_\nu)^m \omega^{p-m}$$

$$= \sum_{m=0}^{p} (-1)^m \binom{p}{m} \omega^{p-m} \sum_{\nu=1}^{n} c_\nu (\omega-l_\nu)^m$$

$$= \sum_{m=0}^{p} (-1)^m \binom{p}{m} \omega^{p-m} C^m(\omega).$$

Next let $\kappa > 0$. Then, as in Theorem 1*,

$$B^\kappa(\omega) = -\int_0^\omega C(\tau) \frac{d}{d\tau} \{\tau^p (\omega-\tau)^\kappa\} d\tau \tag{4}$$

$$= \int_0^\omega C(\tau) \{\kappa \tau^p (\omega-\tau)^{\kappa-1} - p\tau^{p-1}(\omega-\tau)^\kappa\} d\tau.$$

Now, as above,

$$\tau^p = \sum_{m=0}^{p} (-1)^m \binom{p}{m} \omega^{p-m} (\omega-\tau)^m,$$

with a similar formula for τ^{p-1}. Using these and remembering that

$$\kappa \binom{p}{m+1} + p \binom{p-1}{m} = (\kappa+m+1) \binom{p}{m+1},$$

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* Since $l_1 \geqslant 1$, $C(\tau) = 0$ for $0 \leqslant \tau < 1$, and so it is indifferent whether we take 0 or 1 as the lower limit in the integral in (4).

we get

$$\kappa\tau^p(\omega-\tau)^{\kappa-1}-p\tau^{p-1}(\omega-\tau)^\kappa$$

$$=\kappa\sum_{m=0}^{p}(-1)^m\binom{p}{m}\omega^{p-m}(\omega-\tau)^{\kappa+m-1}-p\sum_{m=0}^{p-1}(-1)^m\binom{p-1}{m}\omega^{p-m-1}(\omega-\tau)^{\kappa+m}$$

$$=\kappa\omega^p(\omega-\tau)^{\kappa-1}-\sum_{m=0}^{p-1}(-1)^m\left[\kappa\binom{p}{m+1}+p\binom{p-1}{m}\right]\omega^{p-m-1}(\omega-\tau)^{\kappa+m}$$

$$=\kappa\omega^p(\omega-\tau)^{\kappa-1}-\sum_{m=0}^{p-1}(-1)^m(\kappa+m+1)\binom{p}{m+1}\omega^{p-m-1}(\omega-\tau)^{\kappa+m}$$

$$=\sum_{m=0}^{p}(-1)^m\binom{p}{m}(\kappa+m)\,\omega^{p-m}(\omega-\tau)^{\kappa+m-1}.$$

Using this in (4), we easily find that

$$B^\kappa(\omega)=\sum_{m=0}^{p}(-1)^m\binom{p}{m}\omega^{p-m}(\kappa+m)\int_0^\omega C(\tau)(\omega-\tau)^{\kappa+m-1}\,d\tau$$

$$=\sum_{m=0}^{p}(-1)^m\binom{p}{m}\omega^{p-m}\,C^{\kappa+m}(\omega),$$

which is the result of the lemma.

LEMMA 2. *Suppose that* $b_n=c_n l_n^p$, *where* p *is a positive integer. Let* Σc_n *be summable* $(l, \kappa+1)$ *to sum* C. *Then a necessary and sufficient condition that* Σc_n *should be summable* (l, κ) *is* $B^\kappa(\omega)=o(\omega^{\kappa+p})$.

By the previous lemma,

$$\frac{B^\kappa(\omega)}{\omega^{\kappa+p}}=\sum_{m=0}^{p}(-1)^m\binom{p}{m}\frac{C^{\kappa+m}(\omega)}{\omega^{\kappa+m}}. \tag{5}$$

Now, since Σc_n is summable $(l, \kappa+1)$ to sum C, it follows that, for $m=1, 2, \ldots, p$,

$$\frac{C^{\kappa+m}(\omega)}{\omega^{\kappa+m}}=C+o(1).$$

Using this in (5) we get

$$\frac{B^\kappa(\omega)}{\omega^{\kappa+p}}=\frac{C^\kappa(\omega)}{\omega^\kappa}+C\sum_{m=1}^{p}(-1)^m\binom{p}{m}+o(1) \tag{6}$$

$$=\frac{C^\kappa(\omega)}{\omega^\kappa}-C+o(1),$$

since

$$\sum_{m=1}^{p}(-1)^m\binom{p}{m}=(1-1)^p-1=-1.$$

The result of the lemma follows at once from (6).

7. THEOREM 2. *Suppose that* $A^\kappa(\omega) = o(\omega^{\kappa+\beta})$, *where* $\kappa \geqslant 0$, $\beta > 0$, *and that* $\Sigma a_n l_n^{-\beta}$ *is summable* $(l, \kappa+1)$. *Then it is also summable* (l, κ).

Suppose, first, that β is an integer. Let $c_n = a_n l_n^{-\beta}$. Then, by Lemma 2, a necessary and sufficient condition that Σc_n should be summable (l, κ) is $A^\kappa(\omega) = o(\omega^{\kappa+\beta})$, and this is satisfied by hypothesis.

Suppose next that β is not an integer. Let $\bar{\beta}$ be its integral part and β' its fractional part. Let $b_n = a_n l_n^{1-\beta'}$, $c_n = a_n l_n^{-\beta}$, so that $b_n = c_n l_n^{1+\bar{\beta}}$.

Now, since $A^\kappa(\omega) = o(\omega^{\kappa+\beta})$, it follows from Theorem 1 that

$$B^\kappa(\omega) = o(\omega^{\kappa+\beta+1-\beta'}) = o(\omega^{\kappa+1+\bar{\beta}}).$$

Since $1+\bar{\beta}$ is a positive integer and Σc_n is summable $(l, \kappa+1)$, it follows from Lemma 2 that Σc_n is also summable (l, κ).

8. THEOREM 3. *If* $A^\kappa(\omega) = o(\omega^{\kappa+\beta})$, *where* $\kappa \geqslant 0$, $\beta > 0$, *then* $\Sigma a_n l_n^{-\beta}$ *is either summable* (l, κ) *or is never summable* (l, r) *however large* r *may be.*

To prove this it is clearly sufficient to show that, if $\Sigma a_n l_n^{-\beta}$ is summable $(l, \kappa+\rho)$, where ρ is a positive integer, then the series is summable (l, κ). If $\rho = 1$, the result follows from Theorem 2. Suppose that $\rho > 1$; then, by Lemma 6 of the *Tract*,

$$A^{\kappa+\rho-1}(\omega) = O\left(\left|\int_0^\omega A^\kappa(\tau)(\omega-\tau)^{\rho-2}\,d\tau\right|\right)$$

$$= o(\omega^{\kappa+\beta+\rho-1}).$$

Hence we can apply Theorem 2 with $\kappa+\rho-1$ in place of κ and deduce that $\Sigma a_n l_n^{-\beta}$ is summable $(l, \kappa+\rho-1)$. It is clear that we can repeat the argument successively till we have proved that the series is summable $(l, \kappa+1)$. Then it follows from Theorem 2 that it is also summable (l, κ).

III. *Tauberian Theorems.*

9. Let $\phi(x)$ be defined for $x \geqslant 0$, and let it be bounded and integrable in every finite interval. Let

$$\Phi_r(x) = \frac{1}{\Gamma(r)} \int_0^x \phi(t)(x-t)^{r-1}\,dt,$$

where $r > 0$ is a constant, so that $\Phi_r(x)$ is the Liouville-Riemann integral of order r of the function $\phi(x)$. The following theorems are due to M. Riesz*.

* See pp. 115-117 and 120-124 of Riesz's paper cited in the Introduction. Compare also Lemma 7 (28-29) of the *Tract*.

THEOREM I*. *Let* $0<r\leqslant 1$, $0\leqslant\xi\leqslant x$, *and let*

$$g(x,\ \xi)=\frac{1}{\Gamma(r)}\int_0^{\xi}\phi(t)(x-t)^{r-1}\,dt.$$

Then there exists a τ *in the interval* $0\leqslant\tau\leqslant\xi$ *such that*

$$g(x,\ \xi)=\Phi_r(\tau).$$

THEOREM II. *Suppose that* $r>\kappa>0$, *and that* $V=V(x)$, $W=W(x)$ *are non-decreasing positive functions defined for* $x\geqslant 0$. *Suppose further that* $|\phi(x)|<V$ *for* $x>0$ *and that* $\Phi_r(x)=o(W)$ *as* $x\to\infty$. *Then*

$$\Phi_\kappa(x)=o(V^{1-(\kappa/r)}\,W^{\kappa/r}).$$

It will be noticed that, if $a_1+a_2+\ldots$ is an infinite series, $1\leqslant l_1<l_2<\ldots$ is a sequence tending to infinity, and $A(x)$, $A^r(x)$ are defined as usual, then $A^r(x)$ is, except for a factor depending only on r, the Liouville-Riemann integral of order r of $A(x)$. The following is an easy consequence of Theorem I and will be required in the sequel.

THEOREM A. *Let* $0<r\leqslant 1$, $0\leqslant\xi\leqslant x$, *and let* $A(x)$, $A^r(x)$ *be defined as usual with reference to the infinite series* $a_1+a_2+\ldots$. *Suppose further that* $A^r(x)=o(x^\beta)$ *as* $x\to\infty$, $\beta>0$ *being a constant. Then*

$$\int_0^{\xi}A(t)(x-t)^{r-1}\,dt=o(x^\beta).$$

For, when $0\leqslant\tau\leqslant\xi$, it follows from Theorem I that the modulus of the integral above is equal to

$$\begin{aligned}\left|\int_0^{\tau}A(t)(\tau-t)^{r-1}\,dt\right|&=\frac{1}{r}\,|A^r(\tau)|\\&\leqslant\frac{1}{r}\operatorname*{Max}_{0\leqslant t\leqslant x}|A^r(t)|\\&=o(x^\beta),\end{aligned}$$

since $A^r(x)=o(x^\beta)$ and $\beta>0$.

The following particular case of Theorem II, obtained by taking $V(x)=x^\alpha$, $W(x)=x^\beta$, where $\alpha\geqslant 0$, $\beta>0$, will be required in applications:

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* In Riesz's enunciation (allowing for change of notation) he takes $0<r<1$, $0\leqslant\xi<x$. But the cases when one or both of the conditions $r=1$, $\xi=x$ are satisfied are easily seen to be trivial. The enunciation given here is more convenient for our purpose.

THEOREM B. *Suppose that* $r > \kappa > 0$, *and that, as* $x \to \infty$, $\phi(x) = O(x^{\alpha})$, $\Phi_r(x) = o(x^{\beta})$, *where* $\alpha \geqslant 0$, $\beta > 0$. *Then*

$$\Phi_{\kappa}(x) = o(x^{\alpha+[(\beta-\alpha)\kappa/r]}).$$

10. The first theorem to be proved in this section is the following extension of Theorem 10 of the *Contributions*.

THEOREM 4. *Suppose that* $a_1+a_2+\ldots$ *is an infinite series,* $1 \leqslant l_1 < l_2 < \ldots$ *is a sequence tending to infinity,* $0 < r \leqslant 1$, $\beta > 0$, $\alpha+1 \geqslant 0$. *Suppose further that* $a_n/(l_n - l_{n-1}) = O(l_n{}^{\alpha})$ *and that* $A^r(\omega) = S\omega^r + o(\omega^{\beta})$, *where* S *is a constant. Then*

$$A(\omega) - S = o(\omega^{(\alpha r+\beta)/(r+1)}).$$

We shall first dispose of a case which is comparatively trivial. Suppose that $\beta > \alpha+r+1$, so that

$$\alpha r+\beta > \alpha r+\alpha+r+1 = (\alpha+1)(r+1),$$

$$\frac{\alpha r+\beta}{r+1} > \alpha+1.$$

Let $l_m \leqslant \omega < l_{m+1}$. If $\alpha \geqslant 0$,

$$\begin{aligned} |A(\omega)| = |\sum_{\nu=1}^{m} a_{\nu}| &\leqslant |a_1| + O\left(\sum_{2}^{m} l_{\nu}{}^{\alpha}(l_{\nu} - l_{\nu-1})\right) \\ &= |a_1| + O\left(l_m{}^{\alpha} \sum_{2}^{m} (l_{\nu} - l_{\nu-1})\right) \\ &= |a_1| + O\left[l_m{}^{\alpha}(l_m - l_1)\right] \\ &= O(l_m{}^{\alpha+1}) = O(\omega^{\alpha+1}) = o(\omega^{(\alpha r+\beta)/(r+1)}). \end{aligned}$$

If $-1 < \alpha < 0$, we have, as above,

$$\begin{aligned} |A(\omega)| &< |a_1| + O\left(\sum_{2}^{m} l_{\nu}{}^{\alpha}(l_{\nu} - l_{\nu-1})\right) \\ &< |a_1| + O\left(\sum_{2}^{m} \int_{l_{\nu-1}}^{l_{\nu}} x^{\alpha}\, dx\right) \\ &= |a_1| + O\left(\int_{l_1}^{l_m} x^{\alpha}\, dx\right) \\ &= O(l_m{}^{\alpha+1}) = O(\omega^{\alpha+1}) = o(\omega^{(\alpha r+\beta)/(r+1)}). \end{aligned}$$

If $\alpha = -1$, we have, as in the preceding case*,

$$|A(\omega)| < |a_1| + O\left(\int_{l_1}^{l_m} \frac{dx}{x}\right)$$

$$= O(\log l_m) = O(\log \omega) = o(\omega^{(\alpha r+\beta)/(r+1)}).$$

We may now confine ourselves to the case $\beta \leqslant \alpha+r+1$. We may also suppose†, without loss of generality, that $S = 0$. For brevity let us write $(\alpha r+\beta)/(r+1) = \rho$. If the theorem is false, there is a number $\delta > 0$ such that one of the inequalities (i) $A(\omega) > \delta\omega^\rho$, (ii) $A(\omega) < -\delta\omega^\rho$ at least is satisfied for values of ω increasing beyond all limit. We shall see that either of these suppositions leads to a contradiction. Let us take, for example, the first, and let $\omega > l_1$ be such that $A(\omega) > \delta\omega^\rho$.

By hypothesis there is a constant $C > 0$ such that, for $n \geqslant 2$,

$$\frac{|a_n|}{l_n - l_{n-1}} < C{l_n}^\alpha. \tag{7}$$

Let

$$H = \text{Min}\left(1, \frac{\delta}{C}, \frac{\delta}{2^\alpha C}\right), \quad \lambda = \frac{\beta-\alpha}{r+1};$$

and let

$$l_m \leqslant \omega < l_{m+1}, \quad l_{m+p} \leqslant l_m + H\omega^\lambda < l_{m+p+1},$$

$$l_m \leqslant x \leqslant l_m + H\omega^\lambda, \quad l_{m+q} \leqslant x < l_{m+q+1},$$

so that $p \geqslant 0$, $q \geqslant 0$, $q \leqslant p$. We note that $\alpha+\lambda = \rho$, $\alpha+\lambda(r+1) = \beta$, and $\lambda \leqslant 1$; the last is a consequence of the supposition $\beta \leqslant \alpha+r+1$.

Now, if $\alpha \geqslant 0$, we have, for $\nu = 1, 2, \ldots, q$,

$$l_{m+\nu}^\alpha \leqslant l_{m+q}^\alpha \leqslant l_{m+p}^\alpha \leqslant (l_m + H\omega^\lambda)^\alpha \leqslant (\omega + H\omega^\lambda)^\alpha$$

$$= \omega^\alpha(1 + H\omega^{\lambda-1})^\alpha \leqslant 2^\alpha \omega^\alpha \leqslant \frac{\delta}{CH}\omega^\alpha.$$

On the other hand, if $\alpha < 0$, we have, for $\nu = 1, 2, \ldots, q$,

$$l_{m+\nu}^\alpha \leqslant l_{m+1}^\alpha < \omega^\alpha \leqslant \frac{\delta}{CH}\omega^\alpha;$$

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* The final step is a consequence of the inequality $(\alpha r+\beta)/(r+1) > 0$. For the same reason it was sufficient to deal with $|A(\omega)|$ instead of with $|A(\omega) - S|$.

† If $S \neq 0$, we have only to replace the first term a_1 of the series $a + a_2 + \ldots$ by $a_1 - S$, and argue with the new series.

so that, whether α is positive or negative, we have, for $\nu = 1, 2, \ldots, q$,

$$l_{m+\nu}^{\alpha} \leqslant \frac{\delta}{CH}\omega^{\alpha}. \tag{8}$$

Therefore, for values of x in the interval $l_m \leqslant x \leqslant l_m + H\omega^{\lambda}$, we have*, on using (7) and (8),

$$\begin{aligned} |A(x)-A(\omega)| &= |a_{m+1}+\ldots+a_{m+q}| \leqslant \sum_{\nu=1}^{q} |a_{m+\nu}| \\ &< C \sum_{\nu=1}^{q} l_{m+\nu}^{\alpha}(l_{m+\nu}-l_{m+\nu-1}) \\ &\leqslant C \frac{\delta}{CH}\omega^{\alpha} \sum_{\nu=1}^{q} (l_{m+\nu}-l_{m+\nu-1}) \\ &= \frac{\delta}{H}\omega^{\alpha}(l_{m+q}-l_m) \\ &\leqslant \frac{\delta}{H}\omega^{\alpha}(x-l_m); \end{aligned}$$

and so

$$\begin{aligned} A(x) &\geqslant A(\omega) - |A(x)-A(\omega)| \\ &> \delta\omega^{\beta} - \frac{\delta}{H}\omega^{\alpha}(x-l_m) \\ &= \frac{\delta\omega^{\alpha}}{H}(l_m+H\omega^{\lambda}-x). \end{aligned}$$

Therefore

$$\begin{aligned} \int_{l_m}^{l_m+H\omega^{\lambda}} A(x)(l_m+H\omega^{\lambda}-x)^{r-1}\,dx &> \frac{\delta\omega^{\alpha}}{H}\int_{l_m}^{l_m+H\omega^{\lambda}} (l_m+H\omega^{\lambda}-x)^{r}\,dx \\ &= \frac{\delta\omega^{\alpha}}{H}\,\frac{H^{r+1}\omega^{\lambda(r+1)}}{r+1} \\ &= \frac{\delta H^{r}}{r+1}\omega^{\beta}. \end{aligned} \tag{9}$$

On the other hand, by hypothesis,

$$\int_{0}^{l_m+H\omega^{\lambda}} A(x)(l_m+H\omega^{\lambda}-x)^{r-1}\,dx = \frac{1}{r}A^{r}(l_m+H\omega^{\lambda}) = o\,[(l_m+H\omega^{\lambda})^{\beta}],$$

* If $q = 0$, then $A(x)-A(\omega) = 0$.

and therefore, by Theorem A,

$$\int_0^{l_m} A(x)(l_m+H\omega^\lambda-x)^{r-1}\,dx = o\,[(l_m+H\omega^\lambda)^\beta],$$

and so

$$\int_{l_m}^{l_m+H\omega^\lambda} A(x)(l_m+H\omega^\lambda-x)^{r-1}\,dx = o\,[(l_m+H\omega^\lambda)^\beta] = o(\omega^\beta),$$

since $\lambda \leqslant 1$. Combining this result with (9) we get

$$\frac{\delta H^r}{r+1}\omega^\beta = o(\omega^\beta),$$

$$\frac{\delta H^r}{r+1} = o(1)$$

for a set of values of ω tending to infinity. This is impossible, since $\delta H^r/(r+1)$ is a number greater than 0 and independent of ω. The supposition (i) thus leads to a contradiction. The supposition (ii) can be disposed of either in a similar way, or more easily by changing the signs of $a_1, a_2, \ldots$. The theorem is therefore proved*.

11. THEOREM 5†. *Suppose that* $a_n/(l_n-l_{n-1}) = O(l_n{}^\alpha)$, *where* $\alpha+1 \geqslant 0$, *and that* Σa_n *is summable* (l, r) *to sum* S, *where* $0 < r \leqslant 1$. *Then, if* $0 \leqslant \kappa < r$,

$$(10)\ddagger \qquad \frac{A^\kappa(\omega)}{\omega^\kappa} - S = o\,[\omega^{(\alpha+1)(r-\kappa)/(r+1)}].$$

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* I take this opportunity of pointing out that the following theorem, used by Hardy in connection with some problems in the analytic theory of numbers, *Proc. London Math. Soc.* (2), 18 (1919), 203, is a particular case of Theorem 4:

If $a_n = O(n^\epsilon)$ *and*

$$S_\kappa(x) = \sum_{n \leqslant x} (x-n)^\kappa a_n = O(x^{\gamma+\epsilon}),$$

where $0 < \kappa < 1$, *and* $\gamma > 0$, *for every positive value of* ϵ, *then*

$$S(x) = \sum_{n \leqslant x} a_n = O(x^{[\gamma/(1+\kappa)]+\epsilon})$$

for every positive value of ϵ.

This is obtained by taking, in Theorem 4, $l_n = n$, $\omega = x$, $\alpha = \epsilon$, $\beta = \gamma+\epsilon$, $r = \kappa$, $S = 0$, and then by observing that, since $\epsilon > 0$ is arbitrarily small, $O(x^{\gamma+\epsilon})$ may be replaced by $o(x^{\gamma+\epsilon})$.

† The case of Theorem 5 when $\alpha+1 = 0$ has been already proved by a different method for all values of $r > 0$ without any restriction. See G. H. Hardy, *Proc. London Math. Soc.* (2), 12 (1913), 174–180, and K. Ananda Rau, *Proc. London Math. Soc.* (2), 17 (1918), 334–336.

‡ It may be noticed that the S on the left of (10) is of importance only in the case $\alpha+1 = 0$; when $\alpha+1 > 0$, the S may be omitted since

$$\frac{(\alpha+1)(r-\kappa)}{r+1} > 0.$$

For $\kappa = 0$ the theorem follows by taking $\beta = r$ in the preceding theorem. To prove (10) for $\kappa > 0$ we apply Theorem B to*

$$A(\omega) - S = o\,[\omega^{[(\alpha+1)r]/(r+1)}],$$

$$A^r(\omega) - S\omega^r = o(\omega^r).$$

The required result then follows easily.

The restriction $r \leqslant 1$ in the two preceding theorems appears to be unnecessary, but I have not been able to remove it. In the absence of a proof of Theorem 5 for $r > 1$ we require Theorem 6, which, though it does not give so precise a result as the last theorem, is yet sufficient for the applications in view. I first state an elementary lemma required in the proof of Theorem 6.

LEMMA 3. *Let* $\gamma_0 > 0$, $r > 1$, *and let* $\gamma_1, \gamma_2, \ldots$ *be defined by the recurrence formula*

$$\gamma_{n+1} = \tfrac{1}{2}\gamma_0 + \tfrac{1}{2}\gamma_n\left(1 - \frac{1}{r}\right) \quad (n = 0, 1, 2, \ldots). \tag{11}$$

Then $\gamma_n > 0$ *for all* n, *and* $\gamma_n \to r\gamma_0/(r+1)$ *when* $n \to \infty$.

By successive application of (11) we get

$$\gamma_n = \tfrac{1}{2}\gamma_0\left[1 + \tfrac{1}{2}\left(1 - \frac{1}{r}\right) + \frac{1}{2^2}\left(1 - \frac{1}{r}\right)^2 + \ldots + \frac{1}{2^{n-1}}\left(1 - \frac{1}{r}\right)^{n-1} + \frac{1}{2^{n-1}}\left(1 - \frac{1}{r}\right)^n\right],$$

from which the result that $\gamma_n > 0$ follows at once; also obviously

$$\gamma_n \to \frac{\frac{1}{2}\gamma_0}{1 - \frac{1}{2}\,[1 - (1/r)]} = \frac{r\gamma_0}{r+1}.$$

THEOREM 6. *Suppose that* $a_n/(l_n - l_{n-1}) = O(l_n{}^{\alpha})$, *where* $\alpha + 1 \geqslant 0$, *and that* Σa_n *is summable* (l, r) *to sum* S. *Then, for every* $\epsilon > 0$,

$$A(\omega) - S = o\,[\omega^{[(\alpha+1)r/(r+1)]+\epsilon}]; \tag{12}$$

* If we write $A(\omega) - S = \phi(\omega)$ it is easy to verify that $\Phi_r(\omega)$ is $A^r(\omega) - S\omega^r$, save for a constant multiplier.

† The S on the left of (12) and (13) may be omitted owing to the presence of ϵ in the index of ω on the right. I have retained the S only with a view to preserve the formal relationship of Theorem 6 with Theorem 5.

and, if $0<\kappa<r$,

$$\frac{A^{\kappa}(\omega)}{\omega^{\kappa}}-S=o\,[\omega^{[(\alpha+1)(r-\kappa)/(r+1)]+\epsilon}]. \tag{13}$$

When $0<r\leqslant 1$, Theorem 5 gives this result and more. We may therefore confine ourselves to the case $r>1$. Let $\epsilon>0$ be fixed, and let ϵ_1 be defined by $2\epsilon_1 r=\epsilon(r+1)$. By hypothesis there is a constant $H>0$ such that $|a_n|<Hl_n{}^{\alpha}(l_n-l_{n-1})$ for $n\geqslant 2$. We may suppose, as usual, that $S=0$. We first show that

$$A(\omega)=O(\omega^{\alpha+1+\epsilon_1}).$$

Let $l_n\leqslant\omega<l_{n+1}$. If $\alpha\geqslant 0$,

$$|A(\omega)|=\left|\sum_{\nu=1}^{n}a_\nu\right|\leqslant\sum_{1}^{n}|a_\nu|<|a_1|+H\sum_{2}^{n}l_\nu{}^{\alpha}(l_\nu-l_{\nu-1})$$

$$<|a_1|+Hl_n{}^{\alpha}\sum_{2}^{n}(l_\nu-l_{\nu-1})=|a_1|+Hl_n{}^{\alpha}(l_n-l_1)=O(l_n^{\alpha+1})=O(\omega^{\alpha+1+\epsilon_1}).$$

On the other hand, if $\alpha<0$, let $f(x)$ be defined as follows:

$$f(x)=\frac{a_1}{l_1}\quad\text{for}\quad 0\leqslant x<l_1,$$

$$f(x)=\frac{a_{\nu+1}}{l_{\nu+1}-l_\nu}\quad\text{for}\quad l_\nu\leqslant x<l_{\nu+1}\quad(\nu=1,\,2,\,\ldots).$$

Then $$f(x)=O(l_{\nu+1}^{\alpha})=O(x^{\alpha+\epsilon_1}).$$

Since $\alpha+\epsilon_1>-1$, we have

$$\int_0^{\omega}f(x)\,dx=O\left(\int_0^{\omega}x^{\alpha+\epsilon_1}dx\right)=O(\omega^{\alpha+1+\epsilon_1}).$$

But
$$\int_0^{\omega}f(x)\,dx=\int_0^{l_1}+\int_{l_1}^{l_2}+\ldots+\int_{l_{n-1}}^{l_n}+\int_{l_n}^{\omega}$$

$$=A(\omega)+\frac{a_{n+1}(\omega-l_n)}{l_{n+1}-l_n}$$

$$=A(\omega)+O(\omega\,l_{n+1}^{\alpha})$$

$$=A(\omega)+O(\omega^{\alpha+1+\epsilon_1}).$$

Hence, in any case, we have

$$A(\omega)=O(\omega^{\alpha+1+\epsilon_1}).$$

Let $\gamma_0 = \alpha+1+\epsilon_1$, and let $\gamma_1, \gamma_2, \ldots$ be defined as in Lemma 3. Further, let

$$\beta_n = 1+\gamma_n\left(1-\frac{1}{r}\right) \quad (n = 0, 1, 2, \ldots),$$

so that, by Lemma 3, we have $\beta_n > 0$ for all n.

Now $A(\omega) = O(\omega^{\gamma_0})$, and also, by hypothesis, $A^r(\omega) = o(\omega^r)$. Therefore, by Theorem B,

$$A^1(\omega) = o(\omega^{\beta_0}).$$

Since $a_n/(l_n - l_{n-1}) = O(l_n^{\alpha+\epsilon_1})$, we can apply Theorem 4 with $\alpha+\epsilon_1$ in place of α, β_0 in place of β, and 1 in place of r. We then get

$$A(\omega) = o(\omega^{\gamma_1}).$$

If now we start with this relation instead of $A(\omega) = o(\omega^{\gamma_0})$, we can argue exactly as above and obtain

$$A(\omega) = o(\omega^{\gamma_2}).$$

We can repeat this argument any number of times, and therefore, for any fixed n, however large, we have

$$A(\omega) = o(\omega^{\gamma_n}).$$

But since, by Lemma 3, $\gamma_n \to r\gamma_0/(r+1)$, there is an N such that

$$\gamma_N < \frac{\gamma_0 r}{r+1} + \tfrac{1}{2}\epsilon.$$

Therefore

$$A(\omega) = o(\omega^{\gamma_N}) = o(\omega^{[\gamma_0 r/(r+1)]+\frac{1}{2}\epsilon}) = o(\omega^{[(\alpha+1)r/(r+1)]+\epsilon}), \tag{14}$$

since

$$\frac{\gamma_0 r}{r+1} + \tfrac{1}{2}\epsilon = \frac{(\alpha+1)r}{r+1} + \frac{r\epsilon_1}{r+1} + \tfrac{1}{2}\epsilon = \frac{(\alpha+1)r}{r+1} + \epsilon.$$

We next apply Theorem B to (14) and $A^r(\omega) = o(\omega^r)$ and obtain, for $0 < \kappa < r$,

$$\frac{A^\kappa(\omega)}{\omega^\kappa} = o\left[\omega^{[(\alpha+1)(r-\kappa)/(r+1)]+\epsilon}\right].$$

The theorem is thus completely proved.

IV. *Theorems on convergence and summability.*

12. The following lemma will be required in the course of the proofs of some theorems in this section.

LEMMA 4*. *Suppose that* $1 \leqslant l_1 < l_2 < \ldots$ *is an increasing sequence tending to infinity, and that* $\epsilon > 0$ *is fixed. Then the series*

$$\sum_1^\infty \frac{l_{n+1}-l_n}{l_{n+1} l_n^{\epsilon}}$$

is convergent.

THEOREM 7†. *Suppose that* $a_n/(l_n-l_{n-1}) = O(l_n{}^\alpha)$ *and that* Σa_n *is summable* (l, r). *Let* $0 \leqslant \kappa < r$. *Then* $\Sigma a_n l_n^{-\sigma}$ *is summable* (l, κ) *for*

$$\sigma > \frac{(\alpha+1)(r-\kappa)}{r+1}. \tag{15}$$

(i) Suppose first that $\alpha+1 < 0$ so that

$$\frac{(\alpha+1)(r-\kappa)}{r+1} > \alpha+1. \tag{16}$$

If $\epsilon > 0$, we have, for $n \geqslant 2$,

$$|a_n l_n^{-(\alpha+1+\epsilon)}| < H \frac{l_n-l_{n-1}}{l_n^{1+\epsilon}}, \tag{17}$$

where $H > 0$ is a number independent of n. By Lemma 4 it easily follows that the series

$$\sum_2^\infty \frac{l_n-l_{n-1}}{l_n^{1+\epsilon}}$$

is convergent; and therefore, using (17), we deduce that $\Sigma a_n l_n^{-\sigma}$ is absolutely convergent, and *a fortiori* summable (l, κ) for $\sigma > \alpha+1$ and therefore also for

$$\sigma \geqslant \frac{(\alpha+1)(r-\kappa)}{r+1},$$

using (16). The theorem is therefore proved when $\alpha+1 < 0$.

(ii) Suppose next that $\alpha+1 \geqslant 0$. Then, if $\epsilon > 0$, we have, by Theorem 6,

$$\frac{A^\kappa(\omega)}{\omega^\kappa} = o(\omega^{\mu+\epsilon}), \tag{18}$$

where for shortness μ is written for $(\alpha+1)(r-\kappa)/(r+1)$.

Now $\mu+\epsilon > 0$ and $\Sigma a_n l_n^{-\sigma}$ is summable (l, r) for $\sigma = \mu+\epsilon$. So we may apply Theorem 3 with $\mu+\epsilon$ in place of β and deduce that $\Sigma a_n l_n^{-\sigma}$ is

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* *Cf.*, for example, T. J. I'A. Bromwich, *Introduction to the theory of infinite series* (1908), 32.

† This is the extension of Theorem 18 of the *Contributions* (436).

summable (l, κ) for $\sigma = \mu + \epsilon$. Since ϵ is arbitrary the result of the theorem follows*.

THEOREM 8. *Suppose that* $a_n/(l_n - l_{n-1}) = O(l_n{}^{\alpha})$ *and that* $\Sigma a_n l_n^{-\gamma}$ *is summable* (l, r), γ *being real. Let* $0 \leqslant \kappa < r$. *Then* $\Sigma a_n l_n^{-\sigma}$ *is summable* (l, κ) *for*

$$\sigma > \frac{(\alpha+1)(r-\kappa)+\gamma(\kappa+1)}{r+1}. \tag{19}$$

Let $a_n' = a_n l_n^{-\gamma}$, $\sigma' = \sigma - \gamma$, so that the series $\Sigma a_n l_n^{-\sigma}$ is the same as the series $\Sigma a_n' l_n^{-\sigma'}$. Now

$$\frac{a_n'}{l_n - l_{n-1}} = \frac{a_n l_n^{-\gamma}}{l_n - l_{n-1}} = O(l_n^{\alpha-\gamma}),$$

and also $\Sigma a_n'$ is summable (l, r). Therefore, by the previous theorem, $\Sigma a_n' l_n^{-\sigma'}$ is summable (l, κ) for

$$\sigma' > \frac{(\alpha-\gamma+1)(r-\kappa)}{r+1};$$

and so the series $\Sigma a_n l_n^{-\sigma}$ is summable (l, κ) for

$$\sigma > \frac{(\alpha-\gamma+1)(r-\kappa)}{r+1} + \gamma$$

$$= \frac{(\alpha+1)(r-\kappa)+\gamma(\kappa+1)}{r+1},$$

which is the result to be proved.

13. We are now in a position to proceed to the proof of Theorem 9, which forms an extension of the Schnee-Landau theorem. We require in the course of the proof the following theorem due to M. Riesz†.

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* There appears to be little doubt that we can assert the summability (l, κ) of $\Sigma a_n l_n^{-\sigma}$ for $\sigma = \mu$. When $\alpha + 1 < 0$, this result is trivial, since $\Sigma a_n l_n^{-\mu}$ is absolutely convergent. When $\alpha + 1 = 0$ (so that $\mu = 0$) it is known that Σa_n is convergent (see the references under Theorem 5). When $0 < r \leqslant 1$, $\alpha + 1 > 0$, we can prove the summability (l, κ) of $\Sigma a_n l_n^{-\mu}$, for, using Theorem 5, we can replace (18) by

$$\frac{A^{\kappa}(\omega)}{\omega^{\kappa}} = o(\omega^{\mu}),$$

and, since $\mu > 0$, the result mentioned easily follows on applying Theorem 3 as in the proof of Theorem 7. To prove that $\Sigma a_n l_n^{-\mu}$ is summable (l, κ) when $\alpha + 1 > 0$, $r > 1$, we require to assume the truth of Theorem 5 for $r > 1$.

† See Theorem 41 of the *Tract*, 53–54.

THEOREM C. *Suppose that* $s = \sigma + it$ *is a complex variable and that*

$$f(s) = \Sigma a_n e^{-\lambda_n s} = \Sigma a_n l_n^{-s}$$

is summable for $\sigma > d$ *by typical means (type* $l_n = e^{\lambda_n}$*) of sufficiently high order. Let* $f(s)$ *be regular for* $\sigma > \eta$*, where* $\eta < d$*; and let* k*,* k' *be positive numbers such that* $k' < k$ *and that, for any* $\delta > 0$*, however small,*

$$f(s) = O(|t|^{k'}),$$

uniformly for $\sigma \geqslant \eta + \delta$*. Then the series* $\Sigma a_n l_n^{-s}$ *is summable* (l, k) *for* $\sigma > \eta$*.*

THEOREM 9. *Suppose that, for every* $\delta > 0$*,*

$$\frac{a_n}{l_n - l_{n-1}} = O(l_n^{\delta}),$$

so that the series $\Sigma a_n l_n^{-s}$ *is absolutely convergent for* $\sigma > 1$**. Suppose further that the function* $f(s)$ *defined by the series is regular for* $\sigma > \eta$*, and that there is a number* $r > 0$ *such that*

$$f(s) = O(|t|^r)$$

uniformly for $\sigma > \eta$*. Then* $\Sigma a_n l_n^{-s}$ *is summable* (l, κ)*, where* $0 \leqslant \kappa < r$*, for*

$$\sigma > \frac{r+\eta}{r+1} - \kappa \frac{1-\eta}{r+1}.$$

To prove this we observe that, by Theorem C quoted above, we can infer from the hypotheses of the present theorem that, if $\epsilon > 0$ is arbitrary, then $\Sigma a_n l_n^{-s}$ is summable $(l, r+\epsilon)$ for $\sigma \geqslant \eta + \epsilon$. We can therefore apply Theorem 8 with δ in place of α, $r+\epsilon$ in place of r, and $\eta+\epsilon$ in place of γ. We then get the result that $\Sigma a_n l_n^{-s}$ is summable (l, κ) for

$$\sigma > \frac{(\delta+1)(r+\epsilon-\kappa)+(\eta+\epsilon)(\kappa+1)}{r+\epsilon+1};$$

since δ, ϵ can be taken arbitrarily small, the result is true for

$$\sigma > \frac{(r-\kappa)+\eta(\kappa+1)}{r+1} = \frac{r+\eta}{r+1} - \kappa \frac{1-\eta}{r+1}.$$

It may be worth while to point out in what respects the above theorem forms a generalization of the Schnee-Landau theorem. In the latter theorem the sequence $l_1, l_2, \ldots$ is subjected to a restriction, which may be

* This follows from Lemma 4.

stated in the form* that there exists a finite $h \geqslant 0$ such that

$$\frac{l_{n+1}}{l_{n+1}-l_n} = O(l_n{}^h) \tag{20}$$

when $n \to \infty$; while in Theorem 9 the sequence (l_n) is unrestricted. On the other hand, the condition $a_n = O(l_n{}^\delta)$ which appears in the Schnee-Landau theorem is replaced by

$$\frac{a_n}{l_n-l_{n-1}} = O(l_n{}^\delta)$$

in Theorem 9. Besides these two points of difference, the Schnee-Landau theorem gives us information about convergence only, while Theorem 9 tells us also about the summability of the series. Even so far as the assertion about the convergence goes, Theorem 9 is more general than the Schnee-Landau theorem in the sense that the latter can be deduced from the former. This deduction is made in the following theorem, which is the form that Theorem 9 takes when the sequence (l_n) is subjected to the restriction (20).

THEOREM 10. *Suppose that the sequence (l_n) satisfies the condition* (20), *and that, for every $\delta > 0$, $a_n = O(l_n{}^\delta)$, so that the series $\Sigma a_n l_n^{-s}$ is absolutely convergent for $\sigma > h$. Suppose further that the function $f(s)$ defined by the series is regular for $\sigma > \eta$, and that there is a number $r > 0$ such that*

$$f(s) = O(|t|^r)$$

uniformly for $\sigma > \eta$. Then $\Sigma a_n l_n^{-s}$ is summable (l, κ), where $0 \leqslant \kappa < r$, for†

$$\sigma > \frac{hr+\eta}{r+1} - \kappa\frac{h-\eta}{r+1}.$$

Let $b_n = a_n l_n^{1-h}$, $s' = s+1-h$, $s' = \sigma'+it'$, so that $b_n l_n^{-s'} = a_n l_n^{-s}$, $\sigma' = \sigma+1-h$, $t' = t$, and the series $\Sigma b_n l_n^{-s'}$ is the same as the series $\Sigma a_n l_n^{-s}$. We apply Theorem 9 to $\Sigma b_n l_n^{-s'}$. We observe in the first place that

$$\frac{l_n}{l_n-l_{n-1}} = O(l_{n-1}^h) = O(l_n{}^h),$$

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* The sequence (l_n) was subjected to a more stringent condition in the original form of the Schnee-Landau theorem. Condition (20) was the one substantially adopted by Bohr. *Cf.* the *Tract*, 59, 60, and the first foot-note on the latter page.

† In particular, the series is convergent for $\sigma > (hr+\eta)/(r+1)$; this is the conclusion of the Schnee-Landau theorem.

and so

$$\frac{b_n}{l_n-l_{n-1}} = \frac{a_n l_n^{1-h}}{l_n-l_{n-1}} = a_n l_n^{-h}\frac{l_n}{l_n-l_{n-1}} = O(l_n^{\delta})\,l_n^{-h}\,O(l_n^{h}) = O(l_n^{\delta})$$

for every $\delta > 0$, in consequence of the hypothesis concerning a_n. Secondly, the function $F(s') = f(s)$ defined by $\Sigma b_n l_n^{-s'}$ is regular for $\sigma' > \eta+1-h$, and also

$$F(s') = O(|t'|^r)$$

uniformly for $\sigma' > \eta+1-h$. So, by Theorem 9, $\Sigma b_n l_n^{-s'}$ is summable (l, κ), where $0 \leqslant \kappa < r$, for

$$\sigma' > \frac{r+\eta+1-h}{r+1} - \kappa\frac{1-\eta-1+h}{r+1}$$

$$= \frac{r+\eta+1-h}{r+1} - \kappa\frac{h-\eta}{r+1},$$

and therefore the series $\Sigma a_n l_n^{-s}$ is summable (l, κ) for

$$\sigma > \frac{r+\eta+1-h}{r+1} - \kappa\frac{h-\eta}{r+1} + h - 1$$

$$= \frac{hr+\eta}{r+1} - \kappa\frac{h-\eta}{r+1},$$

which is the result to be proved.

14. We shall suppose throughout the rest of the paper that the sequence (l_n) satisfies condition (20). We proceed now to the proofs of the two following theorems, the first of which is an extension* of Theorem 48 of the *Tract*, and the second is a complement of Theorem 10 above (as regards summability) in much the same way as Satz 55 of Landau's *Handbuch* is a complement of Satz 54.

THEOREM 11. *Suppose that* Σa_n *is summable* (l, r). *Then* $\Sigma a_n l_n^{-s}$ *is summable* (l, κ), *where* $0 \leqslant \kappa < r$, *for* $\sigma > h(r-\kappa)$.

If $h = 0$, so that

$$\frac{l_{n+1}}{l_{n+1}-l_n} = O(1),$$

it follows at once from Theorem 21 of the *Tract* that Σa_n is convergent, from which the result of the theorem follows.

* It is indicated in the *Tract*, 57, that such an extension is true.

We next consider the case $h > 0$. We may suppose that the sum to which Σa_n is summable (l, r) is zero. Let $l_n \leqslant \omega < l_{n+1}$. Then, by Theorems 21 and 22 of the *Tract*, it follows that*

$$A(\omega) = o\left(\frac{l_{n+1}}{l_{n+1}-l_n}\right)^r = o(l_n^{hr}) = o(\omega^{hr}). \tag{21}$$

Also, by hypothesis, $A^r(\omega) = o(\omega^r)$. Therefore, by Theorem B,

$$A^\kappa(\omega) = o(\omega^{h(r-\kappa)+\kappa}). \tag{22}$$

Now, since $h(r-\kappa) > 0$ and $\Sigma a_n l_n^{-h(r-\kappa)}$ is summable (l, r), it follows from (22) and Theorem 3 that the latter series is summable (l, κ), from which the result of the theorem follows†.

THEOREM 12. *Suppose that $\Sigma a_n l_n^{-s}$ is summable by typical means of sufficiently high order for some values of s. Suppose further that the function $f(s)$ defined by the series is regular for $\sigma > \eta$, and that there is a number $r > 0$ such that*

$$f(s) = O(|t|^r)$$

uniformly for $\sigma > \eta$. Then the series is summable (l, κ), where $0 \leqslant \kappa < r$, for‡

$$\sigma > \eta + h(r-\kappa).$$

If $\epsilon > 0$ is given, then, by Riesz's Theorem C quoted above, it follows that the series $\Sigma a_n l_n^{-(\eta+\epsilon)}$ is summable $(l, r+\epsilon)$. If we write $a_n l_n^{-(\eta+\epsilon)} = b_n$,

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* When $l_n < \omega < l_{n+1}$, the result (21) follows directly from Theorem 22 of the *Tract*. When $\omega = l_n$, we have $A(\omega) = a_1 + a_2 + \ldots + a_n$ and (21) follows from Theorem 21 of the *Tract*. The distinction pointed out in the last foot-note on page 36 of the *Tract* for the case $\omega = l_n$ does not apply here if account is taken of the difference in the definitions of $A(l_n)$ adopted here and in the *Tract*. According to the definition in the *Tract* (p. 21), $A(l_n)$ would be $a_1 + a_2 + \ldots + a_{n-1}$, while here it is $a_1 + a_2 + \ldots + a_n$.

It would be sufficient for our purpose here to use the cruder result

$$A(\omega) = O\left(\frac{l_{n+1}}{l_{n+1}-l_n}\right)^r = O(\omega^{hr}).$$

† It may be observed that the conclusion of Theorem 11 is still true if, instead of assuming that Σa_n is summable (l, r), we assume only that Σa_n is finite (l, r), that is

$$A^r(\omega) = O(\omega^r).$$

For then, if $\epsilon > 0$, $\Sigma a_n l_n^{-\epsilon}$ is summable (l, r), and we apply Theorem 11 as proved above to the series Σb_n where $b_n = a_n l_n^{-\epsilon}$. Since ϵ may be arbitrarily small, the required result is asily obtained.

‡ In particular, the series is convergent for $\sigma > \eta + hr$; this is the conclusion of Satz 55 of Landau's *Handbuch*.

then Σb_n is summable $(l, r+\epsilon)$; and so, by Theorem 11, $\Sigma b_n l_n^{-s}$ is summable (l, κ) for $\sigma > h(r+\epsilon-\kappa)$; and therefore $\Sigma a_n l_n^{-s}$ is summable (l, κ) for $\sigma > (\eta+\epsilon)+h(r+\epsilon-\kappa)$. Since ϵ is arbitrary, it follows that $\Sigma a_n l_n^{-s}$ is summable (l, κ) for $\sigma > \eta+h(r-\kappa)$.

We can deduce the following corollary from Theorem 11.

COROLLARY. *Suppose that $\Sigma a_n l_n^{-s}$ is summable (l, r) for some values of s, and that $0 \leqslant \kappa < r$. Let $\sigma(r)$, $\sigma(\kappa)$ denote respectively the abscissae of summability of orders r, κ of the series. Then*

$$\sigma(\kappa)-\sigma(r) \leqslant h(r-\kappa).$$

The proof is immediate. Let $\epsilon > 0$. Then the series $\Sigma a_n l_n^{-s}$ is summable (l, r) for $s = \sigma(r)+\epsilon$. Hence, by Theorem 11, it is summable (l, κ) for* $\sigma > \sigma(r)+\epsilon+h(r-\kappa)$; that is to say,

$$\sigma(\kappa) \leqslant \sigma(r)+\epsilon+h(r-\kappa),$$

or, since ϵ is arbitrary,

$$\sigma(\kappa) \leqslant \sigma(r)+h(r-\kappa),$$

$$\sigma(\kappa)-\sigma(r) \leqslant h(r-\kappa).$$

15. THEOREM 13. *Suppose that, for every $\delta > 0$,*

$$\frac{l_{n+1}}{l_{n+1}-l_n} = O(l_n^{\delta}),$$

and that the series $\Sigma a_n l_n^{-s}$ possesses a region of summability by typical means. Then its line of convergence, line of absolute convergence, and all its lines of summability coincide.

By the corollary to Theorem 11 given above it follows that, if $r > \kappa \geqslant 0$ and r is sufficiently large, then

$$\sigma(\kappa)-\sigma(r) \leqslant \delta(r-\kappa);$$

this is true for every $\delta > 0$, and so

$$\sigma(\kappa)-\sigma(r) \leqslant 0.$$

But obviously

$$\sigma(\kappa)-\sigma(r) \geqslant 0,$$

and so

$$\sigma(\kappa) = \sigma(r).$$

Therefore $\sigma(\kappa)$ is constant for all $\kappa \geqslant 0$; that is to say, all the lines of summability coincide with the line of convergence.

* This follows from Theorem 11 by means of a simple linear transformation on s.

To prove that the line of absolute convergence also coincides with these, it is sufficient to prove that

$$\overline{\lim_{n\to\infty}} \frac{\log n}{\lambda_n} = 0, \tag{23}$$

where $\lambda_n = \log l_n$, for then, by Theorem 9 of the *Tract*, the required result will follow.

To prove (23), let us consider the series Σl_n^{-s}. Let α be its abscissa of convergence. Since Σl_n^{-s} is obviously divergent for $s = 0$, it follows, by Theorem 7 of the *Tract*, that

$$\alpha = \overline{\lim_{n\to\infty}} \frac{\log n}{\lambda_n},$$

so that it is sufficient to prove that $\alpha = 0$.

Now let $\delta > \delta' > 0$. Then, by hypothesis,

$$\frac{l_{n+1}}{l_{n+1} - l_n} < H l_n^{\delta'} \quad (n = 1, 2, 3, \ldots),$$

where $H > 0$ is a number independent of n. Hence

$$l_n^{-\delta} < \frac{H(l_{n+1} - l_n)}{l_{n+1} l_n^{\delta-\delta'}}. \tag{24}$$

By Lemma 4 the series

$$\sum_1^\infty \frac{l_{n+1} - l_n}{l_{n+1} l_n^{\delta-\delta'}}$$

is convergent, since $\delta - \delta' > 0$. Therefore, from (24), $\Sigma l_n^{-\delta}$ is convergent; and, since this is true for every $\delta > 0$, it follows that $\alpha \leqslant 0$. Also obviously $\alpha \geqslant 0$. Therefore $\alpha = 0$ and our theorem is proved.

16. Before concluding this paper I give a theorem which enables us to find the complete solution of the problem of convergence and summability in certain interesting cases.

THEOREM 14. *Suppose that* $a_n = O(l_n^{\delta})$ *for every* $\delta > 0$, *that*

$$f(s) = \Sigma a_n l_n^{-s} \tag{25}$$

is an integral function of finite order for every fixed* σ, *and that*†

* That is to say, corresponding to each fixed σ there is a finite k such that $f(\sigma + it) = O(|t|^k)$ as $|t| \to \infty$.

† This is Lindelöf's μ-function for $f(s)$; *cf. Tract*, 14.

$\mu(\sigma) \sim -\sigma/H$ as $\sigma \to -\infty$, $H > 0$ being a constant. Suppose further that the series $\Sigma a_n l_n^{-s}$ is divergent for $\sigma < 0$, and that

$$\frac{l_{n+1}}{l_{n+1} - l_n} = O(l_n^{H+\epsilon}) \tag{26}$$

for every $\epsilon > 0$. Then $\sigma(\kappa) = -H\kappa$ for $\kappa \geqslant 0$.*

Let κ be fixed and let σ be any real number such that $\mu(\sigma) > \kappa$. Then we can apply Theorem 10 to the series (25) with $\mu(\sigma)+\epsilon$ in place of r, $\sigma+\epsilon$ in place of η, and $H+\epsilon$ in place of h with $\epsilon > 0$. We then get the result

$$\sigma(\kappa) \leqslant \frac{(H+\epsilon)[\mu(\sigma)+\epsilon]+(\sigma+\epsilon)}{\mu(\sigma)+\epsilon+1} - \kappa \frac{H-\sigma}{\mu(\sigma)+\epsilon+1},$$

or, since ϵ can be taken arbitrarily small,

$$\sigma(\kappa) \leqslant \frac{H\mu(\sigma)+\sigma}{\mu(\sigma)+1} - \kappa \frac{H-\sigma}{\mu(\sigma)+1}.$$

This is true whatever σ may be, subject only to the condition $\mu(\sigma) > \kappa$. Hence we can let $\sigma \to -\infty$. We then get

$$\sigma(\kappa) \leqslant \lim_{\sigma \to -\infty} \left[\frac{H\mu(\sigma)+\sigma}{\mu(\sigma)+1} - \kappa \frac{H-\sigma}{\mu(\sigma)+1} \right] \tag{27}$$

provided that this limit exists. Now, since $\mu(\sigma) \sim -\sigma/H$, we have

$$\frac{H\mu(\sigma)+\sigma}{\mu(\sigma)+1} \to 0,$$

$$\frac{H-\sigma}{\mu(\sigma)+1} \to H,$$

as $\sigma \to -\infty$. Therefore (27) gives

$$\sigma(\kappa) \leqslant -H\kappa. \tag{28}$$

Since this is true for every $\kappa \geqslant 0$, it follows in particular that $\sigma(0) \leqslant 0$. But, by hypothesis, $\Sigma a_n l_n^{-s}$ is divergent for $\sigma < 0$, and so $\sigma(0) \geqslant 0$. Therefore $\sigma(0) = 0$.

Now, by the corollary to Theorem 11, we have, if $\kappa > 0$,

$$\sigma(0) - \sigma(\kappa) \leqslant (H+\epsilon)\kappa$$

for every $\epsilon > 0$. Hence

$$\sigma(0) - \sigma(\kappa) \leqslant H\kappa.$$

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* $\sigma(\kappa)$ is the abscissa of summability of order κ of the series $\Sigma a_n l_n^{-s}$.

But we have proved that $\sigma(0) = 0$, and therefore

$$\sigma(\kappa) \geqslant -H\kappa,$$

and this combined with (28) gives

$$\sigma(\kappa) = -H\kappa,$$

and the theorem is completely proved.

As an application of this theorem, let $k > 2$ be an integer and let $\chi(n)$ be a non-principal character (mod k), so that the function

$$L(s, \chi) = \sum_1^\infty \frac{\chi(n)}{n^s} \tag{29}$$

is an integral function. It is easily verified, by means of the functional equation satisfied by $L(s, \chi)$, that its μ-function for $\sigma \leqslant 0$ is $\frac{1}{2} - \sigma$; so that the hypotheses of Theorem 14 are satisfied with $H = 1$. We therefore conclude that the abscissa of summability (n, κ) of the series (29) is $-\kappa$ for all values of $\kappa \geqslant 0$.

ON THE THEORY OF TRIGONOMETRIC SERIES (I)

By S. VERBLUNSKY

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1. Consider the trigonometric series

$$\sum_{n=1}^{\infty} A_n(x), \tag{1}$$

where $$A_n(x) = a_n \cos nx + b_n \sin nx.$$

We write $$P(r, x) = \Sigma A_n(x) r^n \quad (0 < r < 1).$$

The upper and lower Poisson sums of (1), which we denote by $\bar{P}(x)$, $\underline{P}(x)$ respectively, are defined by the equations

$$\bar{P}(x) = \overline{\lim_{r \to 1}} P(r, x), \quad \underline{P}(x) = \underline{\lim_{r \to 1}} P(r, x).$$

If the series (1) is formally integrated p times, where $p = 2, 3, 4$, on writing

$$B_n(x) = b_n \cos nx - a_n \sin nx$$

and neglecting the constants of integration, we obtain the three series

$$\text{(i)} \quad -\Sigma A_n(x)/n^2, \qquad \text{(ii)} \quad \Sigma B_n(x)/n^3, \qquad \text{(iii)} \quad \Sigma A_n(x)/n^4.$$

If the series (i) is the Fourier series of a function $F(x)$, then, for those values of x at which the series (i) is summable by Poisson's method to $F(x)$, there exist certain relations of inequality between the Poisson sums of (1) and the upper and lower limits as $h \to 0$ of $\Delta^2 F(x, h)/h^2$, where

$$\Delta^2 F(x, h) = F(x+h) + F(x-h) - 2F(x);$$

namely, writing

$$D^2 F(x) = \overline{\lim_{h \to 0}} \Delta^2 F(x, h)/h^2, \quad \underline{D}^2 F(x) = \underline{\lim_{h \to 0}} \Delta^2 F(x, h)/h^2,$$

we have the relations

$$\bar{D}^2 F(x) \geqslant \underline{P}(x), \quad \underline{D}^2 F(x) \leqslant \bar{P}(x).$$

This result is due to Rajchman and Zygmund*.

It may be supposed that something analogous is true when (ii) is a Fourier series, and when (iii) is a Fourier series; and this supposition is correct. Concerning the problem of devising a proof which shall establish such analogous theorems, we may observe that the proof of Rajchman and Zygmund furnishes no aid, since they establish their theorem by a proof *ad hoc*—a proof which lacks the necessary adaptability.

In this paper we prove the theorem of Rajchman and Zygmund by a method which is simple and illuminating. The same method, with small changes, establishes the analogous theorem when (ii) is a Fourier series. When (iii) is a Fourier series, the proof remains the same in principle, though not in simplicity.

If the series (i) is the Fourier series of a function $F(x)$, then, at points x at which the series (i) is summable by Poisson's method to $F(x)$, there exist certain relations of inequality between the upper and lower limits as $h \to 0$ of $\Delta^2 F(x, h)/h$, and the upper and lower limits as $r \to 1$ of

$$(1-r)\, P(r, x);$$

namely, the relations

$$\overline{\lim_{h \to 0}} \Delta^2 F(x, h)/h \geqslant \pi \varliminf_{r \to 1} (1-r)\, P(r, x),$$

$$\varliminf_{h \to 0} \Delta^2 F(x, h)/h \leqslant \pi \overline{\lim_{r \to 1}} (1-r) P(r, x).$$

Further, for those values of x at which

$$D^2 F(x) = \lim_{h \to 0} \Delta^2 F(x, h)/h^2$$

exists, we have the equation

$$D^2 F(x) = P(x),$$

where $P(x)$ denotes the Poisson sum of (1); a result which is due to Fatou†.

Both these theorems have their analogues when (ii) is a Fourier series, and when (iii) is a Fourier series.

2. The following results will often be used. If

$$f(x) \sim \tfrac{1}{2}a_0 + \sum_{n=1}^{\infty} (a_n \cos nx + \beta_n \sin nx),$$

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* Rajchman and Zygmund, *Math. Zeitschrift*, 25 (1926), 261–273.

† Fatou, *Acta Math.*, 30 (1906), 335–400 (352).

then $$\tfrac{1}{2}a_0+\Sigma\,(a_n\cos nx+\beta_n\sin nx)\,r^n=\frac{1}{2\pi}\int_{-\pi}^{\pi}f(t+x)\,P\,dt,$$

where $$P=\frac{1-r^2}{\delta},$$

and $$\delta=1-2r\cos t+r^2,$$

so that $$r\frac{d\delta}{dr}=\delta+r^2-1,$$

$$\frac{d}{dr}\left(\frac{r}{\delta}\right)=\frac{1-r^2}{\delta^2}.$$

There exist two positive constants c and C such that

$$c(1-r)^2<\delta<C(1-r)^2 \quad (0\leqslant t\leqslant 1-r),$$

$$ct^2<\delta<Ct^2 \qquad (1-r\leqslant t\leqslant\pi).$$

3. THEOREM I. *If*

$$F(x)\sim C-\Sigma\,A_n(x)/n^2,$$

then, at a point x at which this series is summable (P) to $F(x)$,

$$\overline{\lim_{h\to 0}}\,\Delta^2 F(x,h)/h\geqslant\pi\,\underline{\lim_{r\to 1}}\,(1-r)\,P(r,x),$$

$$\underline{\lim_{h\to 0}}\,\Delta^2 F(x,h)/h\leqslant\overline{\lim_{r\to 1}}\,(1-r)\,P(r,x).$$

We may suppose that $x=0$, $F(0)=0$, $F(x)=F(-x)$. It is sufficient to prove the second inequality, and we may suppose that

$$\underline{\lim_{h\to 0}}\,\Delta^2 F(0,h)/h>-\infty.$$

We need therefore only show that the inequality

$$\underline{\lim_{h\to 0}}\,\frac{2F(h)}{h}>k$$

implies that $$\pi\,\overline{\lim_{r\to 1}}\,(1-r)\,P(r,0)\geqslant k,$$

where k is a finite number.

Consider the even function $f(t)$ defined by

$$f(t)=t \quad (0\leqslant t\leqslant\pi).$$

We have $$\lim \frac{2f(h)}{h} = 2,$$

while $$P(r, 0) = \frac{1}{\pi} \int_0^\pi t \frac{d^2 P}{dt^2} dt,$$

which gives $$\lim \pi(1-r) P(r, 0) = 2.$$

By adding to $F(x)$ a multiple of $f(x)$, we see that it is sufficient to prove that the inequality

$$\lim_{h \to 0} \frac{F(h)}{h} > 0 \tag{2}$$

implies that $$\overline{\lim_{r \to 1}} (1-r) P(r, 0) \geqslant 0. \tag{3}$$

The relation (2) implies that $F(x) \geqslant 0$ in a neighbourhood of the origin.

Since $$P(r, 0) = \frac{1}{\pi} \int_0^\pi F(t) (1-r^2) \frac{d^2}{dt^2} \left(\frac{1}{\delta} \right) dt,$$

it follows that the first member of (3) is unaltered by altering $F(x)$ outside a neighbourhood of the origin. We may therefore suppose that $F(x) \geqslant 0$ everywhere.

We write $$F(t) \sim \tfrac{1}{2}a_0 + \Sigma a_n \cos nt,$$

$$A(r) = \tfrac{1}{2}a_0 + \Sigma a_n r^n$$

$$= \frac{1}{\pi} \int_0^\pi F(t) \frac{1-r^2}{\delta} dt, \tag{4}$$

$$P(r, 0) = -r \frac{d}{dr} \left(r \frac{dA}{dr} \right).$$

If now (3) was false, we should have

$$\overline{\lim} (1-r) P(r, 0) < 0.$$

Hence $$\frac{d}{dr} \left(r \frac{dA}{dr} \right) > \frac{K}{1-r},$$

for some $K > 0$ and for r sufficiently near to 1. This implies that

$$\lim_{r \to 1} \frac{dA}{dr} = +\infty.$$

Thus $A(r)$ tends to 0 as a strictly increasing function, and is therefore negative for r sufficiently near to 1. But this contradicts (4).

4. THEOREM II (Rajchman and Zygmund). *If*

$$F(x) \sim C - \Sigma A_n(x)/n^2,$$

then, at a point x at which this series is summable (P) to $F(x)$,

$$\bar{D}^2 F(x) \geqslant \underline{P}(x), \quad \underline{D}^2 F(x) \leqslant \bar{P}(x).$$

We may suppose that $x = 0$, $F(0) = 0$, $F(x) = F(-x)$. It is sufficient to prove the second inequality, and we may suppose that $\underline{D}^2 F(0) > -\infty$. We need therefore only show that $\underline{\lim}\, 2F(h)/h^2 > k$ implies that $\bar{P}(0) \geqslant k$, where k is a finite number.

Consider the even function $f(t)$ defined by

$$f(t) = 1 - \cos t \quad (0 \leqslant t \leqslant \pi).$$

Then $$D^2 f(0) = P(0) = 1.$$

By adding to $F(x)$ a multiple of $f(x)$, we see that it is sufficient to prove that the inequality

$$\lim_{h \to 0} \frac{F(h)}{h^2} > 0 \tag{5}$$

implies that $$\bar{P}(0) \geqslant 0.$$

As before, we may suppose that $F(x) \geqslant 0$ everywhere. If now $\bar{P}(0) < 0$, then

$$\frac{d}{dr}\left(r \frac{dA}{dr}\right) > 0 \quad (r \geqslant r_0). \tag{6}$$

We observe that

$$\frac{A(r)}{\log 1/r} = \frac{(d/d\rho)\, A(\rho)}{(d/d\rho) \log 1/\rho} \quad (r < \rho < 1)$$

$$= -\rho \frac{d}{d\rho} A(\rho).$$

Hence $$\frac{A(r)}{\log 1/r} - \frac{A(\rho)}{\log 1/\rho} = \sigma \frac{d}{d\sigma} A(\sigma) - \rho \frac{d}{d\rho} A(\rho) \quad (\rho < \sigma < 1),$$

and this expression is positive for $r \geqslant r_0$ by (6). In order to prove that (6) is impossible, we need only show that

$$\lim_{r \to 1} \frac{d}{dr}\left(\frac{A(r)}{\log 1/r}\right) > 0. \tag{7}$$

We have

$$\lim_{r\to 1}\frac{d}{dr}\left(\frac{rA(r)}{1-r^2}\right)=\lim_{r\to 1}\frac{d}{dr}\left(\frac{1}{\pi}\int_0^\pi F(t)\,\frac{r}{\delta}\,dt\right)$$

$$=\lim_{r\to 1}\frac{1}{\pi}\int_0^\pi F(t)\,\frac{1-r^2}{\delta^2}\,dt;$$

and this expression is positive by (5). Finally,

$$\frac{d}{dr}\left(\frac{A(r)}{\log 1/r}\right)=\frac{1-r^2}{r\log 1/r}\,\frac{d}{dr}\left(\frac{rA(r)}{1-r^2}\right)+\frac{rA(r)}{1-r^2}\,\frac{d}{dr}\left(\frac{1-r^2}{r\log 1/r}\right),$$

so that, since

$$\lim\frac{1}{1-r^2}\,\frac{d}{dr}\left(\frac{1-r^2}{r\log 1/r}\right)=-\tfrac{1}{3},$$

the inequality (7) follows.

5. THEOREM III. *If*

$$F(x)\sim C-\Sigma A_n(x)/n^2,$$

then, at a point x at which this series is summable (P) to $F(x)$, $\bar{P}(x)$, $\underline{P}(x)$ lie between the upper and lower limits, as $h\to 0$, of

$$\frac{3}{2h^3}\int_0^h[\Delta^2F(x,h)-\Delta^2F(x,u)]\cos u\,du. \tag{8}$$

We may suppose that $x=0$, $F(0)=0$, $F(x)=F(-x)$. If we write

$$\phi(t)=\sin t\,F(t)-\int_0^t F(u)\cos u\,du,$$

it is sufficient to prove that

$$\underline{P}(0)\geqslant\lim_{t\to 0}3\phi(t)/t^3.$$

We have
$$P(r,0)=\frac{1}{\pi}\int_0^\pi F(t)\,\frac{d^2P}{dt^2}\,dt,$$

and
$$\frac{d^2P}{dt^2}=2r(1-r^2)\left[\frac{4r\sin^2t}{\delta^3}-\frac{\cos t}{\delta^2}\right].$$

Thus

$$P(r,0)=\frac{1}{\pi}\int_0^\pi\frac{(1-r^2)\,8r^2\sin t}{\delta^3}\sin tF(t)\,dt-\frac{1}{\pi}\int_0^\pi\frac{2r(1-r^2)}{\delta^2}F(t)\cos t\,dt.$$

The second term with its sign gives, on integrating by parts,

$$-\frac{1}{\pi}\cdot\frac{2r(1-r^2)}{(1+r)^4}\int_0^\pi F(u)\cos u\,du-\frac{1}{\pi}\int_0^\pi\frac{(1-r^2)\,8r^2\sin t}{\delta^3}\left(\int_0^t F(u)\cos u\,du\right)dt.$$

Hence
$$\underline{P}(0)=\lim_{r\to1}\frac{1}{\pi}\int_0^\pi\frac{(1-r^2)\,8r^2\sin t}{\delta^3}\phi(t)\,dt$$
$$\geqslant\lim_{t\to0}\frac{\phi(t)}{t^3}\cdot\lim_{r\to1}\frac{1}{\pi}\int_0^\pi\frac{(1-r^2)\,8r^2t^3\sin t}{\delta^3}dt,$$

and the value of the second factor is 3.

COROLLARY 1. *At a point x at which*

$$F(x)=\lim_{h\to0}[F(x+h)+F(x-h)]/2, \tag{9}$$

$\bar{P}(x)$, $\underline{P}(x)$ lie between the upper and lower limits, as $h\to0$, of

$$\frac{3}{2h^3}\int_0^h[\Delta^2F(x,h)-\Delta^2F(x,u)]\,du. \tag{10}$$

For, taking $x=0$, $F(0)=0$, $F(x)=F(-x)$, we have $F(h)=o(1)$, so that the difference between (8) and (10) tends to zero with h.

COROLLARY 2 (Fatou). *If $D^2F(x)$ exists, then $P(x)=D^2F(x)$.*

COROLLARY 3. *If*

$$f(x)\sim-\Sigma B_n(x)/n,$$

then $\bar{P}(x)$, $\underline{P}(x)$ lie between the upper and lower limits, as $h\to0$, of

$$\frac{3}{2h^3}\int_0^h u\,[f(x+u)-f(x-u)]\,du.$$

This is a consequence of Corollary 1 and of the equation

$$\int_0^h[\Delta^2F(x,h)-\Delta^2F(x,u)]\,du=\int_0^h u\{f(x+u)-f(x-u)\}\,du.$$

COROLLARY 4 (Grosz*). *If*

$$f(x)\sim-\Sigma B_n(x)/n,$$

then $\bar{P}(x)$, $\underline{P}(x)$ lie between the upper and lower limits, as $h\to0$, of

$$[f(x+h)-f(x-h)]/2h.$$

* Grosz, *Sitzungsberichte d. k. Akad d. Wiss, Wien.*, 124 (1915), 1017–1037.

6. We write

$$DF(x) = \lim_{h\to 0} [F(x+h)-F(x-h)]/2h$$

when the limit exists;

$$\Delta^3 F(x, h) = F(x+3h)-3F(x+h)+3F(x-h)-F(x-3h),$$

$$\underline{D}^3 F(x) = \lim_{h\to 0} \Delta^3 F(x, h)/8h^3, \qquad \overline{D}^3 F(x) = \overline{\lim_{h\to 0}} \Delta^3 F(x, h)/8h^3,$$

and

$$D^3 F(x) = \lim_{h\to 0} \Delta^3 F(x, h)/8h^3$$

when the limit exists.

THEOREM IV. *If*

$$F(x) \sim C+\Sigma B_n(x)/n^3,$$

then, at a point x at which $DF(x)$ exists,

$$\overline{\lim}\, \Delta^3 F(x, h)/h^2 \geqslant 3\pi \underline{\lim}\, (1-r) P(r, x),$$

$$\underline{\lim}\, \Delta^3 F(x, h)/h^2 \leqslant 3\pi \overline{\lim}\, (1-r) P(r, x).$$

We may suppose that $x = 0$, $F(x) = -F(-x)$. It is sufficient to prove the second inequality, and we may suppose that

$$\underline{\lim}\, \Delta^3 F(0, h)/h^2 > -\infty.$$

We need therefore only show that $\underline{\lim}\, 2[F(3h)-3F(h)]/h^2 > k$ implies that $\overline{\lim}\, 3\pi(1-r) P(r, 0) \geqslant k$, where k is a finite number.

Consider the odd function $\phi(t)$ defined by

$$\phi(t) = 2t^2+\mu \sin t \quad (0 \leqslant t \leqslant \pi).$$

Then

$$D\phi(0) = \mu,$$

while

$$\lim 2[\phi(3h)-3\phi(h)]/h^2 = 12\lambda$$

$$= 3\pi \lim (1-r) P(r, 0).$$

By adding to $F(x)$ a suitable $\phi(x)$, we see that it is sufficient to prove that the inequality

$$\underline{\lim}\, [F(3h)-3F(h)]/h^2 > 0 \tag{11}$$

implies that

$$\overline{\lim}\, (1-r) P(r, 0) \geqslant 0, \tag{12}$$

while we may suppose that

$$DF(0) = 0. \tag{13}$$

By (11), $F(3t)/3t > F(t)/t$ for sufficiently small $t > 0$. It follows by (13) that $F(t) > 0$ for such t. Since

$$P(r, 0) = -\frac{1}{\pi}\int_0^\pi F(t)\frac{d^3}{dt^3}\left(\frac{1}{\delta}\right)(1-r^2)\,dt,$$

the value of the first member of (12) is unaltered by altering $F(t)$ outside a neighbourhood of the origin. We may therefore suppose that

$$F(t) \geqslant 0 \quad (0 \leqslant t \leqslant \pi).$$

If now we write

$$F(t) \sim \Sigma \beta_n \sin nt,$$

$$A(r) = \Sigma n\beta_n r^n$$

$$= \frac{1}{\pi}\int_0^\pi F(t)\sin t\,\frac{(1-r^2)\,2r}{\delta^2}\,dt,$$

then, by (13) and § 5, Corollary 4,

$$\lim A(r) = 0,$$

while

$$P(r, 0) = -r\frac{d}{dr}\left(r\frac{dA}{dr}\right).$$

From this point, the proof is the same as for Theorem I.

7. THEOREM V. *If*

$$F(x) \sim C + \Sigma B_n(x)/n^3,$$

then, at a point x at which $DF(x)$ exists,

$$\bar{D}^3 F(x) \geqslant \underline{P}(x), \quad \underline{D}^3 F(x) \leqslant \bar{P}(x).$$

We may suppose that $x = 0$, $F(x) = -F(-x)$. It is sufficient to prove the second inequality, and we may suppose that $\underline{D}^3 F(0) > -\infty$. We need therefore only show that $\underline{\lim}\,[F(3h) - 3F(h)]/4h^3 > k$ implies that $\bar{P}(0) \geqslant k$, where k is a finite number.

Consider the odd function $f(t)$ defined by

$$f(t) = \lambda \sin t + \mu \sin 3t \quad (0 \leqslant t \leqslant \pi).$$

Then

$$Df(0) = \lambda - 3\mu,$$

while

$$D^3 f(0) = \lambda - 27\mu$$

$$= P(0).$$

Thus, by adding to $F(x)$ a suitable $f(x)$, we see that it is sufficient to prove that the inequality

$$\underline{\lim}\,[F(3h)-3F(h)]/4h^3 > 0 \tag{14}$$

implies that $\bar{P}(0) \geqslant 0$, while we may suppose that $DF(0) = 0$. As before, we may suppose that

$$F(t) \geqslant 0 \quad (0 \leqslant t \leqslant \pi).$$

With the notation of § 6, we see that, as in the proof of Theorem II, we need only show that

$$\underline{\lim_{r\to 1}} \frac{d}{dr}\left(\frac{r\,A(r)}{1-r^2}\right) > 0.$$

But

$$\frac{d}{dr}\left(\frac{r\,A(r)}{1-r^2}\right) = \frac{4}{\pi}\int_0^\pi F(t)\sin t\,\frac{r}{\delta}\,\frac{1-r^2}{\delta^2}\,dt;$$

so that, since $\underline{\lim}\, F(t)/t^3 > 0$ by (14), the result follows.

8. THEOREM VI. *If*

$$F(x) \sim C + \Sigma' B_n(x)/n^3,$$

then, at a point x at which $DF(x)$ and $D^3F(x)$ exist,

$$P(x) = D^3F(x).$$

We may suppose that $x = 0$, $F(x) = -F(-x)$, and that

$$DF(0) = D^3F(0) = 0.$$

Then

$$F(t) = o(t), \tag{15}$$

and

$$F(3t) - 3F(t) = o(t^3).$$

Hence

$$F(t/3^n) - 3F(t/3^{n+1}) = o(t/3^{n+1})^3$$

and

$$3^n F(t/3^n) - 3^{n+1} F(t/3^{n+1}) = o(t^3/3^{2n+3})$$

uniformly in the positive integer n. Since, by (15),

$$\lim_{n\to\infty} 3^{n+1} F(t/3^{n+1}) = 0,$$

we get, by addition,

$$F(t) = o(t^3).$$

Further,

$$P(r, 0) = -\frac{1}{\pi}\int_0^\pi F(t)\frac{d^3P}{dt^3}\,dt,$$

and

$$\frac{d^3P}{dt^3} = -2r(1-r^2)\left[\frac{\sin t}{\delta^2} - \frac{6r\sin 2t}{\delta^3} + \frac{24r^2\sin^3 t}{\delta^4}\right],$$

so that

$$P(r, 0) = o\left[(1-r^2)\int_0^\pi \frac{t^4}{\delta^2}\,dt\right]+o\left[(1-r^2)\int_0^\pi \frac{t^4}{\delta^3}\,dt\right]+o\left[(1-r^2)\int_0^\pi \frac{t^6}{\delta^4}\,dt\right]$$
$$= o(1).$$

9. We write

$$\Delta^4 F(x, h) = F(x+2h)-4F(x+h)+6F(x)-4F(x-h)+F(x-2h),$$

$$\bar{D}^4 F(x) = \overline{\lim_{h\to 0}}\, \Delta^4 F(x, h)/h^4, \quad \underline{D}^4 F(x) = \underline{\lim_{h\to 0}}\, \Delta^4 F(x, h)/h^4,$$

and
$$D^4 F(x) = \lim_{h\to 0} \Delta^4 F(x, h)/h^4$$

when the limit exists.

THEOREM VII. *If*

$$F(x) \sim C+\Sigma A_n(x)/n^4,$$

then, at a point x at which $D^2 F(x)$ exists,

$$\overline{\lim}\, 3\Delta^4 F(x, h)/h^3 \geqslant 2\pi \underline{\lim}\, (1-r) P(r, x),$$

$$\underline{\lim}\, 3\Delta^4 F(x, h)/h^3 \leqslant 2\pi \overline{\lim}\, (1-r) P(r, x).$$

We may suppose that $x = 0$, $F(0) = 0$, $F(x) = F(-x)$. It is sufficient to prove the second inequality, and we may suppose that

$$\underline{\lim}\, \Delta^4 F(x, h)/h^3 > -\infty.$$

We need therefore only show that $\underline{\lim}\, 3\,[F(2h)-4F(h)]/h^3 > k$ implies that $\overline{\lim}\, (1-r) P(r, 0) \geqslant k$, where k is a finite number.

Consider the even function $f(t)$ defined by

$$f(t) = \lambda t^3+2\mu(1-\cos t) \quad (0 \leqslant t \leqslant \pi).$$

Then
$$D^2 f(0) = \mu,$$

while
$$3 \lim\, [f(2t)-4f(t)]/t^3 = 12\lambda$$
$$= \lim\, (1-r) P(r, 0).$$

Thus, by adding to $F(x)$ a suitable $f(x)$, we see that it is sufficient to prove that the inequality

$$\underline{\lim}\, [F(2h)-4F(h)]/h^3 > 0 \tag{16}$$

implies that
$$\overline{\lim}\, (1-r) P(r, 0) \geqslant 0, \tag{17}$$

while we may suppose that

$$\lim F(h)/h^2 = 0. \tag{18}$$

By (16), $F(2h)/(2h)^2 > F(h)/h^2$ for sufficiently small h, so that, for such h, $F(h) \geqslant 0$ by (18). Since

$$P(r, 0) = \frac{1}{\pi}\int_0^\pi F(t)(1-r^2)\frac{d^4}{dt^4}\left(\frac{1}{\delta}\right)dt,$$

it follows that the first member of (17) is unaltered by altering $F(t)$ outside a neighbourhood of the origin. We may therefore suppose that $F(t) \geqslant 0$ everywhere.

We write

$$F(t) \sim \tfrac{1}{2}a_0 + \Sigma a_n \cos nt,$$

$$A(r) = \tfrac{1}{2}a_0 + \Sigma a_n r^n$$

$$= \frac{1}{\pi}\int_0^\pi F(t)\frac{1-r^2}{\delta}\,dt,$$

$$B(r) = -r\frac{d}{dr}\left(r\frac{dA}{dr}\right)$$

$$= -\Sigma n^2 a_n r^n,$$

and then

$$P(r, 0) = -r\frac{d}{dr}\left(r\frac{dB}{dr}\right).$$

If now

$$\overline{\lim}\,(1-r)\,P(r, 0) < 0,$$

then

$$\frac{d}{dr}\left(r\frac{dB}{dr}\right) > \frac{K}{1-r},$$

for some $K > 0$ and for r sufficiently near to 1. Hence

$$\lim\frac{dB}{dr} = +\infty.$$

It follows that $B(r)$ tends to its limit as a strictly increasing function. But, by (18) and § 5, Corollary 2, $\lim B(r) = 0$, so that $B(r) < 0$ and

$$\frac{d}{dr}\left(r\frac{dA}{dr}\right) > 0 \tag{19}$$

for r sufficiently near to 1. It follows, as in the proof of Theorem II, that, for such r,

$$\frac{A(r)}{\log(1/r)} - \frac{A(\rho)}{\log(1/\rho)} = \sigma\frac{dA(\sigma)}{d\sigma} - \rho\frac{dA(\rho)}{d\rho} \quad (r < \rho < \sigma < 1)$$

$$> 0.$$

To obtain a contradiction, and hence to prove the theorem, it is sufficient to show that

$$\frac{d}{dr}\left(\frac{A(r)}{\log(1/r)}\right) > 0 \quad (r \geqslant r_0).$$

We have

$$\frac{d}{dr}\left(\frac{rA(r)}{1-r^2}\right) = \frac{1}{\pi}\int_0^\pi F(t)\,\frac{1-r^2}{\delta^2}\,dt,$$

$$\frac{r}{1-r^2}\frac{d}{dr}\left(\frac{1-r^2}{r\log(1/r)}\right) = -\tfrac{1}{3}+O(1-r),$$

so that

$$\frac{1}{1-r^2}\frac{d}{dr}\left(\frac{A(r)}{\log(1/r)}\right)$$

$$= \frac{1}{1-r^2}\frac{d}{dr}\left(\frac{rA(r)}{1-r^2}\,\frac{1-r^2}{r\log(1/r)}\right)$$

$$= \frac{1-r^2}{r\log(1/r)}\,\frac{1}{\pi}\int_0^\pi \frac{F(t)}{\delta^2}\,dt - [\tfrac{1}{3}+O(1-r)]\,\frac{1}{\pi}\int_0^\pi \frac{F(t)}{\delta}\,dt.$$

Now

$$\frac{1-r^2}{r\log(1/r)} = 2+O(1-r),$$

and, by (18),

$$\lim(1-r)\int_0^\pi \frac{F(t)}{\delta}\,dt = 0, \quad \lim(1-r)\int_0^\pi \frac{F(t)}{\delta^2}\,dt = 0,$$

so that

$$\frac{1}{1-r^2}\frac{d}{dr}\left(\frac{A(r)}{\log(1/r)}\right) = \frac{2}{\pi}\int_0^\pi \frac{F(t)}{\delta^2}\,dt - \frac{1}{3\pi}\int_0^\pi \frac{F(t)}{\delta}\,dt + o(1),$$

which, since $\delta < 4$, exceeds a fixed positive number for r sufficiently near to 1.

10. THEOREM VIII. *If*

$$F(x) \sim C + \Sigma A_n(x)/n^4,$$

then, at a point x at which $D^2F(x)$ exists,

$$\bar{D}^4F(x) \geqslant \underline{P}(x), \quad \underline{D}^4F(x) \leqslant \bar{P}(x).$$

We may suppose that $x = 0$, $F(0) = 0$, $F(x) = F(-x)$. It is sufficient to prove the second inequality, and we may suppose that $\underline{D}^4F(0) > -\infty$. We need therefore only show that $\underline{\lim}\, 2[F(2h)-4F(h)]/h^4 > k$ implies that $\bar{P}(0) \geqslant k$, where k is a finite number.

Consider the even function $f(t)$ defined by

$$f(t) = \lambda t^2 + \mu \sin^4 t \quad (0 \leqslant t \leqslant \pi).$$

Then $D^2 f(0) = \lambda$, while

$$\lim 2\,[f(2t) - 4f(t)]/t^4 = 24\mu = P(0).$$

Thus, by adding to $F(x)$ a suitable $f(x)$, we see that it is sufficient to prove that the inequality

$$\lim\,[F(2h) - 4F(h)]/h^4 > 0$$

implies that $\bar{P}(0) \geqslant 0$, while we may suppose that

$$\lim F(h)/h^2 = 0. \tag{20}$$

As in the preceding proof, we may suppose that $F(t) \geqslant 0$ everywhere. We shall in particular take $F(t) = 0$ for $\frac{1}{2}\pi \leqslant t \leqslant \pi$, so that integrals involving $F(t)$ may be taken between the limits 0 and $\frac{1}{2}\pi$ instead of 0 and π. Finally, we may suppose that

$$\lim F(h)/h^5 = +\infty. \tag{21}$$

With the notation of the preceding proof, if $\bar{P}(0) < 0$, then

$$\frac{d}{dr}\left(r\frac{dB}{dr}\right) > 0 \quad (r \geqslant r_0). \tag{22}$$

Now
$$\frac{d}{dr}\left(\frac{B(r)}{\log(1/r)}\right) = \left[r\frac{dB}{dr} + \frac{B(r)}{\log(1/r)}\right] \Big/ \left(r \log\frac{1}{r}\right)$$

$$= \left[r\frac{dB}{dr} - \rho\frac{dB}{d\rho}\right] \Big/ \left(r \log\frac{1}{r}\right) \quad (r < \rho < 1),$$

and this is negative for $r \geqslant r_0$ by (22). Thus

$$\frac{d}{dr}\left(\frac{B(r)}{\log(1/r)}\right) < 0 \quad (r \geqslant r_0). \tag{23}$$

Since

$$r\frac{dA}{dr} = \frac{1}{\pi}\int_0^{\frac{1}{2}\pi} F(t)\left[-\frac{1+r^2}{\delta} + \frac{(1-r^2)^2}{\delta^2}\right] dt, \tag{24}$$

it follows by (20) that

$$\lim r\frac{dA}{dr} = -\frac{1}{\pi}\int_0^{\frac{1}{2}\pi} \frac{F(t)}{1-\cos t}\,dt. \tag{25}$$

We write $\lim r(dA/dr) = l$. Then

$$\frac{d}{dr}\left[\frac{r(dA/dr)-l}{\{\log(1/r)\}^2}\right] = \frac{d/dr\{r(dA/dr)\}}{\{\log(1/r)\}^2} + \frac{2\{r(dA/dr)-l\}}{r\{\log(1/r)\}^3}$$

$$= -\frac{B(r)}{r\{\log(1/r)\}^2} + \frac{2\,(d/d\rho)\{\rho(dA/d\rho)\}}{r\log(1/r)\,.\,2\log(1/\rho)\{-(1/\rho)\}} \qquad (r<\rho<1)$$

$$= \left[\frac{B(\rho)}{\log(1/\rho)} - \frac{B(r)}{\log(1/r)}\right] \Big/ r\log(1/r),$$

and this is negative for $r \geqslant r_0$ by (23). Thus

$$I(r) = \frac{d}{dr}\left[\frac{r(dA/dr)-l}{\{\log(1/r)\}^2}\right] < 0 \quad (r \geqslant r_0). \tag{26}$$

We shall obtain a contradiction, and so prove the theorem, if we show that $I(r) > 0$ for r sufficiently near to 1. We have

$$I(r) = \frac{d}{dr}\left[\frac{r^2\{r(dA/dr)-l\}}{(1-r^2)^2}\left(\frac{1-r^2}{r\log(1/r)}\right)^2\right]$$

$$= \frac{r^2\{r\,(dA/dr)-l\}}{(1-r^2)^2}\,\frac{2(1-r^2)}{r\log(1/r)}\,\frac{d}{dr}\left(\frac{1-r^2}{r\log(1/r)}\right) + \left(\frac{1-r^2}{r\log(1/r)}\right)^2 \frac{d}{dr}\left[\frac{r^2\{r(dA/dr)-l\}}{(1-r^2)^2}\right]$$

$$= I_1(r) + I_2(r).$$

By (24) and (25),

$$\frac{r^2\{r(dA/dr)-l\}}{(1-r^2)^2} = \frac{1}{\pi}\int_0^{\frac{1}{2}\pi} F(t)\left[\frac{r^2\cos t}{(1+r)^2\delta(1-\cos t)} + \frac{r^2}{\delta^2}\right]dt. \tag{27}$$

This is positive; so then is $\{r(dA/dr)-l\}/\{\log(1/r)\}^2$, an expression which, by (26), tends to a finite limit. Hence (27) tends to a finite limit, and so

$$\lim I_1(r)/(1-r^2)$$

is finite. We shall complete the proof by showing that

$$\lim I_2(r)/(1-r^2) = +\infty.$$

By (27),

$$\frac{d}{dr}\left[\frac{r^2\{(dA/dr)-l\}}{(1-r^2)^2}\right] = \frac{1}{\pi}\int_0^{\frac{1}{2}\pi} F(t)\left[\frac{\cos t}{1-\cos t}\left\{\frac{r}{(1+r)^2}\,\frac{1-r^2}{\delta^2} + \frac{r}{\delta}\,\frac{1-r}{(1+r)^3}\right\} + \frac{2r}{\delta}\,\frac{1-r^2}{\delta^2}\right]dt$$

$$> \frac{1}{\pi}\int_0^{\frac{1}{2}\pi} F(t)\,\frac{2r}{\delta}\,\frac{1-r^2}{\delta^2}\,dt.$$

On dividing by $1-r^2$, we obtain by (21) an expression with the property stated.

11. THEOREM IX. *If*

$$F(x) \sim C+\Sigma A_n(x)/n^4,$$

then, at a point x at which $D^2F(x)$ and $D^4F(x)$ exist,

$$P(x) = D^4F(x).$$

We may suppose that $x = 0$, $F(0) = 0$, $F(x) = F(-x)$, and that

$$D^2F(0) = D^4F(0) = 0.$$

Then
$$F(t) = o(t^2),$$

$$F(2t)-4F(t) = o(t^4),$$

whence, by the argument used in the proof of Theorem VI,

$$F(t) = o(t^4).$$

Now
$$P(r, 0) = \frac{1}{\pi}\int_0^\pi F(t)\frac{d^4P}{dt^4}\,dt,$$

and

$$\frac{d^4P}{dt^4} = -2r(1-r^2)\left[\frac{\cos t}{\delta^2} - \frac{4r\sin^2 t}{\delta^3} - \frac{12r\cos 2t}{\delta^3} + \frac{72r^2\sin 2t\sin t}{\delta^4} - \frac{192r^3\sin^4 t}{\delta^5}\right].$$

Hence

$$P(r, 0) = o\left[(1-r^2)\left\{\int_0^\pi\left(\frac{t^4}{\delta^2}+\frac{t^6}{\delta^3}+\frac{t^4}{\delta^3}+\frac{t^6}{\delta^4}+\frac{t^8}{\delta^5}\right)dt\right\}\right]$$

$$= o(1).$$

ON THE THEORY OF TRIGONOMETRIC SERIES (II)

By S. VERBLUNSKY.

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1. Consider a trigonometric series

$$\sum_{n=1}^{\infty} A_n(x), \tag{1}$$

where $A_n(x) = a_n \cos nx + b_n \sin nx$. We write

$$P(r, x) = \Sigma A_n(x) r^n,$$

where $0 < r < 1$, and

$$\bar{P}(x) = \overline{\lim_{r \to 1}} P(r, x), \quad \underline{P}(x) = \underline{\lim_{r \to 1}} P(r, x).$$

The series $$\Sigma n \sin nx,$$

for which $\bar{P}(x) = \underline{P}(x) = 0$, shows that no restrictions on the functions $\bar{P}(x)$, $\underline{P}(x)$ alone will enable us to infer that (1) is a Fourier series. We must introduce conditions of a different nature. These are of three kinds, namely, (*a*) restrictions on the order of magnitude of the coefficients of (1), (*b*) restrictions on the rate of increase of $P(r, x)$ as $r \to 1$, and (*c*) restrictions on the class of series to which (1) belongs. An example of the third kind of condition is that $-\Sigma A_n(x)/n^2$ should be the Fourier series of a continuous function. This particular condition we call condition (F).

Concerning trigonometric series which satisfy the condition (F), it is known that, if $\bar{P}(x)$, $\underline{P}(x)$ are finite, and $\underline{P}(x) \geqslant \psi(x)$ where $\psi(x)$ is integrable, then (1) is a Fourier series. This result is due to Zygmund†, who conjectured that the theorem remains true if the inequality is replaced by $\bar{P}(x) \geqslant \psi(x)$.

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† Zygmund, *Math. Zeitschrift*, 25 (1926), 274–290.

Not only is this conjecture correct, but also the unexpected proposition that the condition "$\bar{P}(x)$ finite" is altogether superfluous. The condition (F) may also be replaced by a condition of type (a); namely that

$$a_n = o(n), \quad b_n = o(n). \tag{2}$$

The condition (F) and the condition (2) may be replaced by less restrictive ones. And, as will be anticipated, these new conditions cannot have the same simplicity as the old ones. They must involve subtle distinctions. Were we to investigate this matter on the assumption that $\underline{P}(x)$ is finite and $\bar{P}(x) \geqslant \psi(x)$, our garden of subtle distinctions would become a jungle of complication into which we could not expect the reader to follow us, had we the recklessness to enter ourselves. It is, therefore, proper to suppose that $\bar{P}(x)$, $\underline{P}(x)$ behave in a simpler way; for example, that $\bar{P}(x) = \underline{P}(x) = P(x)$ is finite and integrable.

Of the widened conditions which we obtain, let us quote the simplest consequence. The condition (2) may be replaced by $a_n = O(n)$, $b_n = O(n)$, together with

$$\lim_{r\to 1}(1-r)\Sigma \frac{B_n(x)}{n} r^n = 0,$$

where $B_n(x) = b_n \cos nx - a_n \sin nx$; a condition which is satisfied trivially when (2) holds.

We establish a series of theorems which are interesting applications of the theorem of uniqueness when (2) holds. The most striking of these is the second, namely if $a_n = o(n^3)$ and

$$\lim_{r\to 1}\Sigma a_n \left(\frac{\sin n\theta}{n\theta}\right)^2 r^n$$

tends to zero with θ, then

$$\lim_{r\to 1}\Sigma a_n r^n = 0.$$

2. We write

$$\Delta^2 F(x, h) = F(x+h)+F(x-h)-2F(x),$$

$$\bar{D}^2 F(x) = \overline{\lim_{h\to 0}} \Delta^2 F(x, h)/h, \quad \underline{D}^2 F(x) = \underline{\lim_{h\to 0}} \Delta^2 F(x, h)/h,$$

and

$$D^2 F(x) = \lim \Delta^2 F(x, h)/h$$

when the limit exists.

In this section we give nine lemmas which are required to prove the first theorem.

LEMMA 1 (Rajchman and Zygmund)†. *If*

$$F(x) \sim -\Sigma A_n(x)/n^2$$

and this series is summable by Poisson's method at a point x, then

$$\bar{D}^2 F(x) \geqslant \underline{P}(x), \quad \underline{D}^2 F(x) \leqslant \bar{P}(x).$$

LEMMA 2‡. *Under the conditions of Lemma* 1,

$$\overline{\lim}\, \Delta^2 F(x, h)/h \geqslant \pi \underline{\lim}\, (1-r) P(r, x),$$

$$\underline{\lim}\, \Delta^2 F(x, h)/h \leqslant \pi \overline{\lim}\, (1-r) P(r, x).$$

LEMMA 3 (de la Vallée Poussin)§. *Let $F(x)$ be continuous for $a \leqslant x \leqslant b$. Let $\bar{D}^2 F(x) \geqslant 0$ for all x satisfying $a < x < b$, except possibly at the points of an enumerable set E. Let the condition*

$$\overline{\lim}\, \Delta^2 F(x, h)/h \geqslant 0 \geqslant \underline{\lim}\, \Delta^2 F(x, h)/h \tag{3}$$

be satisfied at the points of E. Then, for $a \leqslant x < y < z \leqslant b$,

$$\frac{F(x)-F(y)}{x-y} \leqslant \frac{F(y)-F(z)}{y-z}.$$

LEMMA 4 (Steinhaus). *If $F(x)$ satisfies the conditions of Lemma* 3, *then $F(x)$ has increasing derivatives, $F''(x)$ exists almost everywhere in (a, b) and $F''(x)$ is integrable (L) in $(a+\epsilon, b-\epsilon)$ for $0 < \epsilon < (b-a)/2$.*

By Lemma 3, we have, for $a \leqslant x < y < z < w \leqslant b$,

$$\frac{F(x)-F(y)}{x-y} \leqslant \frac{F(z)-F(w)}{z-w}.$$

Letting $y \to x$, $z \to w$ in a suitable manner, we see that $D^+F(x)$, for example, is monotone increasing for $a \leqslant x < b$.

LEMMA 5 (de la Vallée Poussin)‖. *Let $F(x)$ be continuous for*

$$a \leqslant x \leqslant b.$$

Let

$$\bar{D}^2 F(x) \geqslant f(x) \geqslant \underline{D}^2 F(x) \quad (a < x < b),$$

where $f(x)$ is integrable (D), i.e. in the sense of Denjoy-Perron, in (a, b), and finite except possibly at the points of an enumerable set E. Let the

† Verblunsky, "On the theory of trigonometric series (I)", *supra*, 441–456 (Theorem II); I shall refer to this paper as I.

‡ I, Theorem I.

§ de la Vallée Poussin, *Bull. de l'Acad. Roy. de Belg.* (1912), 701–707.

‖ de la Vallée Poussin, *loc. cit.*

condition (3) *be satisfied at the points of* E. *Then*

$$F(x) = \int_a^x dy \int_a^y f(t)\,dt + Ax + B \quad (a \leqslant x \leqslant b).$$

LEMMA 6 (Fatou)†. *Let*

$$F(x) \sim -\Sigma A_n(x)/n^2.$$

If, at a point x, $D^2 F(x)$ *exists, then* $D^2 F(x) = P(x)$.

LEMMA 7 (Grosz)‡. *Let*

$$\phi(x) \sim -\Sigma B_n(x)/n.$$

Then $\bar{P}(x)$, $\underline{P}(x)$ *lie between the upper and lower limits, as* $t \to 0$, *of*

$$[\phi(x+t) - \phi(x-t)]/2t.$$

LEMMA 8. *Let* $\underline{P}(x) > -\infty$ *at the points of a perfect set* Π *with the possible exception of an enumerable set. Then there is a portion* ϖ *of* Π *and a number* $K = K(\varpi)$ *such that, at the points of* ϖ, $P(r, x) \geqslant K$ *for* $0 < r < 1$, *and consequently* $\underline{P}(x) \geqslant K$.

In this lemma it is assumed that $P(r, x)$ is continuous in (r, x) for $0 \leqslant r < 1$ and all x. Let (ϵ_ν) be a monotone diminishing sequence which tends to zero. Let (r_ν) be a monotone increasing sequence of positive numbers which tends to 1. For $r_\nu \leqslant r \leqslant r_{\nu+1}$, $P(r, x)$ is continuous in (r, x), We can find a finite set of numbers $r_{\nu,i}$ $(i = 1, 2, \ldots, n_\nu)$ satisfying

$$r_\nu = r_{\nu,1} < r_{\nu,2} < \ldots < r_{\nu,n_\nu} = r_{\nu+1}$$

such that, for any fixed x, the oscillation of $P(r, x)$ in $(r_{\nu,i}, r_{\nu,i+1})$ is less than ϵ_ν. If we denote the aggregate of numbers $(r_{\nu,i})$ in ascending order of magnitude by (r_n), it follows that, for all x,

$$\overline{\lim_{r \to 1}} P(r, x) = \underline{\lim_{n \to \infty}} P(r_n, x).$$

Let k denote an integer. Let $G_n(k) = \Pi\{P(r_n, x) \geqslant k\}$, the latter symbol denoting the set of points of Π at which the inequality is satisfied. Then $G_n(k)$ is closed. Let E denote the enumerable set at which $\underline{\lim} P(r_n, x) = -\infty$. We write E as $E \equiv (x_1, x_2, \ldots)$. Let

$$F_n(k) = G_n(k) \,.\, G_{n+1}(k) \ldots.$$

† I, Theorem 3, Corollary 2.
‡ I, Theorem 3, Corollary 4.

Then $F_n(k)$ is closed. At a point of $\Pi - E$, $\underline{\lim}\, P(r_n, x) > -\infty$, and hence there is an integer $k+1$ such that $\underline{\lim}\, P(r_n, x) > k+1$. There is an $n(k)$ such that, for $n \geqslant n(k)$, $P(r_n, x) \geqslant k$, and so x belongs to $F_n(k)$ for $n \geqslant n(k)$. The sets $F_n(k)$ for variable n and k can be arranged in a sequence $F_1, F_2, \ldots$ of closed sets, and every point of $\Pi - E$ belongs to at least one of them. But $H_n = F_n + x_n$ is a closed set. Hence $\Pi = \Sigma H_n$. It follows that there is a portion of Π contained in one of the H_n, say in H_m. Hence there is a portion ϖ of Π contained in F_m. On this portion we have

$$P(r_n, x) \geqslant N, \quad N = N(m), \quad n \geqslant n(m).$$

Since the oscillation of $P(r, x)$ in (r_n, r_{n+1}) tends to zero as $n \to \infty$, we have

$$P(r, x) \geqslant M \quad (r \geqslant r_n)$$

for x in ϖ. Finally, since $P(r, x)$ is uniformly bounded for $0 \leqslant r \leqslant r_n$, we have

$$P(r, x) \geqslant K \quad (0 \leqslant r < 1)$$

for x in ϖ.

LEMMA 9. *Let*

$$F(x) \sim -\Sigma A_n(x)/n^2.$$

Let $F(x)$ be continuous in the interval (a, b), and let $\underline{P}(x)$ be finite in this interval except possibly at the points of an enumerable set E. Let $\underline{P}(x) \geqslant \phi(x)$, where $\phi(x)$ is integrable (D) in (a, b). Let

$$\lim (1-r) P(r, x) = 0 \tag{4}$$

at the points of E. Then $D^2 F(x)$ exists almost everywhere in (a, b).

We may suppose that $\phi(x)$ is finite except at the points of E. Let $m(x)$ be a minor function of $\phi(x)$ in (a, b), and let

$$\mu(x) = \int_a^x m(t)\, dt.$$

Let $G(x) = F(x) - \mu(x)$. By Lemma 1,

$$\bar{D}^2 F(x) \geqslant \underline{P}(x) \geqslant \phi(x) \quad (a < x < b).$$

For x not belonging to E, $\bar{D}^2 \mu(x) \leqslant \phi(x)$. Hence, for such x,

$$\bar{D}^2 G(x) \geqslant \bar{D}^2 F(x) - \bar{D}^2 \mu(x) \geqslant 0.$$

For x belonging to E, we have, by (4) and Lemma 2,

$$\overline{\lim}\, \Delta^2 F(x, h)/h \geqslant 0 \geqslant \underline{\lim}\, \Delta^2 F(x, h)/h.$$

Since $m(x)$ is continuous,

$$\lim \Delta^2 \mu(x, h)/h = 0.$$

Hence

$$\overline{\lim}\, \Delta^2 G(x, h)/h \geqslant 0 \geqslant \underline{\lim}\, \Delta^2 G(x, h)/h$$

for x belonging to E. By Lemma 4, $G''(x)$ exists almost everywhere in (a, b). But $\mu''(x)$ exists almost everywhere in (a, b); hence $F''(x)$ and $D^2 F(x)$ also exist almost everywere in (a, b).

3. THEOREM I. *Let*

$$F(x) \sim -\Sigma A_n(x)/n^2,$$

where $F(x)$ is continuous. Let $\underline{P}(x)$ be finite except possibly at an enumerable set E. Let $\bar{P}(x) \geqslant \psi(x)$ where $\psi(x)$ is integrable (D) in $(0, 2\pi)$. Let

$$\lim (1-r) P(r, x) = 0$$

at the points of E. Then (1) is a Fourier series.

We shall find it convenient to use the abbreviation "p.p." for "almost everywhere". We may suppose that $\psi(x)$ is finite except possibly at the points of E. By Lemma 8 there is a non-dense closed set F_1 with the following property. If δ denotes an interval contiguous to F_1, and d denotes an interval completely interior to δ, there is a constant $K = K(d)$ such that, for $x \subset d$, $\underline{P}(x) \geqslant K$. By Lemma 9, $D^2 F(x)$ exists p.p. in d, and therefore p.p. in δ. By Lemma 6, $\underline{P}(x) = \bar{P}(x) \geqslant \psi(x)$ p.p. in δ. Thus p.p. in CF_1, $\underline{P}(x) \geqslant \psi(x)$, while in CF_1, $\underline{P}(x)$ is finite except possibly at an enumerable set at which (4) holds. Let Π_1 be the perfect kernel of F_1. Since $F_1 - \Pi_1$ is enumerable, it follows that, in $C\Pi_1$, $\underline{P}(x)$ has the properties just mentioned.

By Lemma 8 there is a closed set $F_2 \subset \Pi_1$ and non-dense in Π_1, with the following property. If δ denotes an interval contiguous to F_2 and d denotes an interval completely interior to δ and containing points of Π_1, then there is a constant $K = K(d)$ such that, for $x \subset \Pi_1 d$, $\underline{P}(x) \geqslant K$. We thus have in the interval d,

$$\underline{P}(x) \geqslant K \quad (x \subset \Pi_1 d),$$

$$\underline{P}(x) \geqslant \psi(x) \quad (\text{p.p. for } x \subset C \,.\, \Pi_1 d),$$

and $\underline{P}(x)$ is finite except at an enumerable set at which (4) holds. By Lemma 9, $D^2 F(x)$ exists p.p. in d, and hence p.p. in δ. By Lemma 6, $\underline{P}(x) = \bar{P}(x) \geqslant \psi(x)$ p.p. in δ. Thus p.p. in CF_2, $\underline{P}(x) \geqslant \psi(x)$, while in CF_2, $\underline{P}(x)$ is finite except possibly at an enumerable set at which (4) holds.

Let Π_2 be the perfect kernel of F_2. Since $F_2 - \Pi_2$ is enumerable, it follows that, in $C\Pi_2$, $\underline{P}(x)$ has the properties just mentioned.

It is clear how the argument is completed by transfinite induction to give the conclusion that $\underline{P}(x) \geqslant \psi(x)$ p.p. We may replace $\psi(x)$ by an equivalent function which is finite except possibly at the points of E, and such that the inequality is satisfied everywhere. We shall alter the notation and denote this equivalent function by $\psi(x)$.

We consider an arbitrarily assigned interval (a, b). Let $m(x)$ denote a minor function of $\psi(x)$ for this interval. Let

$$\mu(x) = \int_a^x m(t)\,dt,$$

and $G(x) = F(x) - \mu(x)$. Then, as in the proof of Lemma 9, we have for $x \subset CE$, $\bar{D}^2 G(x) \geqslant 0$, while for $x \subset E$,

$$\overline{\lim}\, \Delta^2 G(x, h)/h \geqslant 0 \geqslant \underline{\lim}\, \Delta^2 G(x, h)/h.$$

By Lemma 4, $D^+F(x) - m(x)$ is monotone increasing. Let $m(x)$ tend to

$$\int_a^x \psi(t)\,dt$$

for $a \leqslant x \leqslant b$. It follows that

$$D^+F(x) - \int_a^x \psi(t)\,dt$$

is monotone increasing. If $\lambda(x)$ denotes a monotone increasing function, we can write

$$D^+F(x) = \int_a^x \psi(t)\,dt + \lambda(x). \tag{5}$$

It follows that $-\Sigma B_n(x)/n$ is the Fourier series of $D^+F(x)$; and this function can only have ordinary discontinuities. We show that such discontinuities cannot occur. Suppose, on the contrary, that ξ is a point of discontinuity. If $\xi \subset CE$, $\underline{P}(x)$ is finite and, by Lemma 7,

$$\overline{\lim_{h \to 0}} \frac{D^+F(\xi+h) - D^+F(\xi-h)}{2h} \geqslant \underline{P}(\xi) \geqslant \underline{\lim_{h \to 0}} \frac{D^+F(\xi+h) - D^+F(\xi-h)}{2h};$$

this shows that $\lim D^+F(\xi+h) = \lim D^+F(\xi-h)$. If $\xi \subset E$, we have, by (4) and Lemma 1,

$$\overline{\lim}\, \Delta^2 F(\xi, h)/h \geqslant 0 \geqslant \lim \Delta^2 F(\xi, h)/h;$$

on the other hand,

$$\lim \Delta^2 F(\xi, h)/h = \lim [D^+F(\xi+h) - D^+F(\xi-h)],$$

so that the second member is zero. Thus $D^+F(x) = F'(x)$ is continuous, and we may write (5) in the form

$$F'(x) = \int_a^x \psi(t)\,dt + \lambda(x), \tag{6}$$

where $\lambda(x)$ is continuous and monotone increasing. From this equation it follows that $D^2F(x)$ exists p.p. and is integrable, and so by Lemma 6 that $\underline{P}(x)$ is integrable. We may therefore replace $\psi(x)$ by $\underline{P}(x)$ in the preceding equation. In that equation, $\lambda(x)$ is to be replaced by another continuous monotone increasing function. We shall alter the notation and denote this new function by $\lambda(x)$. We thus obtain

$$F'(x) = \int_a^x \underline{P}(t)\,dt + \lambda(x).$$

Let $M(x)$ denote a major function of $\underline{P}(x)$ for the interval (a, b). Then, since by Lemma 7

$$\underline{\lim} \frac{F'(x+h) - F'(x-h)}{2h} \leqslant \underline{P}(x),$$

on writing $H(x) = F'(x) - M(x)$, we have the relation

$$\underline{\lim} \frac{H(x+h) - H(x-h)}{2h} \leqslant 0, \tag{7}$$

except possibly at the points of the enumerable set E. We can write

$$M(x) = \int_a^x \underline{P}(t)\,dt + \sigma(x),$$

where $\sigma(x)$ is continuous and monotone increasing. Hence

$$H(x) = \lambda(x) - \sigma(x) \tag{8}$$

is continuous and of bounded variation. It follows by (7) that $H(x)$ is monotone diminishing. Let now $M(x)$ tend to

$$\int_a^x \underline{P}(t)\,dt$$

for $a \leqslant x \leqslant b$. Then $\sigma(x)$ tends to zero in this interval, and it follows from (8) that $\lambda(x)$ is monotone diminishing. Since $\lambda(x)$ denotes a monotone

increasing function, it follows that $\lambda(x) = C$, a constant. The equation

$$F'(x) = \int_a^x \underline{P}(t)\,dt + C$$

proves the theorem.

4. In order to prove the second theorem, we require five additional lemmas.

LEMMA 10. *Let*

$$F(x) \sim -\Sigma A_n(x)/n^2,$$

where $F(x)$ is continuous in (a, b). Let $\underline{P}(x)$ be finite in this interval except possibly at the points of an enumerable set E at which

$$\lim (1-r)\,P(r, x) = 0 \tag{9}$$

is satisfied. Let $\overline{P}(x) \geqslant \psi(x)$, where $\psi(x)$ is integrable D in (a, b). Then

$$F(x) = \int_a^x dy \int_a^y \{f(t)+\psi(t)\}\,dt + Ax + B \quad (\alpha < x < \beta),$$

where $f(t) \geqslant 0$ and $a < \alpha < \beta < b$.

The proof of Theorem I establishes the relation

$$F(x) = \int_a^x dy \int_a^y \underline{P}(t)\,dt + Ax + B \quad (\alpha < x < \beta)$$

together with $\underline{P}(x) = \overline{P}(x) \geqslant \psi(x)$ p.p. in (a, b).

LEMMA 11. *Let Π be a perfect set. Let a sequence $\{f_n(x)\}$ of functions continuous on Π converge in the extended sense on Π to a function $f(x)$ which at any point of Π satisfies either $-\infty < f(x) < +\infty$, or else $f(x) = +\infty$. Let the sequence possess the property that, given $\epsilon > 0$, there is an $n(\epsilon)$ such that*

$$f(x) > f_n(x) - \epsilon \quad [n \geqslant n(\epsilon),\ x \subset \Pi].$$

Then $f(x)$ is lower semi-continuous on Π.

By saying that $f(x)$ is lower semi-continuous on Π we mean that, if $\xi \subset \Pi$ and (i) $f(\xi)$ is finite, then $f(\xi) \leqslant \underline{\lim}_{x \to \xi} f(x)$ $(x \subset \Pi)$, and (ii) if $f(\xi) = +\infty$, then $\underline{\lim}_{x \to \xi} f(x) = +\infty$ $(x \subset \Pi)$.

To prove the lemma, consider a point $\xi \subset \Pi$. Then, if $f(\xi)$ is finite and $f(x)$ is not lower semi-continuous at ξ, we can find a sequence

$x_r \to \xi$, $x_r \subset \Pi$, and an $\epsilon > 0$ such that $f(x_r) < f(\xi) - 2\epsilon$ for all r. By hypothesis,

$$f_n(x_r) < f(x_r) + \epsilon \quad [n \geqslant n(\epsilon)]$$

for all r. Hence

$$f_n(x_r) < f(\xi) - \epsilon \quad [n \geqslant n(\epsilon)].$$

Let $r \to \infty$. Since $f_n(x)$ is continuous on Π,

$$f_n(\xi) \leqslant f(\xi) - \epsilon,$$

and, letting $n \to \infty$, we get $f(\xi) \leqslant f(\xi) - \epsilon$, which is a contradiction.

If $f(\xi) = +\infty$ and $f(x)$ is not lower semi-continuous at ξ, we can find a sequence $x_r \to \xi$ and a constant K such that $f(x_r) < K$ for all r. Since

$$f_n(x_r) < f(x_r) + \epsilon \quad [n \geqslant n(\epsilon)],$$

we get $f_n(x_r) < K + \epsilon$ for $n \geqslant n(\epsilon)$ and all r. Letting $r \to \infty$ and then letting $n \to \infty$, we find that $f(\xi) \leqslant K + \epsilon$, again a contradiction.

The result of the lemma remains true for a family of functions depending on a continuous parameter.

LEMMA 12. *If, at the points of a perfect set* Π, $\underline{P}(x) > -\infty$ *except possibly at an enumerable set* E, *then there is a portion* ϖ *of* Π *on which the unique limit* (*finite or infinite*)

$$\lim_{r \to 1} -\Sigma \frac{A_n(x)}{n^2} r^n$$

is upper semi-continuous.

Since

$$\Sigma \frac{A_n(x)}{n} r^n = \int_0^r P(\rho, x) \frac{d\rho}{\rho},$$

it follows that, except possibly at E,

$$Q(x) = \lim_{r \to 1} \Sigma \frac{A_n(x)}{n} r^n$$

has a unique value, and $Q(x) > -\infty$. Writing

$$Q(r, x) = \Sigma \frac{A_n(x)}{n} r^{n-1},$$

we have $Q(x) = \lim_{r \to 1} Q(r, x)$ for $x \subset CE$. By Lemma 8 there is a portion ϖ of Π and a constant $K = K(\varpi)$ such that $Q(r, x) \geqslant K$ for $x \subset \varpi$. The function

$$-F(x) = \lim_{r \to 1} \Sigma \frac{A_n(x)}{n^2} r^n$$

has a unique value for $x \subset \varpi$ and $-F(x) > -\infty$. We have, for such x,

$$-F(x)-\Sigma \frac{A_n(x)}{n^2} r^n = \lim_{\rho \to 1} \int_r^\rho Q(\sigma, x)\, d\sigma$$

$$\geqslant (1-r) K$$

$$> -\epsilon \quad [r \geqslant r(\epsilon, K)].$$

By Lemma 11, $-F(x)$ is lower semi-continuous.

LEMMA 13. *Let* $a_n = o(n)$, $b_n = o(n)$, *and*

$$F(x) \sim -\Sigma A_n(x)/n^2.$$

If, at a point x, $\lim [F(x+h)-F(x-h)] = l$ *has a unique value (finite or infinite), then* $l = 0$.

We have

$$(1-r)\Sigma \frac{B_n(x)}{n} r^n = \frac{1-r}{2\pi} \int_0^\pi [F(x+t)-F(x-t)] \frac{(1-r^2)\, 2r \sin t}{(1-2r\cos t+r^2)^2}\, dt.$$

The limit of the second member is l/π, and that of the first member is zero.

LEMMA 14. *Let* $\Phi(x)$ *be a function which, for each* x *satisfying* $a \leqslant x \leqslant b$, *has a unique value which is finite or* $-\infty$. *Let* $\Phi(x)$ *be upper semi-continuous in* (a, b) *and* $\bar{D}^2\Phi(x) \geqslant 0$ *for* $a < x < b$, *except possibly at the points of an enumerable set* E *at which*

$$\overline{\lim}\, \Delta^2\Phi(x, h)/h \geqslant 0 \geqslant \underline{\lim}\, \Delta^2\Phi(x, h)/h.$$

If α, β *are two points satisfying* $a < \alpha < \beta < b$, *and such that* $\Phi(\alpha)$, $\Phi(\beta)$ *are finite, then*

$$\Phi(x) \leqslant \Phi(\alpha)+\frac{x-\alpha}{\beta-\alpha}\{\Phi(\beta)-\Phi(\alpha)\} \quad (\alpha < x < \beta).$$

The enunciation of the lemma implies that, whenever $a < x < b$, we can form the number $\Delta^2\Phi(x, h)$ for some sequence of h which tends to zero. Values of h for which $\Delta^2\Phi(x, h)$ has no meaning are to be neglected in forming the upper and lower limits in the enunciation.

Suppose that there is a value of x in (α, β) for which

$$\Phi(x) > \Phi(\alpha)+\frac{x-\alpha}{\beta-\alpha}\{\Phi(\beta)-\Phi(\alpha)\}.$$

For suitable $\epsilon > 0$, $G(x) = \Phi(x) + \epsilon(x-a)^2$ satisfies the conditions of the lemma with $\bar{D}^2 G(x) > 0$ except possibly at E; and there is a value of x for which

$$G(x) > G(a) + \frac{x-a}{\beta-a}\{G(\beta)-G(a)\}.$$

If k denotes a number sufficiently near to and greater than

$$\{G(\beta)-G(a)\}/(\beta-a),$$

then there are values of x in (a, β) for which $\phi_k(x) = G(x)-G(a)-k(x-a)$ is positive. Since $\phi_k(x)$ is upper semi-continuous, there is a point $\xi(k)$ at which it attains an absolute maximum. This point is neither a nor β, since $\phi_k(a) = 0$, $\phi_k(\beta) < 0$. Hence $a < \xi(k) < \beta$, and $\bar{D}^2 \phi_k\{\xi(k)\} \leqslant 0$. It follows that $\xi(k) \subset E$. Also $\Delta^2 \phi_k\{\xi(k), h\}/h \leqslant 0$ for sufficiently small h, and therefore $\Delta^2 G\{\xi(k), h\}/h \leqslant 0$ for such h. But

$$\overline{\lim}\, \Delta^2 G\{\xi(k), h\}/h \geqslant 0 \geqslant \underline{\lim}\, \Delta^2 G\{\xi(k), h)/h.$$

Hence $\lim \Delta^2 G\{\xi(k), h\}/h = 0$, and so $\lim \Delta^2 \phi_k\{\xi(k), h\}/h = 0$. Since $\xi(k)$ is a maximum of $\phi_k(x)$, this implies that $\phi_k'\{\xi(k)\} = 0$ and therefore that $G'\{\xi(k)\} = k$. Hence for different k we have different $\xi(k)$. Since k can take an unenumerable set of values, it would follow that E is unenumerable.

5. THEOREM II. *Let $a_n = o(n)$, $b_n = o(n)$. Let $\underline{P}(x)$ be finite except possibly at the points of an enumerable set E at which*

$$\lim (1-r) P(r, x) = 0.$$

Let $\bar{P}(x) \geqslant \psi(x)$, where $\psi(x)$ is integrable D in $(0, 2\pi)$. Then (1) is a Fourier series.

We shall find it convenient to use the abbreviation "u.s.c." for "upper semi-continuous."

For $x \subset CE$,

$$-\lim_{r\to 1} \Sigma \frac{A_n(x)}{n^2} r^n \tag{10}$$

has a unique value which we denote by $F(x)$, and which satisfies $F(x) < +\infty$. For $x \subset E$ we have

$$P(r, x) = o\left(\frac{1}{1-r}\right), \quad \Sigma \frac{A_n(x)}{n} r^{n-1} = o\left(\log \frac{1}{1-r}\right),$$

and (10) has a finite value which we denote by $F(x)$. Further, the series $-\Sigma A_n(x)/n^2$ is the Fourier series of $F(x)$ which is finite p.p.

By Lemmas 8 and 12 there is a non-dense closed set F_1 with the following property. If δ denotes an interval contiguous to F_1, and d denotes an interval completely interior to δ, then there is a constant $K = K(d)$ such that $\underline{P}(x) > K$ for $x \subset d$, and $F(x)$ is u.s.c. in d. We prove that $F(x)$ is continuous in d. Write $\Phi(x) = F(x) - Kx^2/2$, and let S be the set of points in d at which $F(x)$, and therefore $\Phi(x)$, is finite. Then, if α and $\beta > \alpha$ are two points of S, the curve $y = \Phi(x)$ lies below the chord

$$y = \Phi(\alpha) + \frac{x-\alpha}{\beta-\alpha}\{\Phi(\beta) - \Phi(\alpha)\}$$

in (α, β). To see this we observe that, by Lemma 1, at a point at which $F(x)$ is finite, $\bar{D}^2 F(x) > K$, $\bar{D}^2 \Phi(x) > 0$; while, at a point at which $F(x) = -\infty$, $\bar{D}^2 F(x) = +\infty$ and $\bar{D}^2 \Phi(x) = +\infty$. Our assertion concerning $\Phi(x)$ is thus a consequence of Lemma 14.

We can show that $\Phi(x)$ is finite in d. For let ξ be a point of CS and let η be a point of S. We can choose η such that $\frac{1}{2}(\xi+\eta) \subset S$. Since $\xi \subset CS$, $\Phi(\xi) = -\infty$. Since $\Phi(x)$ is u.s.c.,

$$\lim_{x \to \xi} \Phi(x) = -\infty.$$

We can find a sequence $x_n \to \xi$ such that $x_n \subset S$. Then $\lim_{n\to\infty} \Phi(x_n) = -\infty$. Since $x_n \subset S$, $\eta \subset S$, we have, by what has just been proved,

$$\Phi\{\tfrac{1}{2}(x_n+\eta)\} \leqslant \tfrac{1}{2}[\Phi(x_n)+\Phi(\eta)].$$

Hence
$$\lim_{n\to\infty} \Phi\{\tfrac{1}{2}(x_n+\eta)\} = -\infty.$$

The sequence (x_n) may be chosen from all points of S in a neighbourhood of ξ. Hence the points $\frac{1}{2}(x_n+\eta)$ can coincide with any point in a neighbourhood of $\frac{1}{2}(\xi+\eta)$, with the exception of a set of zero measure. Thus, if we neglect a set of zero measure, we can write

$$\lim_{x \to \frac{1}{2}(\xi+\eta)} \Phi(x) = -\infty,$$

and, with the same convention,

$$\lim_{x \to \frac{1}{2}(\xi+\eta)} F(x) = -\infty.$$

This implies that the Fourier series of $F(x)$ is summable by Poisson's method to $-\infty$ at $x = \frac{1}{2}(\xi+\eta)$, *i.e.* by definition, $F\{\frac{1}{2}(\xi+\eta)\} = -\infty$, which contradicts $\frac{1}{2}(\xi+\eta) \subset S$.

Thus $\Phi(x)$ is finite and convex in d; it is also measurable, and therefore continuous; so then is $F(x)$. Since d may be any interval interior to δ, it follows that $F(x)$ is continuous in the open contiguous intervals of F_1.

We next show that $F(x)$ is continuous in the open intervals contiguous to the perfect kernel Π_1 of F_1. Let ξ be an isolated point of F_1. Then there are two intervals (α, ξ), (ξ, β) such that ξ is the only possible discontinuity of $F(x)$ in (α, β). Let (γ, δ) denote an interval interior to (α, ξ). By Lemma 10, $D^2 F(x)$ exists p.p. in (γ, δ). By Lemma 6, $\underline{P}(x) = \bar{P}(x) \geqslant \psi(x)$ p.p. in (γ, δ) and so p.p. in (α, ξ). Similarly, $P(x) \geqslant \psi(x)$ p.p. in (ξ, β). We can therefore find a function $\psi^*(x)$ which is equivalent to $\psi(x)$, which is finite except possibly at the points of E [in (α, β)], and such that $\underline{P}(x) \geqslant \psi^*(x)$ in (α, β). Let $m(x)$ denote a minor function of $\psi^*(x)$ in (α, β), and let

$$\mu(x) = \int_a^x m(t)\, dt.$$

Let $G(x) = F(x) - \mu(x)$. Then, for $x \subset CE$, $\bar{D}^2 G(x) \geqslant 0$, while, for $x \subset E$,

$$\overline{\lim}\, \Delta^2 G(x, h)/h \geqslant 0 \geqslant \underline{\lim}\, \Delta^2 G(x, h)/h,$$

by (4) and Lemma 2. By Lemma 4, $G(x)$ has increasing derivatives in $\alpha < x < \xi$, and therefore $G(x)$ tends, as $x \to \xi$, either to a finite limit or to $+\infty$. Similarly, as $x > \xi$ tends to ξ, $G(x)$ tends to a finite limit or to $+\infty$.

If $G(\xi - 0) = G(\xi + 0) = +\infty$, then, since $F(\xi - 0) = F(\xi + 0) = +\infty$, $F(x)$, which is the Poisson sum of $-\Sigma A_n(x)/n^2$, must satisfy $F(\xi) = +\infty$, which is impossible since, as we saw at the beginning of the proof, $F(x) < +\infty$ for all x. Hence $F(\xi + 0) - F(\xi - 0)$ has a unique value finite or infinite. By Lemma 13, $F(\xi + 0) - F(\xi - 0) = 0$. Hence $F(x)$ is continuous in (α, β).

Let now ξ denote an isolated point of the derivative of $F_1 - \Pi_1$. Then we can find two intervals (α, ξ), (ξ, β) such that ξ is the only possible discontinuity of $F(x)$ in (α, β). It is clear that the preceding argument may be repeated. By transfinite induction we infer that $F(x)$ is continuous in the open intervals contiguous to Π_1.

By Lemmas 8 and 12 there is a closed set F_2 non-dense in Π_1 with the following property. If δ is an interval contiguous to F_2 and d is an interval interior to δ and containing points of Π_1, then to the portion $\varpi = \Pi_1 d$ there corresponds a number $K = K(\varpi)$ such that $\underline{P}(x) > K$ for $x \subset \varpi$, and $F(x)$ is u.s.c. on ϖ. Let a, b denote the extreme points of ϖ. We proceed to show that $F(x)$ is u.s.c. in (a, b). Let (α, β) denote an interval contiguous to ϖ and contained in (a, b). By Lemma 10, if γ is any number satisfying $\alpha < \gamma < \beta$, we have the relation

$$F(x) = \int_\gamma^x dy \int_\gamma^y \{f(t) + \psi(t)\}\, dt + Ax + B \quad (\gamma < x < \beta)$$

with $f(t) \geqslant 0$. It follows that the function

$$H(x) = F(x) - \int_a^x dy \int_a^y \psi(t)\, dt$$

is convex within (α, β). Hence $F(\alpha+0)$, $F(\beta-0)$ have unique values. The relation $F(x) < +\infty$ for all x and the convexity of $H(x)$ imply that those values are finite. For consider the continuous function

$$\mathfrak{F}(x) = \Sigma\, B_n(x)/n^3.$$

Since $-\Sigma\, A_n(x)/n^2$ is summable by Poisson's method for $\alpha < x < \beta$ and its coefficients are $o(1/n)$, it converges for such x. It follows by classical theorems that, for $\alpha < x < \beta$, $\mathfrak{F}'(x) = F(x)$. Hence, for $h > 0$,

$$\lim \frac{\mathfrak{F}(\alpha+h) - \mathfrak{F}(\alpha)}{h} = F(\alpha+0).$$

Also, since the coefficients of the series for $\mathfrak{F}(x)$ are $o(1/n^2)$, we have

$$\lim \Delta^2 \mathfrak{F}(\alpha, h)/h = 0.$$

Hence $\mathfrak{F}'(\alpha) = F(\alpha+0)$. By Lemma 7, with $\phi(x) = \mathfrak{F}(x)$, we have $\mathfrak{F}'(\alpha) = F(\alpha)$. Hence $F(\alpha) = F(\alpha+0)$, and similarly $F(\beta) = F(\beta-0)$. Thus $F(x)$ and $H(x)$ are continuous in the closed contiguous intervals of ϖ. Since $H(x)$ is convex in such closed intervals, it must attain an absolute maximum in each at an end point, *i.e.* a point of ϖ. Hence, since $H(x)$ is u.s.c. in ϖ, it is u.s.c. in (a, b). So then is $F(x)$.

We proceed to show that $F(x)$ is continuous in (a, b). At a point of ϖ at which $F(x)$ is finite, by Lemma 1 we have $\bar{D}^2 F(x) \geqslant \underline{P}(x) > K$. At a point of ϖ at which $F(x) = -\infty$, we have $\bar{D}^2 F(x) = +\infty$. At a point of $C\varpi$ we have $\bar{D}^2 F(x) \geqslant \underline{P}(x)$ and $\underline{P}(x) \geqslant \psi(x)$ p.p. in $C\varpi$, by Lemma 10. We may replace $\psi(x)$ by an equivalent function $\psi^*(x)$ which is finite except possibly at the points of E, $C\varpi$ [in (a, b)], and such that $\bar{D}^2 F(x) \geqslant \psi^*(x)$ for all $x \subset C\varpi$. Let $\lambda(x) = K$ for $x \subset \varpi$ and $\lambda(x) = \psi^*(x)$ for $x \subset C\varpi$. Let $m(x)$ be a minor function of $\lambda(x)$ in (a, b) and let

$$\mu(x) = \int_a^x m(t)\, dt.$$

Let $\Phi(x) = F(x) - \mu(x)$. Then $\bar{D}^2 \Phi(x) \geqslant 0$ in (a, b) except possibly at the points of E at which, by (4) and Lemma 2,

$$\overline{\lim}\, \Delta^2 \Phi(x, h)/h \geqslant 0 \geqslant \underline{\lim}\, \Delta^2 \Phi(x, h)/h.$$

By Lemma 14, the function $\Phi(x)$ has the same property as the function

so denoted at an earlier point in this proof. The argument used there establishes the continuity of $F(x)$ in (a, b).

It now follows that $F(x)$ is continuous within each contiguous interval of F_2. By the argument used for F_1 we can show that $F(x)$ is continuous within each contiguous interval of the perfect kernel Π_2 of F_2. It is clear that we can infer by transfinite induction that $F(x)$ is everywhere continuous, so that the conditions of Theorem I are satisfied.

6. Before considering the problem of replacing condition (F) or the condition $a_n = o(n)$, $b_n = o(n)$ by less restrictive ones, we shall prove the series of theorems, mentioned in § 1, which are applications of what precedes. In these six theorems the conclusion is that a certain series Σa_n is summable by Poisson's method. But, by obvious changes at the end of each proof, it can be shown that this series is summable $(C, 3+\delta)$ in Theorem III, $(C, 4+\delta)$ in Theorem IV, $(C, 5+\delta)$ in Theorem V, $(C, 4+\delta)$ in Theorem VI, $(C, 6+\delta)$ in Theorem VII, and $(C, 5+\delta)$ in Theorem VIII, for all $\delta > 0$.

We require the following lemma.

LEMMA 15. *Let $a_n = o(n)$, $b_n = o(n)$. If (a, b) is contained in an interval at every point of which $\underline{P}(x)$ is finite, and if $\underline{P}(x)$ is integrable in (a, b), then*

$$F(x) \sim -\Sigma A_n(x)/n^2$$

is continuous in (a, b) and has the form

$$F(x) = \int_a^x dy \int_a^y \underline{P}(t)\,dt + Ax + B \quad (a \leqslant x \leqslant b). \tag{11}$$

This is established by the proof of Theorem II.

THEOREM III. *Let $a_n = o(n^2)$. Let*

$$\lim_{r\to 1} \sum_{n=1}^{\infty} a_n \frac{\sin n\theta}{n\theta} r^n = \phi(\theta)$$

and let $\phi(\theta) \to l$ as $\theta \to 0$ where l is finite. Then

$$\lim_{r\to 1} \Sigma a_n r^n = l.$$

We may suppose that $l = 0$. Let

$$F(\theta) \sim -\Sigma \frac{a_n}{n^3} \sin n\theta.$$

Since $$\Sigma \frac{a_n}{n} \sin n\theta$$

is summable by Poisson's method to $\theta\phi(\theta)$ in a neighbourhood of the origin, say in $(-\eta, \eta)$, it follows by Lemma 15 that

$$F(\theta) = \int_0^\theta dy \int_0^y t\phi(t)\,dt + At + B \quad (-\eta < \theta < \eta), \quad (12)$$

where $B = 0$ since $F(\theta)$ is odd. We have

$$-\Sigma \frac{a_n}{n^3} \sin n\theta\, r^n = \frac{1}{2\pi} \int_{-\pi}^{\pi} F(\theta+t)\, P\, dt,$$

where $$P = (1-r^2)/\delta, \quad \delta = 1 - 2r\cos t + r^2.$$

Hence $$\Sigma a_n \cos n\theta\, r^n = -\frac{1}{2\pi} \int_{-\pi}^{\pi} F(\theta+t) \frac{d^3 P}{dt^3}\, dt$$

and $$\Sigma a_n r^n = -\frac{1}{\pi} \int_0^\pi F(t) \frac{d^3 P}{dt^3}\, dt.$$

The limit of the second member is zero by (12).

7. To prove the next theorem we require a further lemma.

LEMMA 16. *Let the series*

$$\Sigma a_n (1-\cos n\theta) \quad [a_n = o(n)]$$

be summable by Poisson's method to zero p.p. in $(-\eta, \eta)$. *Let*

$$\overline{\lim_{r\to 1}} \left| \Sigma a_n (1-\cos n\theta) r^n \right| < \infty$$

for $-\eta < \theta < \eta$. *Then*

$$\lim_{r\to 1} \Sigma a_n r^n$$

has a finite value.

Let $0 < \zeta < \eta$. Let δ satisfy $0 < \delta < \eta - \zeta$. Then, for almost all permissible ζ and δ, we have

$$\lim_{r\to 1} \Sigma a_n \{1-\cos n(\zeta+\delta)\} r^n = 0,$$

$$\lim_{r\to 1} \Sigma a_n \{1-\cos n(\zeta-\delta)\} r^n = 0,$$

so that $$\lim_{r\to 1} \Sigma a_n \sin n\delta \sin n\zeta\, r^n = 0. \quad (13)$$

For those values of ζ and δ for which (13) does not hold, we have

$$\overline{\lim_{r\to 1}} \left| \Sigma a_n \sin n\delta \sin n\zeta r^n \right| < \infty. \tag{14}$$

For a fixed ζ, the relations (13) and (14) hold in the specified circumstances for $0 < \delta < \eta - \zeta$. If we regard

$$\Sigma a_n \sin n\delta \sin n\zeta \tag{15}$$

as a trigonometric series in δ, it follows by Lemma 15 that the series

$$-\Sigma \frac{a_n}{n^2} \sin n\delta \sin n\zeta \tag{16}$$

represents for $0 < \delta < \eta - \zeta$ a function of the form

$$A_\zeta \delta + B_\zeta. \tag{17}$$

On the other hand, if we consider (15) as a trigonometric series in ζ, we infer that, for $0 < \zeta < \eta - \delta$, the series (16) represents a function of the form

$$A_\delta \zeta + B_\delta.$$

We thus have

$$A_\zeta \delta + B_\zeta = A_\delta \zeta + B_\delta \quad (0 < \zeta + \delta < \eta, \ \zeta > 0, \ \delta > 0).$$

Since (16) is odd in δ and converges (as a series in δ) to a linear function on the right of the origin, it follows from (17) that $B_\zeta = 0$ for all permissible ζ, and so $B_\delta = 0$ for all permissible δ. In $A_\zeta \delta = A_\delta \zeta$ let $\zeta = \zeta_0$. Then $A_\delta = A_{\zeta_0} \delta / \zeta_0$. Hence (16) represents a function of the form $K\delta\zeta$ for $0 < \delta + \zeta < \eta$, $\delta > 0$, $\zeta > 0$, and therefore for $-\frac{1}{2}\eta < \delta < \frac{1}{2}\eta$, $-\frac{1}{2}\eta < \zeta < \frac{1}{2}\eta$.

Consider (16) as a series in δ, giving ζ a fixed value. Then

$$\Sigma \frac{a_n}{n^2} \sin n\zeta \sin n\delta r^n = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(t+\delta) P \, dt,$$

where $f(t) = Kt\zeta$ for $-\frac{1}{2}\eta < t < \frac{1}{2}\eta$. Hence

$$\Sigma \frac{a_n}{n} \sin n\zeta \cos n\delta r^n = -\frac{1}{2\pi} \int_{-\pi}^{\pi} f(t+\delta) \frac{dP}{dt} \, dt,$$

and, putting $\delta = 0$,

$$\Sigma \frac{a_n}{n} \sin n\zeta r^n = -\frac{1}{\pi} \int_0^{\pi} f(t) \frac{dP}{dt} \, dt.$$

From this it follows that

$$\Sigma \frac{a_n}{n} \sin n\zeta$$

is summable by Poisson's method to $K\zeta$ in $-\frac{1}{2}\eta < \zeta < \frac{1}{2}\eta$. By Lemma 15,

$$-\Sigma \frac{a_n}{n^3} \sin n\zeta$$

represents a function $F(\zeta)$ which in $-\frac{1}{2}\eta < \zeta < \frac{1}{2}\eta$ has the form $\frac{1}{6}K\zeta^3 + L\zeta + M$, and $M = 0$ since the function is odd. From

$$\Sigma - \frac{a_n}{n^3} \sin n\zeta r^n = \frac{1}{2\pi} \int_{-\pi}^{\pi} F(\zeta + t)\, P\, dt$$

we infer that

$$\Sigma a_n r^n = -\frac{1}{\pi} \int_0^{\pi} F(t) \frac{d^3 P}{dt^3}\, dt,$$

and the limit of the second member is K.

THEOREM IV. *Let* $a_n = o(n^3)$. *Let*

$$\lim_{r \to 1} \Sigma a_n \left(\frac{\sin n\theta}{n\theta}\right)^2 r^n = \phi(\theta),$$

and let $\phi(\theta) \to l$ *as* $\theta \to 0$, *where* l *is finite. Then*

$$\lim \Sigma a_n r^n = l.$$

We may suppose that $l = 0$. We have by hypothesis

$$\lim_{r \to 1} \Sigma \frac{a_n}{n^2} (1 - \cos 2n\theta) r^n = 2\theta^2 \phi(\theta) \tag{18}$$

for $-\eta < \theta < \eta$ say. In this interval, $2\theta^2 \phi(\theta)$ is a bounded even function. Let $f(\theta)$ be a periodic integrable even function which, for $-2\eta < \theta < 2\eta$, has the form $f(\theta) = 2(\frac{1}{2}\theta)^2 \phi(\frac{1}{2}\theta)$. Let

$$f(\theta) \sim \Sigma \frac{a_n}{n^2} \cos n\theta \quad [a_n = o(n^2)].$$

Let $g(\theta) = f(2\theta)$, so that

$$g(\theta) \sim \Sigma \frac{a_n}{n^2} \cos 2n\theta$$

and $g(\theta) = 2\theta^2 \phi(\theta)$ for $-\eta < \theta < \eta$. Since $g(\theta) \to 0$ as $\theta \to 0$,

$$\lim \Sigma \frac{a_n}{n^2} r^n = 0. \tag{19}$$

On the other hand,

$$\lim \Sigma \frac{a_n}{n^2} \cos 2n\theta\, r^n$$

is equal to $g(\theta)$ p.p. in $-\eta<\theta<\eta$, while, since $g(\theta)$ is bounded in that interval,

$$\overline{\lim_{r\to 1}}\left|\Sigma\frac{a_n}{n^2}\cos 2n\theta\, r^n\right|<\infty.$$

Hence $$\lim\Sigma\frac{a_n+a_n}{n^2}(1-\cos 2n\theta)\,r^n=0$$

p.p. for $-\eta<\theta<\eta$, while

$$\overline{\lim}\left|\Sigma\frac{a_n+a_n}{n^2}(1-\cos 2n\theta)\,r^n\right|<\infty$$

in that interval. By Lemma 16,

$$\lim\Sigma\frac{a_n+a_n}{n^2}r^n$$

exists, and so, by (19), $$\lim\Sigma\frac{a_n}{n^2}r^n=A,$$

say. By (18) we have

$$\lim_{r\to 1}\Sigma\frac{a_n}{n^2}\cos 2n\theta\, r^n=A-2\theta^2\phi(\theta)$$

for $-\eta<\theta<\eta$. By Lemma 15,

$$F(\theta)\sim-\Sigma\frac{a_n}{n^4}\cos 2n\theta$$

has the form

$$F(\theta)=\int_0^\theta dy\int_0^y\{A-2t^2\phi(t)\}\,dt+B\theta+C\quad(-\eta<\theta<\eta),\quad(20)$$

and $B=0$ since $F(\theta)$ is even. From

$$-\Sigma\frac{a_n}{n^4}\cos 2n\theta\, r^n=\frac{1}{2\pi}\int_{-\pi}^{\pi}F(\theta+t)\,P\,dt$$

we infer that $$\Sigma a_n r^n=-\frac{1}{16\pi}\int_0^\pi F(t)\frac{d^4P}{dt^4}\,dt,$$

and the limit of the second member is zero by (20).

8. THEOREM V. *If* $a_n=o(n^4)$ *and*

(i) $$\lim_{r\to 1}\Sigma\, a_n\left(\frac{\sin n\theta}{n\theta}\right)^3 r^n=\phi(\theta),\quad\phi(\theta)\to l\ (\textit{finite})\ \textit{as}\ \theta\to 0,\qquad(21)$$

(ii) $$\lim_{r\to 1}\Sigma\frac{a_n}{n^2}\left(\frac{\sin n\theta}{n\theta}\right)r^n=\lambda(\theta),\quad\lambda(\theta)\to m\ (\textit{finite})\ \textit{as}\ \theta\to 0,\qquad(22)$$

then $$\lim\Sigma\, a_n r^n=l.$$

We may suppose that $l = m = 0$. We have

$$4\sin^3\theta = 3\sin\theta - \sin 3\theta.$$

By (21),

$$\lim_{r\to 1} \Sigma \frac{a_n}{n^3}(3\sin n\theta - \sin 3n\theta) r^n = 4\theta^3\phi(\theta),$$

which is, by (22), $\quad 3\theta\lambda(\theta) - 3\theta\lambda(3\theta) = 4\theta^3\phi(\theta).$

Hence $$\lambda\left(\frac{\theta}{3^n}\right) - \lambda\left(\frac{\theta}{3^{n+1}}\right) = -\tfrac{4}{3}\left(\frac{\theta}{3^{n+1}}\right)^2 \phi\left(\frac{\theta}{3^{n+1}}\right).$$

By addition we find $$\lambda(\theta) = \theta^2\mu(\theta) \quad [\mu(\theta)\to 0 \text{ as } \theta\to 0].$$

By (22), $$\lim_{r\to 1} \Sigma \frac{a_n}{n^3} \sin n\theta\, r^n = \theta^3\mu(\theta)$$

for $-\eta < \theta < \eta$, say. By Lemma 15,

$$F(\theta) \sim -\Sigma \frac{a_n}{n^5} \sin n\theta$$

in $-\eta < \theta < \eta$ has the form

$$F(\theta) = \int_0^\theta dy \int_0^y t^3\mu(t)\, dt + A\theta + B,$$

and $B = 0$ since $F(\theta)$ is odd. From

$$-\Sigma \frac{a_n}{n^5} \sin n\theta\, r^n = \frac{1}{2\pi}\int_{-\pi}^{\pi} F(\theta+t)\, P\, dt$$

we infer that $$\Sigma a_n r^n = -\frac{1}{\pi}\int_0^\pi F(t)\, \frac{d^5 P}{dt^5}\, dt,$$

and the limit of the second member is zero.

It may be hoped, but hardly expected, that in Theorem V the condition (ii) is superfluous. We proceed to prove a theorem in which that condition is replaced by one which is formally simpler

THEOREM VI. *If* $\Sigma a_n/n^2$ *converges and*

$$\Sigma a_n \left(\frac{\sin n\theta}{n\theta}\right)^3 = \phi(\theta),$$

where $\phi(\theta)\to l$ *(finite) as* $\theta\to 0$, *then*

$$\lim \Sigma a_n r^n = l.$$

We may suppose that $l = 0$. We have

$$\Sigma \frac{a_n}{n^3} (3 \sin n\theta - \sin 3n\theta) = 4\theta^3 \phi(\theta). \tag{23}$$

If the first member of (23) is arranged as a trigonometric series, we obtain

$$\Sigma b_n \sin n\theta,$$

where $$b_n = 3\frac{a_n}{n^3}$$

when n is not a multiple of 3, and

$$b_n = 3\,\frac{a_{3\nu}}{(3\nu)^3} - \frac{a_\nu}{\nu^3}$$

when $n = 3\nu$. We have, from (23),

$$\lim_{r\to 1} \Sigma \frac{a_n}{n^3} (3 \sin n\theta - \sin 3n\theta) r^n = 4\theta^3 \phi(\theta).$$

The harmonic function

$$\Sigma \frac{a_n}{n^3} (3 \sin n\theta - \sin 3n\theta) r^n$$

can be written $$\Sigma B_n(r, \theta),$$

where $$B_n(r, \theta) = 3\frac{a_n}{n^3} \sin n\theta\, r^n$$

when n is not a multiple of 3, and

$$B_n(r, \theta) = 3\,\frac{a_{3\nu}}{(3\nu)^3} \sin 3\nu\theta\, r^{3\nu} - \frac{a_\nu}{\nu^3} \sin 3\nu\theta\, r^\nu$$

when $n = 3\nu$. Hence

$$\Sigma b_n \sin n\theta\, r^n - \Sigma B_n(r, \theta) = \Sigma \frac{a_\nu}{\nu^3} \sin 3\nu\theta\, r^\nu (1 - r^{2\nu})$$

$$= o(1),$$

since $a_\nu = o(\nu^2)$. Hence

$$\lim_{r\to 1} \Sigma b_n \sin n\theta\, r^n = 4\theta^3 \phi(\theta),$$

and, since $b_n = o(1/n)$,

$$\Sigma b_n \sin n\theta = 4\theta^3 \phi(\theta).$$

This Fourier series may be integrated term by term over the interval $(0, \theta)$. The resulting series being absolutely convergent may be rearranged so as to give the series obtained by integrating term by term the first member of (23). We therefore obtain

$$\Sigma \frac{a_n}{n^4} [3(1-\cos n\theta) - \tfrac{1}{3}(1-\cos 3n\theta)] = o(\theta^4).$$

Let $$F(\theta) = \Sigma \frac{a_n}{n^4} (1-\cos n\theta).$$

Then $$\frac{F(\theta)}{\theta^2} - \frac{F(3\theta)}{(3\theta)^2} = o\left(\frac{\theta}{3}\right)^2.$$

Hence, uniformly in n,

$$\frac{F(\theta/3^n)}{(\theta/3^n)^2} - \frac{F(\theta/3^{n+1})}{(\theta/3^{n+1})^2} = o\left(\frac{\theta}{3^{n+2}}\right)^2.$$

By addition, $$\frac{F(\theta)}{\theta^2} - \frac{F(\theta/3^n)}{(\theta/3^n)^2} = o(\theta^2). \tag{24}$$

Since $\Sigma a_n/n^2$ converges, it follows that

$$\lim_{\theta \to 0} \Sigma \frac{a_n}{n^2} \left(\frac{\sin n\theta}{n\theta}\right)^2 = 2L,$$

say; *i.e.* $\lim F(\theta)/\theta^2 = L$. By (24),

$$F(\theta) = L\theta^2 + o(\theta^4),$$

so that $$\Sigma \frac{a_n}{n^4} \cos n\theta = A - L\theta^2 + o(\theta^4).$$

The proof is completed in the usual way.

9. THEOREM VII. *If* $a_n = o(n^5)$ *and*

(i) $$\lim_{r \to 1} \Sigma a_n \left(\frac{\sin n\theta}{n\theta}\right)^4 r^n = \phi(\theta), \quad \phi(\theta) \to l \ \ (\textit{finite}) \ \textit{as} \ \theta \to 0,$$

(ii) $$\lim_{r \to 1} \Sigma \frac{a_n}{n^2} \left(\frac{\sin n\theta}{n\theta}\right)^2 r^n = \lambda(\theta), \quad \lambda(\theta) \to m \ (\textit{finite}) \ \textit{as} \ \theta \to 0,$$

then $$\lim \Sigma a_n r^n = l.$$

We may suppose that $l = m = 0$. We have

$$8 \sin^4\theta = 8 \sin^2\theta - 2 \sin^2 2\theta.$$

Hence $$\lim_{r\to 1} \Sigma \frac{a_n}{n^4} [8 \sin^2 n\theta - 2 \sin^2 2n\theta] r^n = 8\theta^4 \phi(\theta),$$

or $$8\theta^2 \lambda(\theta) - 8\theta^2 \lambda(2\theta) = 8\theta^4 \phi(\theta).$$

It follows by the argument used in the proof of Theorem V that $\lambda(\theta) = o(\theta^2)$, and so

$$\lim_{r\to 1} \Sigma \frac{a_n}{n^4} (1 - \cos 2n\theta) r^n = o(\theta^4).$$

As in the proof of Theorem IV we infer that

$$\lim \Sigma \frac{a_n}{n^4} r^n = A,$$

so that $$\lim \Sigma \frac{a_n}{n^4} \cos n\theta\, r^n = A + \theta^4 \mu(\theta) \quad [\mu(\theta) = o(1)].$$

It follows by Lemma 15 that

$$F(\theta) \sim -\Sigma \frac{a_n}{n^6} \cos n\theta,$$

in $-\eta < \theta < \eta$ say, has the form

$$F(\theta) = \int_0^\theta dy \int_0^y \{A + t^4 \mu(t)\}\, dt + B\theta + C,$$

and $B = 0$ since $F(\theta)$ is even. The proof is completed in the usual way.

We finally prove the analogue of Theorem VI.

THEOREM VIII. *If $\Sigma a_n/n^2$ converges and*

$$\Sigma a_n \left(\frac{\sin n\theta}{n\theta}\right)^4 = \phi(\theta),$$

where $\phi(\theta) \to l$ (finite) as $\theta \to 0$, then

$$\lim \Sigma a_n r^n = l.$$

We may suppose that $l = 0$. We need the following lemma, which can be proved in a straightforward way.

LEMMA 17. *If Σa_n converges, then*

$$\lim_{\theta\to 0} \Sigma a_n \frac{n\theta - \sin n\theta}{(n\theta)^3} = \tfrac{1}{6} \Sigma a_n.$$

Turning to the proof of the theorem, we have

$$8\sin^4\theta = 4(1-\cos 2\theta)-(1-\cos 4\theta),$$

and hence

$$\Sigma \frac{a_n}{n^4}[4(1-\cos 2n\theta)-(1-\cos 4n\theta)] = o(\theta^4).$$

This may be integrated term by term over $(0, \theta)$ and gives

$$\Sigma \frac{a_n}{n^4}\left[4\left(\theta-\frac{\sin 2n\theta}{2n}\right)-\left(\theta-\frac{\sin 4n\theta}{4n}\right)\right] = o(\theta^5).$$

Writing

$$F(\theta) = \Sigma \frac{a_n}{n^4}\left(\frac{n\theta-\sin n\theta}{n}\right),$$

we have

$$2F(2\theta)-\tfrac{1}{4}F(4\theta) = o(\theta^5).$$

Hence, uniformly in n,

$$\frac{F(\theta/2^n)}{(\theta/2^n)^3}-\frac{F(\theta/2^{n+1})}{(\theta/2^{n+1})^3} = o(\theta/2^{n+1})^2. \tag{25}$$

By the convergence of $\Sigma a_n/n^2$ and by Lemma 17,

$$\lim_{n\to\infty}\frac{F(\theta/2^n)}{(\theta/2^n)^3} = A.$$

Hence and from (25), by addition,

$$F(\theta) = A\theta^3+o(\theta^5).$$

Writing

$$G(\theta) = \Sigma \frac{a_n}{n^5}\sin n\theta,$$

we have

$$G(\theta) = B\theta+A\theta^3+o(\theta^5),$$

and the proof is completed in the usual way.

10. For our subsequent investigations we require five additional lemmas.

LEMMA 18. *Let* $a_n = o(n^2)$ *and*

$$\lim_{r\to 1}\Sigma a_n \frac{\sin n\theta}{n} r^n = \phi(\theta),$$

where $\phi(\theta)\to l$ *as* $\theta\to 0$. *Then*

$$\lim (1-r)\Sigma a_n r^n = \tfrac{1}{2}\pi l.$$

For $a_n = 1$, $\phi(\theta) = \Sigma \sin n\theta/n$ and $l = \frac{1}{2}\pi$, while $(1-r)\Sigma r^n = 1$. It is therefore sufficient to prove the lemma when $l = 0$, in which case the argument is that of Theorem III with obvious changes.

LEMMA 19. *Let* $a_n = o(n^3)$ *and*

$$\lim_{r\to 1} \Sigma a_n \frac{\sin^2 n\theta}{n^2\theta} = \phi(\theta),$$

where $\phi(\theta) \to l$ *as* $\theta \to 0$. *Then*

$$\lim (1-r)\Sigma a_n r^n = \tfrac{1}{2}\pi l.$$

For $a_n = 1$,

$$\phi(\theta) = \Sigma \frac{\sin^2 n\theta}{n^2\theta},$$

$$l = \int_0^\infty \left(\frac{\sin\theta}{\theta}\right)^2 d\theta = \tfrac{1}{2}\pi.$$

It is therefore sufficient to prove the lemma when $l = 0$, in which case the argument is that of Theorem IV with obvious changes.

LEMMA 20. *Let*

$$C_n(x) = c_n \cos nx + d_n \sin nx \quad [c_n = o(n),\ d_n = o(n)].$$

If the series $\Sigma C_n(x)$ *is summable by Poisson's method in an interval* (a, b) *to a finite function* $f(x)$ *which is integrable in* (a, b), *and if*

$$\Sigma E_n(x) \quad [E_n(x) = e_n \cos nx + f_n \sin nx]$$

is the Fourier series of a function which is equal to $f(x)$ *in* (a, b), *then the series*

$$\Sigma C_n(x) - E_n(x)$$

and all the series obtained by formal differentiations are summable by Poisson's method to zero in (a, b); *and the function* $f(x)$ *cannot have a removable discontinuity.*

By Lemma 15 the Fourier series $-\Sigma C_n(x)/n^2$ has in (a, b) the form

$$\int_a^x dy \int_a^y f(t)\,dt + Ax + B,$$

and the Fourier series $-\Sigma E_n(x)/n^2$ has in (a, b) the form

$$\int_a^x dy \int_a^y f(t)\,dt + Cx + D.$$

Hence

$$-\Sigma \frac{C_n(x)-E_n(x)}{n^2} r^n = \frac{1}{2\pi}\int_{-\pi}^{\pi} g(t)\,\frac{1-r^2}{1-2r\cos(t-x)+r^2}\,dt,$$

where $g(t)$ is linear in (a, b). The first part of the lemma follows in an elementary way; and $f(x)$ cannot have a removable discontinuity since it is the Poisson sum of a Fourier series.

A finite function $f(x)$ defined in an interval (a, b) will be said to possess the property R in that interval, if, given any perfect set $\Pi \subset (a, b)$, there is a portion of Π on which $f(x)$ is continuous.

LEMMA 21. *If*

$$\lim_{r\to 1}(1-r)\Sigma \frac{B_n(x)}{n} r^n = 0,$$

then the series

$$\Sigma \frac{B_n(x)}{n^3}$$

is summable by Poisson's method to a function $\mathfrak{F}(x)$ *which possesses the property R.*

From

$$\Sigma \frac{B_n(x)}{n} r^{n-1} = o\left(\frac{1}{1-r}\right)$$

we infer that

$$\Sigma \frac{B_n(x)}{n^2} r^{n-1} = o\left(\log \frac{1}{1-r}\right),$$

so that

$$\lim_{r\to 1}\Sigma \frac{B_n(x)}{n^3} r^n$$

exists because

$$\int_r^1 \log \frac{1}{1-\rho}\,d\rho$$

exists. The family of continuous functions of x,

$$\phi_r(x) = \frac{1}{\log(1/1-r)}\Sigma \frac{B_n(x)}{n^2} r^{n-1},$$

converges to zero as $r \to 1$ for all x. Hence, given any perfect set Π, there is a portion ϖ of Π and a constant $K = K(\varpi)$ such that $|\phi_r(x)| < K$ for $x \subset \varpi$ and $r(\varpi) \leqslant r < 1$. Hence

$$\left|\mathfrak{F}(x) - \Sigma \frac{B_n(x)}{n^3} r^n\right| = \left|\int_r^1 \log \frac{1}{1-\rho}\,\phi_\rho(x)\,d\rho\right|$$

$$< K\int_r^1 \log \frac{1}{1-\rho}\,d\rho$$

for $x \subset \varpi$. The family

$$\Sigma \frac{B_n(x)}{n^3} r^n$$

of continuous functions of x converges uniformly on ϖ as $r \to 1$, so that $\mathfrak{F}(x)$ is continuous on ϖ.

LEMMA 22. *Let $\phi(x)$ be finite in the interval (a, b). Let*

$$D\phi(x) = \lim_{h \to 0} \frac{\phi(x+h)-\phi(x-h)}{2h}$$

have the value $f(x)$ for $a < x < b$, where $f(x)$ is continuous in (a, b). If α and $\beta > \alpha$ are two points of continuity of $\phi(x)$, then

$$\phi(\beta)-\phi(\alpha) = \int_\alpha^\beta f(x)\,dx.$$

In particular, if $\phi(x)$ is continuous, then

$$\phi(x) = \int_a^x f(t)\,dt + C.$$

It is sufficient to consider the case $f(x) = 0$. If $\phi(\alpha)$ and $\phi(\beta)$ are unequal, we may suppose that $\phi(\alpha) > \phi(\beta)$. Let

$$2k = [\phi(\alpha)-\phi(\beta)]/(\beta-\alpha),$$

and

$$\phi_k(x) = \phi(x)+k(x-\alpha).$$

Then

$$\phi_k(\alpha) > \phi_k(\beta). \tag{26}$$

Consider the set G of points in the closed interval (α, β) at which

$$\phi_k(x) \geqslant \tfrac{1}{2}\{\phi_k(\alpha)+\phi_k(\beta)\}.$$

Then $\alpha \subset G$. Since $\phi_k(x)$ is continuous at α, it follows by (26) that there is a greatest interval (α, ξ) such that, for $\alpha \leqslant x \leqslant \xi$, we have $x \subset G$. Further, $\xi < \beta$ since $\beta \subset CG$ and $\phi_k(x)$ is continuous at β. Hence ξ is a limiting point of CG on its right and therefore $D\phi_k(\xi) \leqslant 0$, $D\phi(x) \leqslant -k$, which is a contradiction.

11. We now proceed to replace the condition (F) and the condition (2) by less restrictive conditions. As mentioned in §1, we shall suppose that $P(x)$ exists and is integrable.

The condition

$$\lim (1-r)\,\Sigma \frac{B_n(x)}{n} r^n = 0 \tag{27}$$

is satisfied when $a_n = o(n)$, $b_n = o(n)$. When condition (F) holds, *i.e.* when

$$F(x) \sim -\Sigma\, A_n(x)/n^2$$

and $F(x)$ is continuous, we have

$$(1-r)\Sigma \frac{B_n(x)}{n} r^n = \frac{(1-r)}{\pi} \int_0^\pi \{F(x+t)-F(x-t)\} \frac{(1-r^2) r \sin t}{\delta^2} dt,$$

and the limit of the second member is zero.

When (27) holds, the series $\Sigma B_n(x)/n^3$ is by Lemma 21 summable by Poisson's method to a function $\mathfrak{F}(x)$. When condition (F) holds, $\mathfrak{F}(x)$ is the integral of $F(x)$, and $D\mathfrak{F}(x) = F(x)$. If $a_n = O(n)$, $b_n = O(n)$, then $-\Sigma A_n(x)/n^2$ converges to a function $F(x)$, and $D\mathfrak{F}(x) = F(x)$ by a theorem of Hardy and Littlewood†.

We now have

THEOREM IX. *If* $a_n = o(n^4)$, $b_n = o(n^4)$, $P(x)$ *is finite and integrable,*

$$\lim (1-r)\Sigma \frac{B_n(x)}{n} r^n = 0, \tag{28}$$

and $D\mathfrak{F}(x)$ *exists, then* (1) *is the Fourier series of* $P(x)$.

Since $P(x)$ is of Baire's first class, given any perfect set Π there is a portion ϖ of Π on which $P(x)$ is bounded. Hence

$$\Sigma \frac{A_n(x)}{n} r^n$$

and

$$\Sigma \frac{A_n(x)}{n^2} r^n$$

tend to their limits as $r \to 1$ uniformly for $x \subset \varpi$. The series

$$-\Sigma \frac{A_n(x)}{n^2} \tag{29}$$

is everywhere summable by Poisson's method to a function $F(x)$ which possesses the property R.

12. At this point we require a further lemma.

LEMMA 23. *Under the conditions of Theorem IX, if* (a, b) *is interior to an interval in which* $F(x)$ *is continuous, then*

$$F(x) = \int_a^x dy \int_a^y P(t)\, dt + Ax + B \quad (a \leqslant x \leqslant b),$$

$$\mathfrak{F}(x) = \int_a^x F(t)\, dt + C \quad (a \leqslant x \leqslant b).$$

† Hardy and Littlewood, *Proc. London Math. Soc.* (2), 22 (1924), 18.

By definition, $$\mathfrak{F}(x) = \lim_{r\to 1} \Sigma \frac{B_n(x)}{n^3} r^n.$$

Hence $$\frac{\mathfrak{F}(x+h)-\mathfrak{F}(x-h)}{2h} = -\lim_{r\to 1} \Sigma \frac{A_n(x)}{n^2} \frac{\sin nh}{nh} r^n. \tag{30}$$

Since $D\mathfrak{F}(x)$ exists, the second member of (30) tends to that number as $h\to 0$; and thus, by Theorem III,

$$-\lim_{r\to 1} \Sigma \frac{A_n(x)}{n^2} r^n = D\mathfrak{F}(x).$$

Hence for all x we have the equation

$$D\mathfrak{F}(x) = F(x). \tag{31}$$

By Lemma 21, $\mathfrak{F}(x)$ possesses the property R. Hence the points of discontinuity of $\mathfrak{F}(x)$ in (a, b) form a non-dense set G. And by (31) and Lemma 22, if x is a point of continuity of $\mathfrak{F}(x)$, we have

$$\mathfrak{F}(x) = \int_a^x F(t)\,dt + C, \tag{32}$$

where C is the same constant for all permissible x. It follows that $\mathfrak{F}(x)$ cannot have an isolated discontinuity. For, in virtue of (32), such a discontinuity could only be a removable one. This, however, is impossible by Lemma 20, with $f(x) = \mathfrak{F}(x)$.

The set G is therefore dense in itself and everywhere dense in a non-dense perfect set Π. Since $\mathfrak{F}(x)$ possesses the property R, there is a portion ϖ of Π on which $\mathfrak{F}(x)$ is continuous. Let A, B denote the extreme points of ϖ, and let (α, β) denote a contiguous interval of ϖ. Then, since $D\mathfrak{F}(x)$ exists, we have $\lim \mathfrak{F}(\alpha+h)-\mathfrak{F}(\alpha-h) = 0$. But $\mathfrak{F}(\alpha+0)$ exists, since, within (α, β), $\mathfrak{F}(x)$ is given by (32). If we let $h\to 0$ so that $\alpha-h \subset \varpi$, we infer that $\mathfrak{F}(\alpha) = \mathfrak{F}(\alpha+0)$, and similarly $\mathfrak{F}(\beta) = \mathfrak{F}(\beta-0)$. Since $\mathfrak{F}(x)$ has the form (32) in the closed contiguous intervals of ϖ and is continuous on ϖ, it follows that (32) holds for $\mathrm{A} \leqslant x \leqslant \mathrm{B}$. This contradicts the supposition that G is everywhere dense in Π. Hence $\mathfrak{F}(x)$ is continuous in (a, b). By Lemma 22, the equation (32) holds in (a, b). This establishes the second part of the lemma.

We can find a trigonometric series

$$\Sigma B_n'(x)/n^3,$$

where $$B_n'(x) = b_n' \cos nx - a_n' \sin nx \quad [b_n' = o(n^2),\ a_n' = o(n^2)],$$

which is the Fourier series of an integral, and which converges in (a, b) to $\mathfrak{F}(x)$. By Lemma 20,

$$(P) \quad \Sigma \frac{A_n(x)-A_n'(x)}{n^2} = 0 \quad (a \leqslant x \leqslant b),$$

$$(P) \quad \Sigma A_n(x)-A_n'(x) = 0 \quad (a \leqslant x \leqslant b), \tag{33}$$

the symbol (P) denoting that the Poisson sum of the series is to be taken. But

$$-\Sigma A_n'(x)/n^2$$

is the Fourier series of a function which is equal to $F(x)$ in (a, b); and by (33),

$$(P) \quad \Sigma A_n'(x) = P(x) \quad (a \leqslant x \leqslant b).$$

By Lemma 1, $$\bar{D}^2 F(x) \geqslant P(x) \geqslant \underline{D}^2 F(x)$$

in (a, b), so that the first part of the lemma follows from Lemma 5.

13. We can now complete the proof of Theorem IX. Our immediate object is to show that $F(x)$ is continuous. Let E denote the set of points of discontinuity of $F(x)$. Then E cannot have an isolated point. For suppose that ξ is an isolated discontinuity of $F(x)$. Choose (α, ξ), (ξ, β) so that ξ is the only point of E in (α, β). By Lemma 23, $F(\xi+0)$, $F(\xi-0)$, $\mathfrak{F}(\xi+0)$, $\mathfrak{F}(\xi-0)$ all exist. Since $D\mathfrak{F}(\xi) = F(\xi)$ we have

$$\mathfrak{F}(\xi+0) = \mathfrak{F}(\xi-0).$$

Since, by Lemma 20 with $f(x) = \mathfrak{F}(x)$, $\mathfrak{F}(x)$ cannot have a removable discontinuity, it follows that $\mathfrak{F}(x)$ is continuous in (α, β). We now have

$$\frac{\mathfrak{F}(\xi+h)-\mathfrak{F}(\xi)}{h} = \lim_{\delta \to 0} \frac{\mathfrak{F}(\xi+h)-\mathfrak{F}(\xi+\delta)}{h-\delta}.$$

Since, in $(\xi+\delta, \xi+h)$, $\mathfrak{F}(x)$ is the integral of $F(x)$, it follows by the mean value theorem that

$$\lim [\mathfrak{F}(\xi+h)-\mathfrak{F}(\xi)]/h = F(\xi+0).$$

On the other hand,

$$\lim [\mathfrak{F}(\xi+h)-\mathfrak{F}(\xi-h)]/(2h) = F(\xi).$$

Hence $$\lim \Delta^2 \mathfrak{F}(\xi, h)/(2h) = F(\xi+0)-F(\xi).$$

But, by definition,

$$\Delta^2 \mathfrak{F}(\xi, h)/(2h) = -\lim_{r \to 1} \Sigma \frac{B_n(x)}{n} \frac{\sin^2 \frac{1}{2}nh}{\frac{1}{2}(n^2 h)} r^n,$$

so that, by Lemma 19,

$$\lim (1-r)\Sigma \frac{B_n(x)}{n} = \tfrac{1}{2}\pi\{F(\xi)-F(\xi+0)\}.$$

Hence, by (28), $F(\xi) = F(\xi+0)$, and similarly $F(\xi) = F(\xi-0)$.

Thus E is dense in itself and everywhere dense in a non-dense perfect set Π. There is a portion ϖ of Π on which $F(x)$ is continuous. Let A, B denote the extreme points of ϖ. Let (α, β) denote a contiguous interval of ϖ. Then, by Lemma 23, $F(\alpha+0)$, $\mathfrak{F}(\alpha+0)$ exist. Since $D\mathfrak{F}(\alpha) = F(\alpha)$, $\mathfrak{F}(\alpha-0)$ exists and is equal to $\mathfrak{F}(\alpha+0)$. Thus $\mathfrak{F}(x)$ is bounded in a neighbourhood of α, so that, by the conclusion of Lemma 20, $\mathfrak{F}(\alpha) = \mathfrak{F}(\alpha+0)$. From this we infer that $F(\alpha) = F(\alpha+0)$ by the argument just used in showing that $F(\xi) = F(\xi+0)$. Similarly $F(\beta) = F(\beta-0)$. Since now $F(x)$ is continuous on ϖ, and by Lemma 23 has the form

$$F(x) = \int_{a_n}^{x} dy \int_{a_n}^{y} P(t)\,dt + A_n x + B_n \quad (a_n \leqslant x \leqslant b_n),$$

where (a_n, b_n) are the contiguous intervals of ϖ, it follows that $F(x)$ is continuous in (A, B). This contradicts the supposition that E is everywhere dense in Π. Hence $F(x)$ is everywhere continuous. By Lemma 23 it follows that

$$\mathfrak{F}(x) = \int_a^x dy \int_a^y dt \int_a^t P(\theta)\,d\theta + Ax^2 + Bx + C$$

in an arbitrary interval (a, b). Since the coefficients of $\Sigma B_n(x)/n^3$ are $o(n)$, it follows by Theorem II that this series is the Fourier series of $\mathfrak{F}(x)$.

14. The condition

$$\lim (1-r)\Sigma \frac{B_n(x)}{n} r^n = 0$$

is expressed by saying that

$$B_n(x)/n$$

tends to zero as $n \to \infty$ by Poisson's method. Consider the following definition. If

$$\lim_{r\to 1} \Sigma \frac{B_n(x)}{n} \frac{\sin nh}{n} r^n$$

tends to zero as $h \to 0$, then we write

$$\lim_{n\to\infty} \frac{B_n(x)}{n} = 0 \quad (\mathfrak{R}, 1). \tag{34}$$

It appears from Lemma 18 that, when $a_n = o(n^3)$, $b_n = o(n^3)$, Poisson's method of passage to a limit is more powerful than the method $(\mathfrak{R}, 1)$. We

may therefore expect that if, in Theorem IX, we replace the condition (28) by the condition (34), and the condition $a_n = o(n^4)$, $b_n = o(n^4)$ by the condition $a_n = o(n^3)$, $b_n = o(n^3)$, then the explicit condition that $D\mathfrak{F}(x)$ should exist will prove to be superfluous.

THEOREM X. *If $a_n = o(n^3), b_n = o(n^3)$, $P(x)$ is finite and integrable, and*

$$\lim \frac{B_n(x)}{n} = 0 \quad (\mathfrak{B}, 1),$$

then (1) *is the Fourier series of $P(x)$.*

The series
$$-\Sigma \frac{A_n(x)}{n^2} \tag{35}$$
is summable by Poisson's method to a function $F(x)$ which possesses the property R. We require the following lemma.

LEMMA 24. *Under the conditions of Theorem X, if (a, b) is contained in an interval in which $F(x)$ is continuous, then*

$$F(x) = \int_a^x dy \int_a^y P(t)\,dt + Ax + B \quad (a \leqslant x \leqslant b).$$

Let $-\Sigma A_n'(x)/n^2$, where

$$A_n'(x) = a_n' \cos nx + b_n' \sin nx \quad [a_n' = o(n^2),\ b_n' = o(n^2)],$$

be the Fourier series of a function $g(x)$ which is equal to $F(x)$ in (a, b). Then, since the coefficients of (35) are $o(n)$, it follows by Lemma 20 that

$$(P) \quad \Sigma \frac{A_n(x) - A_n'(x)}{n^2} = 0 \quad (a \leqslant x \leqslant b),$$

$$(P) \quad \Sigma A_n(x) - A_n'(x) = 0 \quad (a \leqslant x \leqslant b).$$

Hence
$$(P) \quad -\Sigma A_n'(x)/n^2 = F(x),$$
$$(P) \quad \Sigma A_n'(x) = P(x),$$

for $a \leqslant x \leqslant b$. Since the first of these series is a Fourier series, it follows by Lemma 1 that

$$\bar{D}^2 F(x) \geqslant P(x) \geqslant \underline{D}^2 F(x)$$

for $a \leqslant x \leqslant b$, so that the lemma follows from Lemma 5.

We proceed to prove Theorem X. Since $F(x)$ has the property R, its points of discontinuity form a non-dense set E. We first show that E

cannot have an isolated point. Suppose on the contrary that ξ is an isolated discontinuity of $F(x)$. Then, by Lemma 24, $F(\xi+0)$, $F(\xi-0)$ both exist. By definition,

$$F(\xi+h)-F(\xi-h) = -2\lim_{r\to 1}\Sigma\frac{B_n(x)}{n}\frac{\sin nh}{n}r^n,$$

and by (34) the second member tends to zero as $h\to 0$. Thus

$$F(\xi+0) = F(\xi-0).$$

By the conclusion of Lemma 20 with $f(x) = F(x)$, $F(x)$ cannot have a removable discontinuity at ξ. Hence $F(x)$ is continuous at ξ.

Thus E is dense in itself and everywhere dense in a non-dense perfect set Π. There is a portion ϖ of Π on which $F(x)$ is continuous. Let A, B denote the extreme points of ϖ. Let (α, β) denote a contiguous interval of ϖ. Then $F(\alpha+0)$ exists by Lemma 24. By (34),

$$\lim\{F(\alpha+h)-F(\alpha-h)\} = 0.$$

Let h tend to zero so that $\alpha-h \subset \varpi$. It follows that $F(\alpha) = F(\alpha+0)$, and similarly $F(\beta) = F(\beta-0)$. Hence

$$F(x) = \int_{a_n}^{x} dy \int_{a_n}^{y} P(t)\,dt + A_n x + B_n \quad (a_n \leqslant x \leqslant b_n),$$

where (a_n, b_n) denotes a contiguous interval of ϖ. It follows that $F(x)$ is continuous in (A, B), which contradicts the assumption that E is everywhere dense in Π. Hence $F(x)$ is everywhere continuous. By Lemma 24,

$$F(x) = \int_{a}^{x} dy \int_{a}^{y} P(t)\,dt + Ax + B$$

in an arbitrary interval (a, b). Since the coefficients of (35) are $o(n)$, it follows by Theorem II that (35) is the Fourier series of $F(x)$.

15. In the preceding theorem the condition $a_n = o(n^3)$, $b_n = o(n^3)$ may be replaced by the condition that $\Sigma A_n(x)/n^4$ should be the Fourier series of a continuous function.

THEOREM XI. *Let*

$$G(x) \sim \Sigma A_n(x)/n^4,$$

where $G(x)$ is continuous. Let $P(x)$ be finite and integrable, and

$$\lim\frac{B_n(x)}{n} = 0 \quad (\mathfrak{B}, 1). \tag{36}$$

Then (1) is the Fourier series of $P(x)$.

The series (35) is summable by Poisson's method to a function $F(x)$ which has the property R. We require two lemmas.

LEMMA 25. *Let* $C_n(x) = c_n \cos nx + d_n \sin nx$. *Let*

$$\Phi(x) \sim -\Sigma \frac{C_n(x)}{n^2},$$

where $\Phi(x)$ *is continuous. Let*

$$\Sigma C_n(x)$$

be summable by Poisson's method in (a, b) *to a finite function* $f(x)$ *which is integrable in* (a, b). *Let*

$$\Sigma E_n(x) \quad [E_n(x) = e_n \cos nx + f_n \sin nx]$$

be the Fourier series of a function which is equal to $f(x)$ *in* (a, b). *Then*

$$\Sigma\{C_n(x) - E_n(x)\}$$

and all the series obtained by formal differentiation are summable by Poisson's method to zero in (a, b); *and* $f(x)$ *cannot have a removable discontinuity.*

This is the analogue of Lemma 20. In order to prove the result by the method of that lemma, we must show that

$$\Phi(x) = \int_a^x dy \int_a^y f(t)\, dt + Ax + B \quad (a \leqslant x \leqslant b).$$

This, however, is established by the proof of Theorem I.

LEMMA 26. *Under the conditions of Theorem XI, if* (a, b) *is contained in an interval in which* $F(x)$ *is continuous, then*

$$F(x) = \int_a^x dy \int_a^y P(t)\, dt + Ax + B \quad (a \leqslant x \leqslant b).$$

This is proved by the method of Lemma 24, Lemma 25 being used instead of Lemma 20.

The rest of the proof of Theorem XI is precisely similar to that of Theorem X. Lemmas 25 and 26 replace Lemmas 20 and 24. At the end of the proof, Theorem I is used instead of Theorem II.

OCTADIC SURFACES AND PLANE QUARTIC CURVES

By W. L. EDGE

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Introduction.

It is well known how Hesse, in a classical paper*, established a (1, 1) correspondence between a general plane quartic δ, without nodes or cusps, and the twisted sextic ϑ which is the locus of the vertices of the cones which belong to a net of quadric surfaces. He used the correspondence to obtain results concerning the bitangents of δ; but it can also be used to obtain other geometrical results concerned both with δ and with ϑ, and it is proposed so to make use of it in what follows.

The paper is divided into two sections. The first section opens with a description of the contact conics and cubics of δ. The net of quadric surfaces is then introduced (§ 2), and certain planes, denoted by ϖ and associated with the points of ϑ, are obtained (§ 3); the cubic complex formed by the generators of the quadrics is introduced in § 4. The representation of the quadrics of the net by the points of a plane leads naturally, by way of the conics of the plane, to those quartic surfaces having nodes at the eight base points of the net and called by Cayley *octadic surfaces*. There is an octadic surface associated with each conic of the plane; if the conic touches δ the octadic surface must have an additional node at the corresponding point of ϑ (§§ 5–6).

The following paragraphs (§§ 7–10) give an account of the octadic surfaces which have nodes additional to the eight at the base points of the net of quadrics; the results are obtained by making use of the correspondence between these surfaces and the conics which touch δ. An octadic surface can have one, two, three, or four additional nodes just as a conic can touch δ in one, two, three, or four points; those surfaces which have four additional nodes fall into sixty-three systems, of which twenty-eight systems are of one kind and thirty-five systems of a second

* *Journal für Math.*, 49 (1855), 279–332; *Gesammelte Werke*, 345.

kind. In § 11 the correspondence between conics and octadic surfaces is used in connection with those sets of six bitangents of δ (commonly called Brianchon sets of bitangents) which touch a conic.

We then proceed, in the second part of the paper, to consider the trisecants of ϑ; § 12 describes the conjugacy between the points and trisecants of ϑ; in § 13 the existence of an octadic surface with nodes at the three intersections of ϑ with any one of its trisecants is established; and § 14 is concerned with a singly-infinite family of contact cubics of δ which is associated with a trisecant of ϑ. When, in § 15, those trisecants of ϑ are considered which are also generators of quadrics of the net, it is found that they consist of eight sets of three coplanar trisecants, and that there can be no planes, other than these eight, which contain three trisecants of ϑ. The six points of δ which correspond to the six points in which ϑ is met by any one of these eight planes are the points of contact of δ with a contact cubic which breaks up into three lines; the lines form a triangle whose vertices lie on δ and whose sides touch δ. Thus eight in-and-circumscribed triangles of δ are obtained; they all belong to a definite system of contact cubics of δ. There are thirty-five other similar systems of contact cubics to each of which there belong eight in-and-circumscribed triangles of δ, so that there are in all 288 triangles both inscribed and circumscribed to δ.

The next paragraphs give some account of the different kinds of twisted sextics of genus 3 and of the way in which they can be put into birational correspondence with plane quartics. When a twisted sextic is the locus of vertices of the cones which belong to a net of quadrics *consisting of the polar quadrics of the points of a plane in regard to a cubic surface* it cannot be put into birational correspondence with a general plane quartic but only with a plane quartic in which an infinity of pentagrams can be inscribed. But, although this twisted sextic and plane quartic are particularized, the correspondence between the two curves can be made to serve the same purposes as can Hesse's correspondence between the more general curves, and properties of the two curves can be obtained by means of it (§§ 20–23).

The paper closes (§ 24) with an allusion to Cayley's paper on the problem of the in-and-circumscribed triangle.

I.

1. A plane quartic curve without double points has twenty-eight bitangents. It has also sixty-three systems of quadritangent conics; each system consists of ∞^1 conics and has the quartic curve for its envelope.

There is a conic of each system touching the curve at any point, and two quadritangent conics belong to the same or different systems according as their eight points of contact with the quartic do or do not all lie on the same conic.

The quartic curve has, further, sixty-four systems of contact cubics; each cubic touches the quartic in six points, and the systems are of two kinds according as the six contacts do or do not lie on a conic. The contacts of a cubic belonging to a system of the first kind do *not* lie on a conic; there are thirty-six systems of this kind. The remaining twenty-eight systems are of the second kind; the six points of contact of any cubic belonging to one of these systems are on a conic. Moreover, the two remaining intersections of the conic with the quartic are the two points of contact of a bitangent, and, conversely, any conic through the two points of contact of a bitangent meets the quartic further in six points which are points of contact of a contact cubic belonging to a system of the second kind. All the contact cubics of any one system of the second kind are obtainable by means of the conics through the two points of contact of a given bitangent, and the twenty-eight systems of the second kind arise from the twenty-eight bitangents in this way. The twelve points of contact of two contact cubics which belong to the same system, whether of the first or second kind, form the complete intersection of the quartic with a cubic curve, and, conversely, if the points of contact of two contact cubics all lie on the same cubic curve the two contact cubics belong to the same system. Each system consists of ∞^3 cubic curves; through three points of general position on the quartic there passes one and only one contact cubic of each system. Each system of the first kind contains fifty-six cubics which break up into three bitangents; each system of the second kind contains forty-five cubics which so break up. Every one of the ${}^8C_3$ $(= 36\times 56+28\times 45)$ triplets of bitangents can be regarded as a contact cubic*.

2. Three fixed quadric surfaces

$$Q_0 = 0, \quad Q_1 = 0, \quad Q_2 = 0$$

in three-dimensional space have in common eight "associated" points and determine a net $\mathfrak{N}$ of quadric surfaces with the eight points as base points; the quadrics of $\mathfrak{N}$ are given by

$$xQ_0+yQ_1+zQ_2 = 0,$$

* Concerning the systems of contact curves of a plane quartic without double points see Clebsch, *Journal für Math.*, 63 (1864), 209–213.

where $x:y:z$ are varying parameters*. We can, following Hesse, represent the quadrics of $\mathfrak{N}$ by the points of a plane σ; to any given quadric of $\mathfrak{N}$ corresponds a definite set of values of $x:y:z$ which may be taken as coordinates of the representative point in σ. There are, among the quadrics of $\mathfrak{N}$, ∞^1 cones; these are represented in σ by the points of a plane quartic δ without double points. The equation of δ is obtained by equating to zero the discriminant of a quadric of $\mathfrak{N}$; this discriminant is a symmetrical determinant of four rows and columns whose elements are linear in x, y, z. The vertices of the cones themselves lie on a twisted sextic ϑ of genus 3, and the representation establishes a (1, 1) correspondence between δ and ϑ. Conversely: given any plane quartic curve δ without multiple points, its points can be regarded as representing the cones which belong to a net of quadrics in space. For, in order that this should be possible, it is only necessary that the left-hand side of the equation of δ should be capable of being reduced to a symmetrical determinant of four rows and columns whose elements are linear functions of the coordinates; and such a reduction can, in fact, be carried out in thirty-six different ways.

The (1, 1) correspondence between δ and ϑ is examined by Hesse in his paper, which is devoted to the study of the twenty-eight bitangents of δ. He shows that the line which joins any pair of base points of $\mathfrak{N}$ is a chord of ϑ, and that its two intersections with ϑ correspond, in the (1, 1) correspondence, to the two points of contact of δ with a bitangent. The twenty-eight bitangents of δ are thus put into correspondence with the twenty-eight joins of the pairs of eight associated points in space. Hesse shows also that the points of σ which represent the ∞^1 quadrics of $\mathfrak{N}$ which touch a given plane α are the points of a contact cubic S_α of δ which belongs to a certain one, say S, of the thirty-six systems of the first kind; the ∞^3 cubics of the system S are associated in this way with the ∞^3 planes of space. The six points in which ϑ is met by α correspond to the six points of δ in which it is touched by S_α. The locus of the poles of α in regard to the quadrics of $\mathfrak{N}$ is a cubic surface containing ϑ; the cubic curve in which α meets this surface is the locus of points of contact of α with quadrics of $\mathfrak{N}$ and is in (1, 1) correspondence with S_α.

The system S is not special in any way. A homogeneous quartic polynomial in three variables can be expressed, in thirty-six different ways, as a symmetrical determinant of four rows and columns whose

* The properties of a net of quadrics and of curves and surfaces associated with it are investigated by Sturm in a paper entitled "Untersuchungen über das Flächennetz zweiter Ordnung", *Journal für Math.*, 70 (1869), 212. Sturm does not make use of the representation of the quadrics of the net by the points of a plane.

elements are linear functions of the variables. Each of these thirty-six ways of writing the equation of a given quartic curve selects one of the thirty-six systems of contact cubics of the first kind to play the part of the system S in the above representation; the contact cubics of any one of these thirty-six systems can be associated with the planes of space in this manner*. Any property of the system S which is obtained by means of the representation can also be predicated of each of the other thirty-five systems of the first kind.

3. We shall denote the cone of $\mathfrak{N}$ whose vertex is at the point P of ϑ by (P).

The quadrics of a pencil belonging to $\mathfrak{N}$ are represented in σ by the points of a line; this line meets δ in four points which represent the four cones belonging to the pencil. Let us consider in particular those quadrics of $\mathfrak{N}$ which pass through a point P of ϑ; they form a pencil, since the condition of passing through P is linear, and (P) is one of them. But when a pencil of quadrics contains a cone whose vertex P is on every quadric of the pencil, the quartic base curve of the pencil has a double point at P; the quadrics all touch the plane ϖ which contains the two tangents of the base curve at P, and there are only two other cones belonging to the pencil, their vertices A and B both lying in ϖ. Thus there is, associated with each point P of ϑ, a plane ϖ which contains P; ϖ is touched at P by all the quadrics of $\mathfrak{N}$ which pass through P. These quadrics are represented in σ by the points of a tangent of δ; the tangent

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* If the equation to the plane quartic is

$$\begin{vmatrix} a & h & g & u \\ h & b & f & v \\ g & f & c & w \\ u & v & w & d \end{vmatrix} = 0,$$

where the elements of the determinant are linear functions of x, y, z, then the cubic curve

$$\begin{vmatrix} a & h & g & u & \alpha \\ h & b & f & v & \beta \\ g & f & c & w & \gamma \\ u & v & w & d & \delta \\ \alpha & \beta & \gamma & \delta & 0 \end{vmatrix} = 0$$

is a contact-cubic. The system of contact cubics to which it belongs is of the first kind, and all the ∞^3 contact cubics of this system are obtained by giving different values to the constants α, β, γ, δ.

touches δ at the point p which corresponds to P and meets δ again in the two points a and b corresponding to A and B.

There are, on δ, twenty-four points of inflection i; the tangent to δ at such a point has three-point contact, and only meets δ again in one further point j. Correspondingly we have, on ϑ, twenty-four points I; the pencil of quadrics of $\mathfrak{N}$ which pass through I contains only two cones, one whose vertex is I and another whose vertex is a point J of ϑ. The quartic base curve of such a pencil of quadrics has a cusp at I, and the quadrics all touch the plane which contains J and the cuspidal tangent of the quartic curve. This is a limiting case of the former when one of A and B coincides with P; the plane ϖ which is associated with a point I not only passes through I but contains the tangent of ϑ at I. We have thus twenty-four points of ϑ at which the tangents lie in the associated planes ϖ.

4. Since three linear conditions have to be imposed upon a quadric of $\mathfrak{N}$ in order to make it contain a line, and the quadrics of $\mathfrak{N}$ have only freedom two, an arbitrary line is not a generator of a quadric of $\mathfrak{N}$. But each of the ∞^2 quadrics of $\mathfrak{N}$ has ∞^1 generators, so that the lines which are generators of quadrics of $\mathfrak{N}$ are ∞^3 in aggregate and form a complex V. V may also be defined as the complex of lines which are cut in involution by the quadrics of $\mathfrak{N}$. It is a cubic complex*.

The lines of V which pass through an arbitrary point generate a cubic cone and those which lie in an arbitrary plane envelop a curve of the third class. The cubic cone generated by the lines of V which pass through a point P of ϑ breaks up into the quadric cone (P) and the pencil of lines through P in the associated plane ϖ. Hence, if a line of V is not a generator of a cone of $\mathfrak{N}$ and yet meets ϑ, it must lie in the plane ϖ associated with its intersection with ϑ. The envelope of the third class formed by the lines of V which lie in ϖ consists of the pencil of lines through P and the tangents of some conic.

Through an arbitrary point O there pass eighteen lines of V which meet ϑ, these being the common generators of the sextic cone which projects ϑ from O and of the cubic cone of lines of V which pass through O. Of these eighteen lines four are generators of cones of $\mathfrak{N}$ which pass through O; they join O to the vertices of the four cones which belong to the pencil of quadrics of $\mathfrak{N}$ passing through O. The remaining fourteen

* Sturm, *loc. cit.*, 217; Reye, *Die Geometrie der Lage*, 3 (Leipzig, 1923), 137; Cayley, *Proc. London Math. Soc.* (1), 3 (1871), 19–69 (61–63); *Papers*, 7, 173–175. The complex is studied in detail by Montesano, *Memorie della R. Acc. di Bologna* (5), 3 (1892), 549.

lines are not generators of cones of $\mathfrak{N}$ and therefore lie in the planes ϖ associated with their intersections with ϑ. Hence, fourteen of the planes ϖ pass through O, and these are the only planes ϖ which can do so. *The planes ϖ associated with the points of ϑ form a developable* of class fourteen (and genus 3).*

It can now be ascertained at exactly how many points P of ϑ the tangent lies in the associated plane ϖ. For the planes ϖ form a developable of class fourteen and the tangents of ϑ, a curve of order six and genus 3, form a ruled surface of order sixteen; the planes of the developable and the generators of the ruled surface are in (1, 1) correspondence, and the locus of the points of intersection of pairs of corresponding planes and generators is the sextic curve ϑ. Hence† the number of tangents of ϑ which lie in the associated planes ϖ is $14+16-6=24$. We have already observed that the tangent of ϑ at any one of the twenty-four points I lies in the associated plane ϖ; hence *the twenty-four points of ϑ at which quadrics of $\mathfrak{N}$ have stationary contact, and only these points, are such that the tangent of ϑ lies in the associated plane ϖ.*

5. The coordinates of the points of a conic in σ are given by

$$x:y:z=a_0\theta^2+2b_0\theta+c_0 : a_1\theta^2+2b_1\theta+c_1 : a_2\theta^2+2b_2\theta+c_2,$$

where θ is a parameter; they represent the singly-infinite family of quadrics

$$(a_0\theta^2+2b_0\theta+c_0)\,Q_0+(a_1\theta^2+2b_1\theta+c_1)\,Q_1+(a_2\theta^2+2b_2\theta+c_2)\,Q_2=0.$$

This system of quadrics is such that there are two of its members passing through an arbitrary point of space; it is *a system of index* 2 and may be denoted by Σ_2. Conversely, any singly-infinite system of quadrics of index 2 which belongs to $\mathfrak{N}$ is represented by a conic in σ.

The envelope of the quadrics of Σ_2, obtained by the elimination of θ from the equations

$$(a_0\theta+b_0)\,Q_0+(a_1\theta+b_1)\,Q_1+(a_2\theta+b_2)\,Q_2=0,$$

$$(b_0\theta+c_0)\,Q_0+(b_1\theta+c_1)\,Q_1+(b_2\theta+c_2)\,Q_2=0,$$

is

$$(a_0Q_0+a_1Q_1+a_2Q_2)\,(c_0Q_0+c_1Q_1+c_2Q_2)=(b_0Q_0+b_1Q_1+b_2Q_2)^2.$$

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* *Cf.* Sturm, *loc. cit.*, 230.

† *Cf.* Edge, *The theory of ruled surfaces* (Cambridge, 1931), § 23; or *Proc. London Math. Soc.* (2), 33 (1931), 53.

This is a quartic surface having nodes at the eight base points of $\mathfrak{N}$; such a quartic surface, with nodes at eight *associated* points, is called by Cayley an *octadic surface*.

The tangential equation of the conic which represents the quadrics of Σ_2 is

$$(la_0+ma_1+na_2)(lc_0+mc_1+nc_2) = (lb_0+mb_1+nb_2)^2.$$

Hence we have the following:

The envelope of those quadrics of $\mathfrak{N}$ which are represented in σ by the points of the conic γ whose equation is

$$ax^2+by^2+cz^2+2fyz+2gzx+2hxy = 0$$

is the octadic surface Γ whose equation is

$$AQ_0^2+BQ_1^2+CQ_2^2+2FQ_1Q_2+2GQ_2Q_0+2HQ_0Q_1 = 0,$$

where $A = bc-f^2$, ...

The envelope of the quadrics of any system Σ_2, of index 2, belonging to $\mathfrak{N}$ is an octadic surface whose nodes are the base points of $\mathfrak{N}$ and, conversely, any octadic surface is the envelope of a singly-infinite system, of index 2, of quadrics of the net determined by its nodes.*

Through any point of a quadric Q which belongs to Σ_2 there passes, in general, one other quadric of Σ_2 meeting Q in an elliptic quartic curve through the eight base points; we thus have a pencil of elliptic quartic curves on Q to which the curve of contact of Q with Γ belongs. Through an arbitrary point of space there pass two quadrics of Σ_2; Γ is the locus of points for which the two quadrics coincide.

The quadrics of $\mathfrak{N}$ which pass through an arbitrary point of space form a pencil and include two quadrics of Σ_2, but if the point lies on Γ there is only one quadric belonging to Σ_2 and it touches Γ along the base curve of the pencil. There is a pencil of these elliptic quartic curves on Γ; any two of them form the complete intersection of Γ with a quadric of $\mathfrak{N}$ Conversely, any quadric Q which passes through the nodes of Γ must contain the whole of the elliptic quartic curve which passes through the nodes and through any other point common to Γ and Q; hence any quadric which passes through the nodes of an octadic surface meets it in two elliptic quartic curves. The quadric Q is represented by a point q of σ; the two tangents from q to γ touch γ in the two points which

* The generators of the quadrics of Σ_2 are all bitangents of the surface; they form a congruence of order 4 and class 12, which is therefore part of the congruence formed by the bitangents of the surface. Concerning this property of the octadic surface see Kummer, *Berliner Monatsberichte* (1872), 474.

represent the two quadrics touching Γ along the two elliptic quartic curves in which it is met by Q.

The points of a tangent of γ represent the quadrics of a pencil whose base curve is one of the quartic curves* on Γ; the four cones of the pencil are represented by the four intersections of the tangent with δ. If the tangent of γ also touches δ two of the four cones must coincide; the base curve of the pencil of quadrics acquires a double point, this double point lying necessarily on ϑ. Hence *the twenty-four intersections of* Γ *with* ϑ *correspond to the points of contact of* δ *with the twenty-four common tangents of* γ *and* δ. Of the elliptic quartic curves of the pencil on Γ twenty-four acquire double points and so are rational.

To any two octadic surfaces Γ and Γ' associated with $\mathfrak{N}$ there correspond in σ two conics γ and γ'; these conics have four common points, so that Γ and Γ' have four common tangent quadrics. Moreover, γ and γ' have four common tangents, so that *any two octadic surfaces with the same nodes intersect in four elliptic quartic curves all passing through the nodes.*

As a particular example of a system Σ_2 we may mention the quadrics of $\mathfrak{N}$ which touch an arbitrary line; the envelope of such a system is an octadic surface which contains the line. There is thus an octadic surface containing any given line, and such octadic surfaces correspond to a set of ∞^4 conics in σ.

The points of a curve in σ whose tangential equation is $\psi(l, m, n) = 0$ represent quadrics of $\mathfrak{N}$ whose envelope is $\psi(Q_0, Q_1, Q_2) = 0$; thus the quadrics which are represented by the points of a curve of class μ have for their envelope a surface of order 2μ with μ-ple points at the eight base points of $\mathfrak{N}$. Two such surfaces of orders 2μ and $2\mu'$ intersect in $\mu\mu'$ elliptic quartic curves. In particular, the cones of $\mathfrak{N}$ are represented by the points of δ and therefore have for their envelope† a surface D^{24} of order 24.

6. Suppose now that we take the general octadic surface Γ whose nodes are the base points of $\mathfrak{N}$, and make it pass through a point P of ϑ;

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* If we take two lines in σ and establish a projective relation between them (in which the intersection of the two lines does not correspond to itself), the lines which join corresponding pairs of points touch a conic γ which touches the two lines. Hence an alternative generation of the octadic surface. If we take two pencils, say π and π', of quadrics which belong to $\mathfrak{N}$, and establish a projective relation between them (in which the quadric common to π and π' does not correspond to itself), the surface generated by the quartic curves in which pairs of corresponding quadrics of π and π' intersect is an octadic surface containing the base curves of π and π'.

† Several properties of the surface D^{24} are given in Sturm's paper.

this imposes one linear condition upon Γ. The quadrics of $\mathfrak{N}$ which pass through P all touch the same plane ϖ at P and form a pencil whose base curve has a node at P and passes through the eight nodes of Γ; this quartic curve must lie entirely on Γ. Hence ϖ is the tangent plane of Γ at P and *the condition which is imposed on the octadic surface to make it pass through P causes it also to have a definite tangent plane at P.* It follows that there can be no octadic surface touching ϑ at P unless the tangent of ϑ at P lies in the associated plane ϖ. If it does, as at the twenty-four points I, the octadic surface, if made to pass through the point, touches ϑ there without any further condition being imposed upon it; but if we impose two linear conditions on the surface to make it touch ϑ at a general point P we obtain a surface with a ninth node at P. In order to make an octadic surface have an extra node at a point of ϑ we need only impose *two* linear conditions.

If the conic γ touches δ at a point p, two of the twenty-four common tangents of γ and δ coincide with the tangent at p, so that two of the intersections of the corresponding octadic surface with ϑ coincide with the point P. Hence the octadic surface has a ninth node at P, at least unless P happens to be one of the points of ϑ at which the tangent lies in the associated plane ϖ. But this proviso is, in fact, unnecessary; *if a conic γ touches δ at any point, then the octadic surface Γ has the corresponding point of ϑ for a node.* For we have seen that the only points of ϑ at which the tangent lies in the associated plane ϖ are the twenty-four points I corresponding to the inflections of δ; it is thus quite certain that, if a conic γ touches δ at a point which is not an inflection, the octadic surface Γ has a ninth node at the corresponding point of ϑ. If an octadic surface is made to pass through one of the points I, it touches ϑ there without any further condition being imposed upon it; this corresponds to the fact that, if a conic γ is made to touch an inflectional tangent of δ, this inflectional tangent counts for two among the common tangents of γ and δ. But if γ actually touches δ at a point of inflection i, the tangent of δ at i counts for three among the common tangents of γ and δ; the octadic surface Γ not only touches ϑ at I but has a node at I, the tangent of ϑ being a generator of the nodal cone*.

Since a conic meets δ in eight points there are eight cones of the net $\mathfrak{N}$ which touch an octadic surface along quartic curves. If the conic touches δ at p two of the eight cones coincide with the cone (P). Hence, if an octadic surface Γ_P has a ninth node at the point P of ϑ the cone (P) is part of the cone of tangents which can be drawn from P to Γ_P. A plane

* This may easily be verified algebraically.

through P meets Γ_P in a quartic curve with a node at P, and there are six tangents from P to this curve. The tangent cone from P to Γ_P is therefore a sextic cone, so that it must consist of (P) and a quartic cone; the eight common generators of (P) and the quartic cone are the eight lines which join P to the base points of $\mathfrak{N}$. The tangent cone to Γ_P from any one of the other eight nodes is a non-degenerate sextic cone with eight double generators*.

Octadic surfaces with additional nodes.

7. If an octadic surface passes through an arbitrary point T, which is supposed not to lie on ϑ, it must contain the whole of the elliptic quartic curve which passes through T and the eight nodes; hence the condition which is imposed on the surface in order that it should pass through T causes the surface also to touch a given line at T; wherefore, in order that the surface should have a ninth node at T, it is only necessary to impose three, and not four linear conditions. We should then expect there to be ∞^2 of the surfaces with a ninth node at T, but these all break up into pairs of quadrics. If an octadic surface has a ninth node and does not break up, this node must be on ϑ. When a conic touches δ the corresponding octadic surface has an additional node at the corresponding point of ϑ; thus we can have octadic surfaces with one, two, three, or four additional nodes just as we can have conics touching δ in one, two, three, or four points. This seems to be a direct and conclusive way of obtaining the surfaces.

There are ∞^3 conics which touch δ at a given point p; hence there are ∞^3 octadic surfaces Γ_P which have a ninth node at a given point P of ϑ. They form a linear system, since the two conditions which are imposed on a conic to make it touch the tangent of δ at p are linear in the coefficients of the tangential equation of the conic. They are all touched by the cone (P) along the same quartic curve, this being the base curve of the pencil of quadrics of $\mathfrak{N}$ represented by the tangent of δ at p.

There are ∞^1 conics which touch δ at two given points p and q; they form a pencil of conics having double contact; their equation, whether in point coordinates or tangential coordinates, is a linear combination of the equations of two of them. Hence there are ∞^1 octadic surfaces Γ_{PQ} which have additional nodes at two given points P and Q of ϑ, all the surfaces belonging to the same pencil. The cone (P) touches all the surfaces of the pencil along the quartic curve which is the base curve of the pencil of quadrics of $\mathfrak{N}$ which pass through P; this quartic curve has

* *Cf.* Rohn, *Leipziger Berichte*, 36 (1884), 55.

a double point at P and passes through the eight nodes of the octad. The cone (Q) touches all the surfaces of the pencil along a similar quartic curve. At any point, other than one of the ten nodes, of either of these quartic curves all the surfaces of the pencil touch each other; the base curve of the pencil, in all of order 16, consists of these two quartic curves each counted twice.

The pencil of conics which touch δ at p and q cuts out a linear series g_4^1 on δ which, since δ is of genus 3, has twelve double points. These include the two remaining intersections of the line pq with δ, since the line pq taken twice is a conic of the pencil. There are, therefore, ten points of δ at which it is touched by conics which touch it at p and q; of the pencil of conics which touch δ at p and q ten are tritangent conics of δ. These tritangent conics give rise to octadic surfaces with three additional nodes; *of the pencil of octadic surfaces* Γ_{PQ} *there are ten which have an eleventh node.* The possibility of any surface Γ_{PQ} having an additional node is denied by Cayley*; he argues that, since the pencil of surfaces has only freedom 1 and two conditions have to be imposed on the surface in order that it should have an eleventh node at a given point of ϑ, the two conditions cannot simultaneously be satisfied. But there might well be points R of ϑ which are such that, while the two conditions which are to be imposed on a surface Γ_{PQ} in order that it should have a node at R are independent of each other, yet they form, with the four conditions imposed upon the surface in order that it should have nodes at P and Q, a set of six conditions of which only five are linearly independent. It is clear, from the study of the tritangent conics of δ, that there are ten such points R; there is then a unique octadic surface Γ_{PQR} with nodes at the three points P, Q, R of ϑ. These three points P, Q, R are symmetrically related to each other; an octadic surface which is made to have nodes at two of them and to pass through the third has this third point also as a node.

The result that ten of the surfaces Γ_{PQ} have an eleventh node is given by Rohn†; but in a paper published later he makes the incorrect statement that twelve surfaces of the pencil have an eleventh node, and he is followed in this mistake by Jessop‡. It may therefore be advisable to verify the result, which we may do by considering the linear series g_{20}^1 cut out on ϑ by the pencil of surfaces Γ_{PQ}; each surface of the pencil meets ϑ in twenty points other than P and Q. The linear series has

* *Loc. cit.*, 41; *Papers*, 7, 153.

† *Loc. cit.*, 57.

‡ Rohn, *Math. Annalen*, 29 (1887), 87. Jessop, *Quartic surfaces* (Cambridge, 1916), 16.

forty-four double points, since ϑ is of genus 3; but these include the ten points, other than P and Q, in which ϑ is met by the quadric—$\Phi=0$, say—of $\mathfrak{N}$ which passes through P and Q, $\Phi^2=0$ being a surface of the pencil. There are thirty-four other double points of g^1_{20}; the surface Γ_{PQ} which passes through one of these points either has a node there or else touches ϑ. The surface of the pencil which passes through one of the twenty-four points I necessarily touches ϑ there, and these are the only points at which surfaces of the pencil can touch ϑ. The remaining ten points determine the surfaces of the pencil which have an eleventh node, which surfaces are therefore ten in number.

8. Since there are conics which touch δ in four points, there are octadic surfaces with four additional nodes on ϑ; these surfaces have each twelve nodes in all. No octadic surface can have more than twelve nodes if the base points of $\mathfrak{N}$ are of general position. The quadritangent conics of δ form sixty-three systems, each containing ∞^1 conics; any point of δ is the point of contact of one conic of each system. Hence, of the ∞^3 octadic surfaces which have a ninth node at a given point of ϑ, sixty-three are twelve-nodal surfaces.

The line joining any pair of base points of $\mathfrak{N}$ is a chord of ϑ; a plane through it meets ϑ in four other points which correspond* to the points of contact of a quadritangent conic of δ; this quadritangent conic and the bitangent which corresponds to the join of the two base points make up a contact cubic of δ of the system S. Hence *the four other intersections P, Q, R, S of ϑ with any plane through any one of the joins of the pairs of base points are nodes of a twelve-nodal octadic surface* Γ_{PQRS}. The plane contains six nodes of the surface, since it contains two points of the octad, say 7 and 8, as well as P, Q, R, S. These six nodes lie on a conic, and the plane is a trope of the surface, touching it all along the conic. That the six nodes lie on a conic follows from the fact that the section of $\mathfrak{N}$ by the plane is a net of conics of which 7 and 8 are base points; the Jacobian curve of this net consists of the line 78 and a conic through 7 and 8, and this Jacobian passes through P, Q, R, S. The tangent cone from P to Γ_{PQRS} includes the plane $PQRS$ and the quadric cone (P), these having in common the lines $P7$ and $P8$; the remainder of the tangent cone is the cubic cone whose vertex is P and which contains the nine lines PQ, PR, PS, $P1$, $P2$, $P3$, $P4$, $P5$, $P6$. The tangent cone from Q, R, or S breaks up in a similar way. When P is taken arbitrarily on ϑ, we can at

* Hesse, *Journal für Math.*, 49 (1855), 302.

once identify twenty-eight of the sixty-three sets of three points of ϑ which, when taken with P, are nodes of twelve-nodal octadic surfaces; they are given by the planes joining P to the twenty-eight joins of the pairs of base points. Having given a geometrical construction for these twenty-eight sets of four points to which P belongs, it remains to give the construction for the other thirty-five sets.

9. Let us divide the eight points of the octad into two tetrads; this can be done in thirty-five ways. Take, for example, the two tetrads 1234 and 5678.

We shall use the symbol 5(1234) to denote those quadric cones whose vertex is at 5 and which contain the four lines 51, 52, 53, 54; with corresponding symbols for other similar sets of cones. These cones 5(1234) form a pencil; since 51, 52, 53, 54 are all chords of ϑ, all the cones of the pencil have eight fixed intersections with ϑ and meet ϑ further in sets of four points forming a g_4^1. Now one cone of the pencil consists of the pair of planes 512, 534; the set of the g_4^1 given by this pair of planes consists of the two intersections of ϑ with 12 and of the two intersections of ϑ with 34. Similarly the two intersections of ϑ with 13 and the two intersections of ϑ with 24 also make a set of the g_4^1, while the two intersections of ϑ with 14 and the two intersections of ϑ with 23 make up a third set of the g_4^1. But it follows, in precisely the same way, that the pencil of cones 6(1234) also cuts out a g_4^1 on ϑ to which these same three sets of four points belong; the two linear series g_4^1 must therefore be the same. Further, the same g_4^1 is cut out on ϑ by the pencil of cones 7(1234) and by the pencil of cones 8(1234). Hence, we have the following: if an arbitrary point P on ϑ is taken, then the four cones 5(1234P), 6(1234P), 7(1234P), and 8(1234P) have in common three further points Q, R, S of ϑ. The set of points $PQRS$ is that set of the g_4^1 which contains P. The eight points 1, 2, 3, 4, P, Q, R, S are therefore eight associated points, and the sextic curve which is the locus of vertices of cones through these points passes through 5, 6, 7, and 8.

It is a well-known property of eight associated points that the twisted cubic through any six of them has the line joining the other two as a chord, the two intersections of the cubic with this line being the two points in which the line meets the sextic curve which is the locus of vertices of cones through the points. Hence the cone 5(12346), which contains the twisted cubic 123456, meets 78 in its two intersections with ϑ; thus the pair of points in which ϑ is met by 56 and the pair of points in which ϑ is met by 78 form a set of the g_4^1. Similarly there is a set of the g_4^1 which consists of the intersections of ϑ with 57 and 68 and another set of the g_4^1 which

consists of the intersections of ϑ with 58 and 67. It now follows that this same g_4^1 is cut out on ϑ by the pencil of cones 1(5678), by the pencil of cones 2(5678), by the pencil of cones 3(5678), and by the pencil of cones 4(5678). Whence the eight points 5, 6, 7, 8, P, Q, R, S are also eight associated points, and the sextic curve which is the locus of vertices of cones through them passes through 1, 2, 3, and 4.

The twelve intersections of ϑ with any quadric correspond to the twelve intersections of δ with a plane cubic; the cubics of the plane are associated in this way with the quadrics of space*. Consider, in particular, the intersections of ϑ with the cone $5(1234P)$; they consist of the four pairs of points in which ϑ is met by its chords 51, 52, 53, 54 and of the four points P, Q, R, S. The corresponding points on δ are the eight points of contact of four bitangents and four points p, q, r, s; these twelve points must form the complete intersection of δ with a cubic curve. Hence, since the points of contact of any three bitangents of δ are the points of contact of a contact cubic (consisting of the three bitangents), the points of contact of the fourth bitangent and the four points p, q, r, s are also the points of contact of a contact cubic (belonging to the same system as does that formed by the other three bitangents); this latter contact cubic must break up into the fourth bitangent and a quadritangent conic touching δ at p, q, r, s. Whence, since p, q, r, s are the points of contact of a quadritangent conic of δ, P, Q, R, S *are the four additional nodes of a twelve-nodal octadic surface* Γ_{PQRS}. We thus see how, given an arbitrary point P of ϑ, we can find, for any division of the eight points 1, 2, 3, 4, 5, 6, 7, 8 into two tetrads, three points Q, R, S on ϑ which make up with P a set of additional nodes of a twelve-nodal octadic surface Γ_{PQRS}. There are thirty-five such sets of three points, one set arising for each of the thirty-five possible divisions of the octad into two tetrads.

10. We have now accounted for all the sixty-three twelve-nodal octadic surfaces Γ_{PQRS} which have a ninth node at a given point P of ϑ; twenty-eight of these surfaces have six coplanar nodes and the other thirty-five are such that their twelve nodes consist of three tetrads of points, each pair of tetrads forming a set of eight associated points†. This latter type of surface can therefore be regarded as octadic in three different ways; it is "triply octadic", and it contains three pencils of elliptic quartic curves. Of the sixty-three different systems of octadic surfaces

* Hesse, *loc. cit.*, 289.

† Kummer, *Abhandlungen der Akademie zu Berlin* (1866), 100–101; Stahl, *Journal für Math.*, 95 (1883), 301; Rohn, *Math. Annalen*, 29 (1887), 88; Cayley, *B.A. Report* (1886), 540–541; *Papers*, 13, 1–2.

corresponding to the sixty-three systems of quadritangent conics of δ, twenty-eight systems of surfaces have six coplanar nodes and the remaining thirty-five systems consist of triply octadic surfaces. The eight points of contact of two quadritangent conics of δ of the same system lie on a conic, and the eight intersections of δ with any conic correspond to eight associated points* on ϑ. Hence, if we take two triply octadic surfaces which belong to the same system, their two sets of four nodes on ϑ form eight associated points.

These two types of twelve-nodal octadic surfaces were first obtained by Kummer† as the focal surfaces of line congruences of the second order and sixth class. A line congruence of the second order and sixth class has six double rays which may either all pass through the same point or else be the six edges of a tetrahedron; in the former case the focal surface is the twelve-nodal quartic surface of which eight nodes are associated points and the remaining four are coplanar; in the latter case the focal surface is a triply octadic surface. The former type of twelve-nodal octadic surface (with six coplanar nodes) is the apparent contour of a six-nodal three-dimensional cubic locus in [4] from a point of itself; it was so obtained by Castelnuovo‡ and by Segre§. The dual surface, of the fourth class, with one node and twelve singular tangent planes, was obtained by Reye‖.

Brianchon sets of bitangents.

11. We may also consider briefly octadic surfaces which correspond to conics which touch bitangents of δ. Let 1, 2, 3, 4, 5, 6, 7, 8 be the base points of $\mathfrak{N}$; the line 78, for example, is a chord of ϑ, the two intersections corresponding to the two points of contact of δ with a bitangent. We call this bitangent 78, and we have all the twenty-eight bitangents in this way. If an octadic surface has its nodes at 1, 2, 3, 4, 5, 6, 7, 8 and meets ϑ in one of its two intersections with 78 the surface must contain¶ the

* Hesse, *loc. cit.*, 292.

† *Loc. cit.*, 94–107.

‡ *Atti del R. Istituto Veneto* (6), 5 (1887), 1249.

§ *Memorie Torino* (2), 39 (1888), 3–48 (15).

‖ *Journal für Math.*, 86 (1879), 94.

¶ A line joining a pair of nodes of a quartic surface has two intersections with it at each node; if a point of the line other than the two nodes is on the surface the line must lie entirely on the surface as meeting it in more than four points. In order that an octadic surface should contain the line joining a pair of its nodes, one linear condition must be imposed upon the surface.

line 78, and hence also meets ϑ in its other intersection with 78; this corresponds to the fact that any conic which touches the tangent of δ at the point corresponding to one of the two intersections of 78 with ϑ must also touch the tangent of δ at the point corresponding to the other intersection, and may be taken as a proof that the two intersections of ϑ with 78 correspond to the points of contact of δ with a bitangent.

The relationship between conics γ and octadic surfaces Γ lends itself in particular to the consideration of Brianchon sets of bitangents; a Brianchon set of bitangents is defined as a set of six bitangents which all touch the same conic.

The conic which touches the five bitangents 12, 13, 14, 15, 16 corresponds to an octadic surface containing the five lines 12, 13, 14, 15, 16; there is just one such surface*. The quadric cone containing these five lines must be the nodal cone of Γ at 1; it meets Γ further in a twisted cubic which must pass through the vertex 1 and through the five points 2, 3, 4, 5, 6 which are nodes of Γ. But this cubic is none other than 123456, which therefore lies on Γ. In particular its two intersections† with 78 lie on Γ, so that Γ contains the line 78, and the conic which touches the five bitangents 12, 13, 14, 15, 16 also touches the bitangent 78. The six bitangents therefore form a Brianchon set. We obtain the same result if we take the conic to touch the five bitangents 12, 13, 14, 15, 78. For, in order to prove that the octadic surface Γ which contains the five lines 12, 13, 14, 15, 78 also contains the line 16, it is sufficient to prove that Γ contains the cubic curve 234578, since this curve has 16 as a chord. Suppose that the plane 678 meets the lines 12, 13, 14, 15 in α, β, γ, δ; it meets Γ in the line 78 and in a cubic curve which has a node at 6 and passes through the six points 7, 8, α, β, γ, δ. Now the cubic curve 234578 is projected from 1 on to the plane 678 into a cubic curve which also has a node at 6 and passes through the six points 7, 8, α, β, γ, δ; and these two cubic curves in the plane 678 must coincide as having a common node and six common points. Whence it follows that the intersection, other than 7 and 8, of the cubic curve 234578 with the plane 678 lies on the surface Γ; and, further, that the cubic curve 234578 lies entirely on Γ as it passes through six of its nodes and meets it in a further point. Thus Γ contains the line 16.

A set of six bitangents of the type 12, 13, 14, 56, 57, 58 also forms a Brianchon set. For consider the octadic surface Γ which contains the

* See foot-note ¶ on p. 507.

† The cubic through any six of a set of eight associated points has the line joining the other two as a chord.

five lines 12, 13, 14, 56, 57. If the lines 12, 13, 14 meet the plane 567 in points α, β, γ the surface Γ meets the plane in the lines 56, 57 and the conic $\alpha\beta\gamma 67$. Now the cubic curve 123467 is projected from 1 on to the plane 567 into this conic $\alpha\beta\gamma 67$; hence the intersection, other than 6 and 7, of the cubic with the plane 567 lies on Γ. Whence it follows that Γ contains the cubic curve 123467, and therefore the line 58.

To the conic touching the five bitangents 12, 23, 31, 45, 46 there corresponds an octadic surface containing the five lines 12, 23, 31, 45, 46. The plane 123 meets this surface in the lines 12, 23, 31 and another line which must pass through the two points in which the plane is met by the lines 45 and 46; this line is the intersection of the two planes 123 and 456, and therefore meets 56. Hence the octadic surface contains the line 56, so that the conic which touches the five bitangents 12, 23, 31, 45, 46 also touches 56. The six bitangents 12, 23, 31, 56, 64, 45 therefore form a Brianchon set.

II.

12. It has already (§ 3) been pointed out that the plane ϖ associated with a point P of ϑ contains the vertices A and B of two cones (A) and (B) which both pass through P, these being the only cones of $\mathfrak{N}$ other than (P) which do so. The plane ϖ meets ϑ in three points X, Y, Z other than P, A, B; since all lines in ϖ which pass through P are generators of quadrics of $\mathfrak{N}$ the three lines PX, PY, PZ belong to the complex V. Thus *through any point of ϑ there pass three of its chords which are generators of quadrics of $\mathfrak{N}$ which are not cones*; the chords of ϑ which are generators of quadrics of $\mathfrak{N}$ which are not cones therefore form a ruled surface on which ϑ is a triple curve. Now a line of V which meets ϑ either lies in the plane ϖ associated with its intersection with ϑ or else is a generator of the cone of $\mathfrak{N}$ whose vertex is at that point; in the former case its intersection with ϑ is one of the double points of the involution which is cut out on the line by the quadrics of $\mathfrak{N}$. It follows that, if a chord of ϑ belongs to V and is not a generator of any cone of $\mathfrak{N}$, its two intersections with ϑ are the double points of the involution on the line and are conjugate in regard to every quadric of $\mathfrak{N}$. Thus the three points X, Y, Z are all conjugate to P in regard to every quadric of $\mathfrak{N}$; so that the polar plane of P in regard to each quadric of $\mathfrak{N}$ passes through X, Y, and Z. Thus *X, Y, Z are collinear, and the line XYZ is a trisecant of ϑ*. There is thus established a (1, 1) correspondence* between the points and trisecants of ϑ; we shall say that corresponding point and trisecant are *conjugate* to each other. If Q, R, S are three points of ϑ which, taken with P, are the four

* Sturm, *loc. cit.*, 231–232; Reye, *Die Geometrie der Lage*, 3 (Leipzig, 1923), 138.

vertices of the cones of a pencil of quadrics belonging to $\mathfrak{N}$, the plane QRS is the polar plane of P in regard to all the quadrics of the pencil and must, therefore, pass through XYZ; hence the planes of the triads of points of ϑ which form, together with a given point of ϑ, sets of four points which are the vertices of cones of a pencil all pass through the trisecant conjugate to the given point. If two points of ϑ are conjugate in regard to all quadrics of $\mathfrak{N}$, they are symmetrically related to each other; the trisecant conjugate to either point passes through the other. Thus the three trisecants conjugate to X, Y, and Z all pass through P; there are exactly three trisecants passing through any point of ϑ, because the projection of ϑ from the point on to a plane is a quintic of genus 3, having therefore three double points.

Let X be the sixth point of intersection with ϑ of the plane containing two of its trisecants PP_1P_2, PP_3P_4 which pass through P. The polar line of this plane in regard to the cone (X) must contain the point Y conjugate to the trisecant PP_1P_2 and the point Z conjugate to the trisecant PP_3P_4; it is therefore the trisecant XYZ conjugate to P. Hence, if we take a plane which contains two trisecants through a point P of ϑ, the sixth intersection of the plane with ϑ is the point conjugate to the third trisecant through P.

13. The six intersections of ϖ with ϑ correspond to six points of contact of δ with a contact cubic of the system S, and we have seen (§ 3) that, if p on δ corresponds to P on ϑ, then the tangent of δ at p meets it again in the two points a and b which correspond to A and B. The contact cubic must therefore contain the line pab as a part; it will consist of this line and a conic passing through a and b and touching δ at the three points x, y, z which correspond to X, Y, Z. There are, in fact, thirty-six conics which pass through a and b and touch δ at each of three other points*; these, when taken with the line pab, form the contact cubics three of whose points of contact are p, a, b and which belong to the thirty-six systems of the first kind. The points x, y, z are the points of contact of that tritangent conic passing through a and b which, when taken with the line pab, forms a contact cubic of the system S. We thus see† how to pass from the point of δ which corresponds to P to the three points of δ which correspond to the three points of ϑ on the trisecant conjugate to P.

Since there is a conic touching δ at the three points which correspond to the three intersections of ϑ with any one of its trisecants *there is an*

* *Cf.* W. P. Milne, *Journal London Math. Soc.*, 6 (1931), 95.

† *Cf.* Scorza, *Math. Annalen*, 52 (1899), 457–8.

octadic surface with nodes at the three intersections of ϑ with any one of its trisecants. Such a surface has eleven nodes in all and the trisecant lies on the surface.

The lines of V which lie in a plane ϖ either pass through P or touch some conic y. Each quadric of $\mathfrak{N}$ which touches ϖ at a point other than P meets ϖ in a pair of tangents of y; hence the tangents of y are paired in an involution, and the locus of intersections of the pairs of tangents is a line. Hence the quadrics of $\mathfrak{N}$ which touch ϖ at points other than P do so at the points of a line; this line must be XYZ, for the three cones (X), (Y), (Z) must be considered as touching ϖ at X, Y, Z.

The quadrics which touch ϖ at P are represented in σ by the points of the line pab. Hence the quadrics which touch ϖ at the points of XYZ are represented in σ by the points of the tritangent conic xyz, and their envelope is the octadic surface Γ_{XYZ} which has nodes at X, Y, and Z. Thus ϖ touches Γ_{XYZ} all along XYZ (in fact, if any quartic surface has three collinear nodes it must have the same tangent plane at every point of the line joining the nodes). Through any point O of ϖ there pass two quadrics which touch Γ_{XYZ} along quartic curves; of the two tangents from O to y one lies on each quadric. For a point of y itself the two quadrics coincide, so that y is on Γ_{XYZ}. The complete intersection of ϖ with Γ_{XYZ} consists of the line XYZ counted twice and the conic y.

The cones (A) and (B), being represented by the points a and b of δ, touch ϖ at P and also at points of the conic y; in fact, they touch ϖ all along their respective generators AP and BP. Now the quadric which touches ϖ at any point of XYZ meets ϖ in the two tangents of y which pass through the point; hence PA and PB must be the two tangents of y at its intersections with XYZ. PA and PB are flecnodal tangents of Γ_{XYZ}, each of them having four-point contact with the surface.

14. Since there is one and only one plane passing through three arbitrary points of ϑ, there is one and only one contact cubic of the system S having three arbitrary contacts with δ; but through three points of ϑ which lie on a trisecant there pass the planes of a pencil, so that there must be a singly-infinite family of contact cubics of the system S which touch δ at the three points which correspond to the intersections of ϑ with any one of its trisecants. Moreover, since a plane through a trisecant of ϑ meets ϑ again in three points which, together with the point conjugate to the trisecant, are the vertices of four cones belonging to a pencil of quadrics, the three remaining contacts of any one of the cubics with δ are collinear, and the line on which they lie meets δ again in the point which corresponds to that point of ϑ which is conjugate to the trisecant.

Let $C_4 = 0$ be the equation of δ, and let $T = 0$ be the equation of its tangent at p. Suppose also that $C_2 = 0$ is the equation of the conic which passes through a and b and is such that $TC_2 = 0$ is a contact cubic of the system S; then $C_2 = 0$ touches δ at x, y, and z. If $L = 0$ is any line through p the equation

$$C_4 - \lambda L^2 C_2 = 0$$

represents, for varying λ, a pencil of quartic curves; each curve of the pencil passes through a and b and touches δ at x, y, z, and its four intersections with $L = 0$. There is, therefore, a curve of the pencil which contains, as a part of itself, the tangent of δ at p, so that we may write

$$C_4 - L^2 C_2 \equiv TC_3,$$

where $C_3 = 0$ is a contact cubic of δ whose six points of contact are x, y, z, and the three intersections, other than p, of δ with $L = 0$. Moreover, this contact cubic belongs to the system S, since the six points of contact of $C_3 = 0$ and the six points of contact of $TC_2 = 0$ form together the twelve intersections of δ and the cubic $LC_2 = 0$. Hence, since $L = 0$ may be any line through p, we obtain a singly-infinite family of contact cubics which all belong to the system S and touch δ at x, y, and z. Moreover, the three remaining contacts of any one of the cubics with δ are collinear and the line joining them meets δ again in p.

All the cubics of the family touch $C_2 = 0$ at x, y, and z; hence, owing to the properties of residual sets of points on a cubic, the three remaining intersections of any two of them are collinear. Corresponding to any two of the cubics, say $C_3 = 0$ and $C_3' = 0$, we have identities of the form

$$C_4 - L^2 C_2 \equiv TC_3, \quad C_4 - L'^2 C_2 \equiv TC_3',$$

whence

$$(L'^2 - L^2) C_2 \equiv T(C_3 - C_3');$$

taking $T \equiv L' + L$, we must have $C_3 - C_3' \equiv (L' - L) C_2$. Hence the three intersections, other than x, y, and z, of any two cubics of the family lie on a line passing through p; and this line is the harmonic conjugate of the tangent of δ at p in regard to the two lines on which the collinear triads of contacts lie. Whence also there are three quadrics which belong to $\mathfrak{N}$, which are not cones and which touch both of two planes passing through the same trisecant of ϑ; the three quadrics belong to a pencil and one of the cones of the pencil has its vertex at the point of ϑ which is conjugate to the trisecant. The plane containing the vertices of the other three cones of the pencil passes through the trisecant and is the harmonic conjugate, in regard to the two given planes, of the plane which joins the trisecant to its conjugate point.

There is a unique plane which contains a trisecant and an arbitrary point of ϑ; hence there is a unique cubic of the family which touches δ at x, y, z, and at one other arbitrary point. Any line through p meets δ in three points which are the points of contact of one and only one cubic of the family, which also touches δ at x, y, and z. Since there are ten tangents from p to δ, there are ten cubics of the family whose contacts with δ, other than x, y, and z, consist of an ordinary contact and a contact of the third order or four-point contact. Correspondingly there are ten planes which pass through a trisecant of ϑ and touch ϑ elsewhere.

Through any trisecant of ϑ there pass two tangent planes of any quadric of $\mathfrak{N}$ which is not a cone; but for a cone the two planes coincide with the plane joining the trisecant to the vertex of the cone. Hence through an arbitrary point O of σ there pass two cubics of the family which touch δ at x, y, and z and belong to the system S; the two cubics have two other intersections on the line Op. If O lies on δ, the two cubics coincide.

15. We have seen in § 12 that, if a line of V meets ϑ and is not a generator of the cone of $\mathfrak{N}$ whose vertex is at its intersection with ϑ, then this intersection must be a double point of the involution which is cut out on the line by the quadrics of $\mathfrak{N}$. If a trisecant of ϑ is a line of V, then, since an involution has only two double points, it must be* a generator of a cone of $\mathfrak{N}$ whose vertex is at one of its three intersections with ϑ.

Suppose then that EMN is a trisecant of ϑ and a generator of (E). M and N are conjugate points in regard to every quadric of the net, so that the trisecant conjugate to M passes through N and the trisecant conjugate to N passes through M. The quadrics of $\mathfrak{N}$ which pass through M all touch the same plane ϖ_M at M, and the quadrics of $\mathfrak{N}$ which pass through N all touch the same plane ϖ_N at N; the planes ϖ_M and ϖ_N are therefore the same plane—the tangent plane of (E) along its generator EMN. This plane, as being associated with M, must contain the trisecant conjugate to M, the point E, and one other point G of ϑ which is the vertex of a cone passing through M; as being associated with N it must contain the trisecant conjugate to N, the point E, and one other point F of ϑ which is the vertex of a cone passing through N. The trisecants conjugate to M and N both lie in the plane and intersect in a point L of ϑ, which is

* A trisecant of $\mathfrak{F}$ cannot be a generator of *two* cones of $\mathfrak{N}$. Two cones with a common generator determine a pencil of quadrics whose base curve consists of this common generator and a twisted cubic; the only such pencils of quadrics which belong to $\mathfrak{N}$ are those which contain a line joining a pair of base points of $\mathfrak{N}$.

therefore conjugate to the trisecant EMN; the plane is the ϖ-plane associated with each of the points L, M, N and is a triple plane of the developable formed by the ϖ-planes. It touches the three cones (E), (F), (G) along their respective generators EMN, FNL, GLM; these three lines are all trisecants of ϑ which belong to V. *The trisecants of ϑ which belong to V are therefore distributed in coplanar sets of three.*

It is also true, conversely, that, if three trisecants of ϑ are coplanar, they all belong to V. For suppose that EMN, FNL, GLM are three coplanar trisecants of ϑ. Since FNL is a trisecant the three points G, E, M are the vertices of three cones of $\mathfrak{N}$ which belong to the same pencil; hence the two lines in which (E) meets the plane are harmonically conjugate in regard to EG and EM. Similarly, the same two lines are harmonically conjugate in regard to EF and EN. Hence, since EM and EN are the same line, the cone (E) must touch the plane* along the line EMN, which therefore belongs to V. Similarly, FNL and GLM also belong to V.

The ruled surface generated by the trisecants of ϑ is a surface R_2^8 of order eight†. Hence, since V is a cubic complex, there are twenty-four trisecants of ϑ which belong to V. These trisecants lie three in each of eight planes. These eight planes are tritangent planes of R_2^8 and, since three coplanar trisecants of ϑ are necessarily lines of V, are the only tritangent planes which R_2^8 possesses.

The envelope of the third class, which is formed by the lines of V lying in the plane LMN, consists of the three pencils of lines through L, M, N respectively. The net of conics in which the quadrics of $\mathfrak{N}$ meet the plane consists of those conics for which LMN is a self conjugate triangle.

16. The grouping of the twenty-four trisecants of ϑ, which belong to V, into coplanar triplets is also seen by considering the plane quartic δ. For to the three points E, M, N on a trisecant of ϑ correspond three points e, m, n of δ which are points of contact of a tritangent conic whose two other intersections f, g with δ are such that their join touches δ at some

* The only alternative is that E, F, G should be collinear, which they are not; *cf.* §19.

† Sturm, *loc. cit.*, 231. The order of the surface also follows from a general formula given by Zeuthen for the degree of the ruled surface formed by the trisecants of a curve, *Annali di Mat.* (2), 3 (1869), 183–185. When we know the order of the ruled surface formed by the trisecants the class of the developable formed by the planes ϖ can be obtained at once. For the planes ϖ join the points of $\mathfrak{d}$ to their conjugate trisecants, and so join the points of a curve of order six to corresponding generators of a ruled surface of order eight, there being a (1, 1) correspondence between the points of the curve and the generators of the ruled surface. Hence, since no point of $\mathfrak{d}$ lies on its conjugate trisecant, the class of the developable is $6 + 8 = 14$.

point l. But, if EMN is a generator of (E), the points m and n of δ must be points of contact of tangents which pass through e; hence the tritangent conic must break up into the pair of lines em, en which touch δ at m and n and meet it again in g and f respectively; δ passes through e, f, g and touches fg, ge, ef respectively at l, m, n. The six points are the points of contact of a contact cubic of the system S which breaks up into three lines. Since fl, fn touch δ at l and n, the point F of ϑ is the vertex of a cone (F) passing through L and N; moreover, since ge touches δ at m and the six points are the points of contact of a contact cubic of the system S, FLN is a trisecant of ϑ. Similarly, GLM is a trisecant of ϑ and a generator of (G); and we again obtain a plane containing the three trisecants EMN, FNL, GLM.

The triangle efg has its vertices on δ, and each of its sides touches δ; it is both inscribed and circumscribed to δ. Thus the system S of contact cubics contains eight triangles of this kind; there are thus eight in-and-circumscribed triangles belonging to each of the thirty-six systems of contact cubics of the first kind. A contact cubic which belongs to a system of the second kind has its six points of contact on a conic and so cannot break up into a triangle of this nature. Hence there are in all 288 *triangles both inscribed and circumscribed to* δ.

17. A digression may now be made in order to mention some results obtained by Schur* and to speak of their connection with the present work.

We may call a twisted sextic of genus 3 which is the locus of the vertices of cones which belong to a net of quadrics a Hesse sextic. It is determined by the net of quadrics, and therefore by the assignation of seven of its base points; the Hesse sextics in space are ∞^{21} in aggregate. But a twisted sextic of genus 3 in space (defined, for example, as the residual intersection of two cubic surfaces through a twisted cubic curve) is *not* in general the locus of the vertices of cones belonging to a net of quadrics. We shall call the general curve a Schur sextic and denote it by $\mathfrak{S}$, using ϑ as before to denote a Hesse sextic. The Schur sextics in space are ∞^{24} in aggregate and include all the Hesse sextics; Hesse himself obtains ϑ as the curve common to two cubic surfaces through a twisted cubic†. A Hesse sextic remains a Hesse sextic when it is subjected to a projective transformation; it is not projectively equivalent to the general Schur curve and, considered from this point of view, is a specialization of the Schur curve. If, however, we consider two curves to be equivalent when

* *Math. Annalen*, 18 (1881), 1.

† *Journal für Math.*, 49 (1855), 281–282.

they can be placed in *birational* correspondence with each other, then the Hesse sextic is *not* a specialization of the Schur curve. For any curve $\mathfrak{S}$, being of genus 3, can be put into (1, 1) correspondence with a plane quartic—the canonical curve of genus 3—and the plane quartic can always be put into (1, 1) correspondence with a Hesse sextic. Thus a Schur sextic is always birationally equivalent to a Hesse sextic.

18. We have seen that there is a relation of conjugacy between the points and trisecants of ϑ; there is also a (1, 1) correspondence between the points and trisecants of $\mathfrak{S}$. This correspondence was established by Schur* by means of certain quadruplets of points on $\mathfrak{S}$. These quadruplets are, in fact, simply the sets of the canonical series $g_4{}^2$ on $\mathfrak{S}$.

Take any two cubic surfaces through $\mathfrak{S}$; they determine a pencil H of cubic surfaces whose base curve consists of $\mathfrak{S}$ and a twisted cubic c^3 meeting $\mathfrak{S}$ in eight points†. The net of quadrics which pass through c^3 cuts out the canonical series‡ $g_4{}^2$ on $\mathfrak{S}$. We consider those sets of $g_4{}^2$ which contain a given point P of $\mathfrak{S}$; they are cut out on $\mathfrak{S}$ by the pencil of quadrics passing through c^3 and P, these quadrics also containing the chord g of c^3 which passes through P. There is one cubic surface K of the pencil H which contains g. A quadric $Q=0$ which contains c^3 and P meets K in c^3, g, and a conic q; the three intersections, other than P, and which do not lie on c^3, of $Q=0$ with $\mathfrak{S}$ lie on q. The plane of q, $\alpha=0$ say, meets K further in a line and $\mathfrak{S}$ in three points which must lie on this line. *Hence the plane which contains any set of three points of $\mathfrak{S}$ which belong to the same canonical set meets $\mathfrak{S}$ further in three collinear points.* Moreover, we obtain the same set of three collinear points whichever of the ∞^1 sets of three points we take which make up canonical sets with P. For let $Q_1=0$ and $Q_2=0$ be the equations of any two quadrics which pass through c_3 and P, and let $\alpha_1=0$ and $\alpha_2=0$ be the planes of the two conics q_1 and q_2 in which the quadrics respectively meet K. Then the two cubic surfaces $Q_1\alpha_2=0$ and $Q_2\alpha_1=0$ have in common fifteen points of $\mathfrak{S}$, viz. P, the

* *Loc. cit.*, 17.

† If two curves of orders n_1 and n_2, with h_1 and h_2 apparent double points, form the complete intersection of two surfaces of orders μ and ν, then, if i is the number of intersections of the two curves,

$$i=n_1(\mu+\nu-n_1-1)+2h_1=n_2(\mu+\nu-n_2-1)+2h_2.$$

‡ If two curves together form the complete intersection of two surfaces of orders μ and ν, the canonical series on either curve is cut out by the surfaces of order $\mu+\nu-4$ which pass through the other.

eight intersections of $\mathfrak{S}$ with c_3, the three intersections of $\mathfrak{S}$ with q_1, and the three intersections of $\mathfrak{S}$ with q_2. But, since $\mathfrak{S}$ is of genus 3, when fifteen of the intersections of $\mathfrak{S}$ with a cubic surface are given the remaining three are thereby determined*; the two cubic surfaces $Q_1\alpha_2 = 0$ and $Q_2\alpha_1 = 0$ therefore meet $\mathfrak{S}$ further in the same set of three points. But $Q_1\alpha_2 = 0$ meets $\mathfrak{S}$ further in three collinear points in the plane $\alpha_2 = 0$ and $Q_2\alpha_1 = 0$ meets $\mathfrak{S}$ further in three collinear points of the plane $\alpha_1 = 0$; whence it follows that the line of intersection of the two planes $\alpha_1 = 0$ and $\alpha_2 = 0$ is a trisecant of $\mathfrak{S}$.

Many properties of $\mathfrak{S}$ are best obtained by regarding it as a section of a sextic surface in [4]; the canonical sets on $\mathfrak{S}$ are then given by ∞^2 rational quartic curves which lie on the surface.

The sets of the canonical series on a plane quartic δ are the sets of four points in which it is met by the lines of the plane; hence the sets of the canonical series on ϑ are the sets of four vertices of cones which belong to the same pencil of quadrics. Schur's result thus includes that of §12; but it should be noticed that, whereas, in the conjugacy between points and trisecants of ϑ, if P' is a point of ϑ on the trisecant conjugate to P, then the trisecant conjugate to P' passes through P, this is not true for a curve $\mathfrak{S}$.

Any two sets of the canonical series on ϑ form a set of eight associated points†. Conversely, if two canonical sets on a twisted sextic of genus 3 form a set of eight associated points, then any two canonical sets do so and the curve is a Hesse curve‡.

It has been stated that the plane containing any three points of $\mathfrak{S}$ which belong to the same canonical set meets $\mathfrak{S}$ further in three collinear points; it is also true, conversely, that any plane containing a trisecant of $\mathfrak{S}$ meets $\mathfrak{S}$ further in three points which belong to the same canonical set. For the planes through a trisecant cut out a g_3^1 on $\mathfrak{S}$ and, on a curve of genus 3, a g_3^1 is, by the Riemann-Roch theorem, necessarily a special series. The sets of the g_3^1 form canonical sets with a fixed point of $\mathfrak{S}$, this being the point associated with the particular trisecant of $\mathfrak{S}$ chosen. We therefore have the following: *in any birational correspondence between a curve $\mathfrak{S}$ and a plane quartic three points of $\mathfrak{S}$ whose plane contains a trisecant always correspond to three collinear points of the plane quartic, and conversely.* Those sets of three points of $\mathfrak{S}$ whose planes contain the same tri-

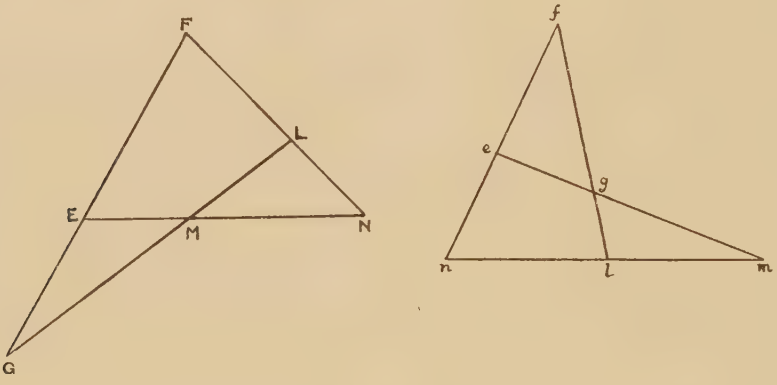
* The surfaces of any given order $l \geqslant 3$ cut out on $\mathfrak{S}$ a non-special complete linear series; hence the postulation of $\mathfrak{S}$ for surfaces of order l is $6l-2$. The postulation of $\mathfrak{S}$ for cubic surfaces is 16; all cubic surfaces cut out a g_{18}^{15} on $\mathfrak{S}$.

† Hesse, *loc. cit.*, 288.

‡ Schur, *loc. cit.*, 22.

secant correspond to sets of three collinear points whose joins meet the quartic again in the same point.

If there should happen to be a plane which meets $\mathfrak{S}$ in the six vertices of a quadrilateral, these six points must correspond to six points on the plane quartic which are also the six vertices of a quadrilateral; the four sets of three collinear points of $\mathfrak{S}$ correspond to the four sets of three non-collinear points on the plane quartic, and conversely, as indicated in the figure.



19. The Hesse curve ϑ is the locus of vertices of the cones which belong to a net of quadrics, and, in general, a net of quadrics cannot be regarded as the net of polar quadrics of points of a plane in regard to a cubic surface*. If, however, we have a net of quadrics which are the polar quadrics of the points of a plane in regard to a cubic surface, we shall call the twisted sextic of genus 3, which is the locus of vertices of the cones belonging to the net, a Reye sextic† and denote it by $\mathfrak{R}$. The necessary and sufficient condition that a Hesse sextic should be a Reye sextic is that there should be a plane which meets it in the six vertices of a quadrilateral.

Suppose that there is a plane which contains four trisecants EMN, FNL, GLM, EFG of a Hesse sextic; and let O be the point conjugate to the trisecant EFG. The trisecants conjugate to E, F, G pass through L, M, N respectively (§ 12), and they all pass through O; hence they are the lines OL, OM, ON. Let X, Y, Z be the respective further intersections of these lines with the curve; they are conjugate to the trisecants EMN, FNL, GLM. Then, since Y is conjugate to FNL and Z to GLM, the trisecant conjugate to L passes through Y and Z; hence, since the

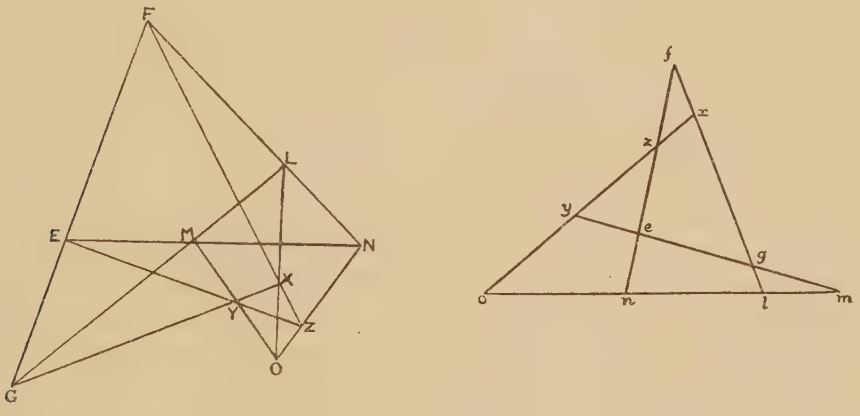
* *Cf.* Frahm, *Math. Annalen*, 7 (1874), 635; Töplitz, *ibid.*, 11 (1877), 434; Salmon, *Geometry of three dimensions* (Dublin, 1882), 208.

† *Cf.* Reye, *Die Geometrie der Lage*, 3 (Leipzig, 1923), Vorträge 15, 16.

trisecant conjugate to L passes through E (§ 12), the line YZ passes through E. Similarly, ZX passes through F and XY through G; the trisecants EYZ, FZX, GXY are conjugate respectively to the points L, M, N. We have thus five planes forming a pentahedron; each plane of the pentahedron meets the curve in the six vertices of a quadrilateral, each pair of planes has a trisecant of the curve as its line of intersection, and any three planes meet in a point of the curve. The vertices of the pentahedron are ten points of the curve, while the edges of the pentahedron are ten trisecants of the curve, each vertex being conjugate to the opposite edge.

There are ∞^1 such pentahedra inscribed in $\mathfrak{R}$; the faces of the pentahedra all osculate the same twisted cubic*. Any plane which contains two trisecants meeting in a point of $\mathfrak{R}$ contains two other trisecants and meets $\mathfrak{R}$ in the six vertices of a quadrilateral, pairs of opposite vertices being conjugate points in regard to every quadric of the net. The ruled surface formed by the trisecants of $\mathfrak{R}$ has now not eight tritangent planes, but an infinity of quadritangent planes—the faces of the pentahedra.

Now let Λ be any plane quartic which is in birational correspondence with $\mathfrak{R}$. The six points E, F, G, L, M, N of $\mathfrak{R}$ correspond to



six points e, f, g, l, m, n of Λ which are six vertices of a quadrilateral. Similarly to the six intersections of $\mathfrak{R}$ with a face of any one of its inscribed pentahedra there correspond six points of Λ which are vertices of a quadrilateral. Hence to the ten points of $\mathfrak{R}$ which are the vertices of any one of the pentahedra there correspond ten points of Λ which are the intersections of five lines; the correspondence is as indicated in the figure. Thus, associated with any pentahedron inscribed in $\mathfrak{R}$ we have a

* *Cf.* Schur, *loc. cit.*, 23–27.

pentagram inscribed in Λ; Λ is therefore circumscribed to ∞^1 pentagrams and is a Lüroth quartic*. Thus *any plane quartic which is birationally equivalent to a Reye sextic is a Lüroth quartic*†.

It is, of course, by no means true, conversely, that any twisted sextic which is birationally equivalent to a Lüroth quartic is a Reye sextic; but, given any Lüroth quartic Λ, we can always find at least one Reye sextic with which it is in birational correspondence. In order to do this we have only to take a certain system, of the first kind, of contact cubics of Λ (namely the system that we shall meet with in § 22 below) to play the part of the system S of § 2.

20. A plane quartic has not, in general, an inscribed pentagram; but, if a plane quartic has an inscribed pentagram, then it has an infinity of inscribed pentagrams‡. That the sides of all these pentagrams touch a conic k was proved by Lüroth; it is interesting, in the present context, to deduce this from the space figure. A side of a pentagram meets Λ in four points corresponding to four points of $\mathfrak{R}$ which are vertices of a tetrahedron whose four faces all belong to the same pentahedron. Hence, in order to find how many lines there are which are sides of pentagrams and which pass through an arbitrary point of the plane, we have to find the number of pentahedra which are such that four of their planes form a self-conjugate tetrahedron in regard to an arbitrary quadric Q of the net. These planes are common to two developables; they belong to the developable formed by the faces of the pentahedra and also to the developable formed by the faces of the ∞^1 tetrahedra which are inscribed in $\mathfrak{R}$ and are self-conjugate in regard to Q. Both these developables consist of tangent planes of the surface R_2^8 formed by the trisecants of $\mathfrak{R}$. The first developable is of class 3; each of its planes contains four generators of R_2^8, and each generator of R_2^8 lies in two of its planes. The second developable is the polar reciprocal of $\mathfrak{R}$ in regard to Q; it is of class 6 and is such that each of its planes contains one generator of R_2^8, and each generator of R_2^8 lies in one of its planes. The number of planes common to the two

* *Cf.* Lüroth, *Math. Annalen*, 1 (1869), 37.

† *Cf.* Frahm, *loc. cit.*

‡ If $x = 0$, $y = 0$, $z = 0$, $u = 0$, $v = 0$ are the equations of five lines, the quartic curve

$$x^{-1} + y^{-1} + z^{-1} + u^{-1} + v^{-1} = 0$$

passes through the ten intersections of pairs of the five lines. The equation of a plane quartic cannot, in general, be reduced to this form, although the form contains an apparently sufficient number of arbitrary constants.

developables is therefore*, by the formula dual to that which gives the number of intersections of two curves on a ruled surface,

$$3.4.1+6.1.2-8.2.1=8.$$

These eight planes fall into two sets of four. For take one of the planes; it meets $\aleph$ in three points on a trisecant and in three other points U, V, W which, together with the point X conjugate to the trisecant, form a self-conjugate tetrahedron in regard to Q. The lines VW, WU, UV meet the trisecant at its intersections with $\aleph$ and the planes VWX, WUX, UVX, as well as belonging to a tetrahedron self-conjugate in regard to Q, all belong to the same pentahedron as UVW. Thus we account for four of the eight planes common to the two developables; the remaining four also form a tetrahedron self-conjugate in regard to Q and belong to one of the pentahedra. It follows that through the point of the plane which represents the quadric Q there pass *two* lines which are sides of pentagrams; or the sides of the pentagrams all touch a conic k.

Any plane section of the Hessian of a cubic surface is a Lüroth quartic; the plane meets the faces of the Sylvester pentahedron in the five sides of a pentagram inscribed in the curve. The polar quadrics of points of the quartic curve are, since the curve is on the Hessian, all cones; these belong to the net of polar quadrics of all points of the plane, and their vertices, which all lie on the Hessian, all lie on a Reye sextic in birational correspondence with the Lüroth quartic. There are thus ∞^3 Reye sextics on the Hessian; they all pass through the ten nodes of the Hessian.

21. The argument of § 15, concerning the trisecants of ϑ which belong to V, can also be used in the particular case when the Hesse sextic is a Reye sextic $\aleph$. We find in exactly the same way that the twenty-four trisecants of $\aleph$ which belong to V consist of eight coplanar triplets. But now, if EMN, FNL, GLM are three coplanar trisecants and generators of the cones (E), (F), (G) respectively, the three points E, F, G must be collinear; the trisecant on which they lie does not belong to V. The plane ϖ which contains these four trisecants EMN, FNL, GLM, EFG is the face of a pentahedron inscribed in $\aleph$; the point of intersection of any three of the planes of the pentahedron is conjugate to the trisecant which is the line of intersection of the other two. But we know, from § 15,

* If we have, on a ruled surface of order 8, a quadruple curve of order 3 meeting each generator in two points, and a curve of order 6 meeting each generator in one point, the number of intersections of the two curves is 8; the above result is dual to this. *Cf.* Edge, *The theory of ruled surfaces* (Cambridge, 1931), 311 *et seq.*

that the points L, M, N are conjugate to the respective trisecants EMN, FNL, GLM; hence *two* faces of the pentahedron coincide with ϖ. Hence, of the ∞^1 pentahedra eight are such that two of their faces coincide; this is to be expected, since the pentahedra are formed by the sets of an involution of sets of five planes among the osculating planes of a twisted cubic, and eight sets of the involution will have double elements just as a g_5^1 on a rational curve has eight double points. The remaining three planes of the pentahedron to which ϖ belongs join the three trisecants EMN, FNL, GLM to the point O of $\Re$ which is conjugate to the trisecant EFG. *The tangents of* $\Re$ *at* L, M, *and* N *all pass through* O. The point O of $\Re$ is therefore such that there are three tangents of $\Re$ passing through it, and there are eight such points on the curve; the projection of $\Re$ on to a plane from any one of these points is a quintic with three cusps. Of any twisted sextic of genus 3 there are twenty-four tangents which meet it again*; if the sextic is a Reye sextic these consist of eight sets of three tangents, the three tangents of a set all meeting the curve in the same point.

22. When the curve $\Re$ is transformed birationally into a Lüroth quartic Λ the points E, F, G are transformed, as in the case of the Hesse sextic, into three points e, f, g of Λ which are the vertices of a triangle whose sides touch Λ in those points l, m, n which correspond to the points L, M, N of $\Re$. But now, since the plane LMN contains the trisecant EFG of $\Re$, the three points l, m, n on Λ are collinear. The line on which they lie meets Λ again in the point o which corresponds to the point O of $\Re$. *There are thus eight in-and-circumscribed triangles of* Λ, *each triangle being such that the three points of contact of its sides with* Λ *are collinear*. These eight triangles were noticed by Lüroth†; they all belong to the same system of contact cubics of Λ. Since the six intersections of $\Re$ with any plane correspond to six points of Λ which are points of contact of a contact cubic of this system, we have, in particular, the following: the six vertices of the quadrilateral formed by any four sides of any one of the pentagrams inscribed in Λ are points of contact of Λ with a contact cubic. Further, since the points in which two contact cubics of the same system touch Λ all lie on a cubic curve, we have the following: two sides of any one of the pentagrams meet Λ each in three points other than the point of inter-

* There is a general formula giving the number of tangents of a curve which meet it again, or, what is the same thing, the number of trisecants which touch the curve; see Zeuthen, *Annali di Mat.* (2), 3 (1870), 186.

† *Loc. cit.*, 43.

section of the two sides; these six points lie on a cubic curve which touches Λ at each vertex of the triangle formed by the three remaining sides of the pentagram.

Three points of Λ which correspond to three points on a trisecant of R are points of contact of a tritangent conic; the line which joins the other two intersections of the conic with Λ touches Λ at the point which corresponds to the point of R conjugate to the trisecant. We have, therefore, the following: three lines belonging to the same pentagram form a triangle whose vertices are points of contact of a tritangent conic of Λ; the join of the other two intersections of the conic with Λ touches Λ at the point of intersection of the two remaining sides of the pentagram*.

23. Each of the pentahedra inscribed in R has associated with it a pentagram inscribed in Λ; the correspondence between the vertices of the pentahedron and those of the pentagram is as indicated in the figures in § 19. Consider, in particular, one of the pentahedra two of whose faces coincide; we imagine that the plane XYZ in the figure turns about the line EFG until it coincides with the plane LMN, the points X, Y, Z coinciding with the points L, M, N respectively. Then the line xyz must turn about the point o until it coincides with the line lmn, the points x, y, z coinciding with the points l, m, n respectively. Each of the pentahedra two of whose faces coincide is therefore associated with a pentagram two of whose sides coincide. If O is one of the eight points of R in which three of its tangents meet it, and if these tangents touch R at L, M, N, the plane LMN meets R again in three collinear points E, F, G on the lines MN, NL, LM respectively; we have a pentahedron whose five planes consist of the three planes $OMNE$, $ONLF$, $OLMG$, and of the plane $EFGLMN$ counted twice. Correspondingly there are three points e, f, g on Λ such that the lines fg, ge, ef touch Λ in three collinear points l, m, n, the line lmn meeting Λ again in o; we have a pentagram whose five sides consist of the three lines fgl, gem, efn, and of the line $lmno$ counted twice.

Through an arbitrary point of the plane there pass two tangents of k; these lines are sides of two different pentagrams inscribed in Λ unless the point lies on Λ, in which case the two lines are sides of the same pentagram. But through a point of k there passes only one line which is a side of a pentagram. Now we have seen that the point o on Λ is such that only one line, viz. $olmn$, passes through it which is a side of a pentagram; hence o must be on k and $olmn$ is the tangent of k at o. We have seen also that

* For these results *cf.* Lüroth, *loc. cit.*, 46.

there are eight points such as O on $\mathfrak{R}$; hence there are eight points such as o on Λ, these being the eight intersections of Λ with k. The tangent of k at any one of its intersections with Λ meets Λ in three further points; the tangents of Λ at these three points are such that the triangle formed by them has its vertices on Λ.

The points of the conic k represent quadrics of the net whose envelope is an octadic surface K. Hence the eight cones of the net whose vertices are at the eight points where $\mathfrak{R}$ is met by triads of its tangents all touch the same octadic surface K along quartic curves. Moreover, since the twenty-four points where Λ is touched by the common tangents of Λ and k are the eight sets of three points such as l, m, n, the twenty-four intersections of the octadic surface K with $\mathfrak{R}$ consist of the points of contact of $\mathfrak{R}$ with those of its tangents which meet it again. Furthermore, we saw in §6 that the tangent plane of an octadic surface at any one of its intersections with ϑ is the plane ϖ associated with the point; hence the eight planes such as LMN are tritangent planes of K.

There are, besides the eight in-and-circumscribed triangles of Λ that we have noticed, 280 other in-and-circumscribed triangles of Λ (§16).

24. The problem of obtaining the number of triangles which are both inscribed and circumscribed to a plane algebraic curve was solved by Cayley*. He approached the problem in two ways. His first solution begins by obtaining the number of triangles whose vertices lie one on each of three given curves and whose sides touch three other given curves, these six curves following each other in a definite order. The cases where the curves are not all distinct are then considered successively; finally, when the six curves are all identical, the number of in-and-circumscribed triangles of a curve is obtained. There thus arise, corresponding to what Cayley calls "the different relations of identity between the different curves", fifty-two different problems†. The problems become more complicated as we proceed, that of the in-and-circumscribed triangle being the last of the series. The solution of the problem given by Cayley‡ gives 288 as the number of in-and-circumscribed triangles of a non-singular plane quartic.

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* "On the problem of the in-and-circumscribed triangle", *Phil. Trans. Royal Soc.*, 161 (1871), 369–412; *Papers*, 8, 212–257.

† One of these problems, that of finding the number of triangles inscribed in one curve and circumscribed to another, has been solved by Schubert, *Math. Annalen*, 17 (1880), 183.

‡ The number of in-and-circumscribed triangles is given on p. 221 (case 52). To obtain the number for the non-singular plane quartic we put $x = 4$, $X = 12$, $\xi = 36$.

Cayley's second attempt to solve the problem is by means of the theory of correspondence. The vertices of the in-and-circumscribed triangles are included among the united points of a certain correspondence on the curve, but these united points also include other points called by Cayley "heterotypic" solutions of the problem. This second attempt to solve the problem does not succeed, owing to the difficulty of accounting for these heterotypic solutions, and has to be left unfinished. In the case of the non-singular plane quartic, however, it would have been comparatively easy to identify all the heterotypic solutions; they are simply the intersections of the quartic with those of its tangents which meet it again in the points of contact of its bitangents. The problem can therefore be solved by this method, and the number 288 is again obtained; but for curves with singularities and for curves of higher order than the fourth the heterotypic solutions are by no means so easily dismissed. The matter will be considered in a subsequent paper*.

The number of heterotypic solutions which Cayley says are to be accounted for is incorrect, because of a mistake in his calculations†.

Arguments which are based on correspondence theory usually obtain merely numerical results; they do not disclose geometrical relations. Neither of the two ways just indicated of obtaining the number 288 gives any inkling of the fact that these triangles consist of thirty-six sets of eight, the eight triangles of any one set being contact cubics of one of the thirty-six systems of the first kind.

* This paper, entitled "Cayley's problem of the in-and-circumscribed triangle", has been accepted for publication in these *Proceedings*.

† *Loc. cit.*, 257. The expression which is printed as

$$X^3-20X^2-8Xx-4x^2+125X+44x-486$$

ought really to be

$$X^3-20X^2-8Xx-4x^2+157X+76x-486,$$

and, furthermore, the term $-4x^2$ has been overlooked in the subsequent calculations.

ON THE THEORY OF TRIGONOMETRIC SERIES (III)

By S. VERBLUNSKY.

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1. Consider the continuous function of bounded variation $f(x)$ which is defined in the closed interval $(0, 2\pi)$, and which satisfies the condition

$$\lim_{n\to\infty} \int_0^{2\pi} e^{inx}\,df = 0$$

when n is a natural number. It has been shown by Rajchman* that,

(a) if $t(x)$ is bounded and B-measurable in $(0, 2\pi)$, then

$$\lim_{n\to\infty} \int_0^{2\pi} e^{inx} t(x)\,df = 0;$$

(b) if $g(x)$ is periodic and has a finite number of discontinuities in $(0, 2\pi)$, then

$$\lim_{n\to\infty} \int_0^{2\pi} g(nx)\,df = \frac{f(2\pi)-f(0)}{2\pi} \int_0^{2\pi} g(x)\,dx.$$

Rajchman† has further observed that these two propositions enable us to infer that the total variation of $f(x)$ on a set of type H is zero. A set of type H is defined as follows. Let (n_p) denote an increasing sequence of positive integers. Let $E(x)$, where $x \geqslant 0$, denote the excess of x over the greatest integer multiple of 2π which does not exceed x. Let σ be a set in $(0, 2\pi)$. As x describes σ, $E(n_p x)$ describes a set σ_p. If $\Sigma\sigma_p$ is not everywhere dense in $(0, 2\pi)$, then σ is of type H.

Rajchman establishes the propositions (a) and (b) by means of his theory of formal multiplication of trigonometric series. That theory, in its present state, often obscures the question which it answers. What may be called a natural way of establishing the propositions (a) and (b)

* Rajchman, *Math. Annalen*, 101 (1929), 686–700.

† Rajchman, *Comptes rendus*, 187 (1928), 1026–1028.

is first to prove them, say, when $t(x)$ and $g(x)$ are trigonometric polynomials; and then to consider uniformly convergent or boundedly convergent sequences of functions for which the propositions are known to hold. If this is done, one is led to some significant generalizations of the propositions.

The theorem that the total variation of $f(x)$ on a set of type H is zero reminds one of the theorem that, if a trigonometric series converges to zero except at the points of a set of type H, then all its coefficients are zero. That generalization of the first theorem which is deducible from generalizations of propositions (a) and (b) is considered in this paper. It is possible to prove a corresponding generalization of the second theorem; this will be considered in a subsequent paper, where the connection between the two theorems will appear. The many new definitions which it has been found desirable to introduce preclude a more particular mention in this introductory section of the contents of this paper.

2. In this section we give the definition and principal properties of the Lebesgue-Stieltjes integral.

Let $\psi(x)$ be monotone increasing in the closed interval (a, b). Let $t(x)$ be bounded and B-measurable in that interval. We define a function $T(\xi)$ for $\psi(a) \leqslant \xi \leqslant \psi(b)$ by the conditions

(i) if $a < x < b$, then

$$T(\xi) = t(x), \qquad \psi(x-0) \leqslant \xi \leqslant \psi(x+0);$$

(ii)
$$T(\xi) = t(a), \qquad \psi(a) \leqslant \xi \leqslant \psi(a+0);$$

$$T(\xi) = t(b), \qquad \psi(b-0) \leqslant \xi \leqslant \psi(b).$$

The function $T(\xi)$ is bounded and B-measurable as a function of ξ in $[\psi(a), \psi(b)]$. We define the Lebesgue-Stieltjes integral

$$\int_a^b t(x)\, d\psi \tag{1}$$

by the equation

$$\int_a^b t(x)\, d\psi = \int_{\psi(a)}^{\psi(b)} T(\xi)\, d\xi. \tag{2}$$

If $\psi(x)$ is of bounded variation in the closed interval (a, b), we can write

$$\psi(x) = \psi_1(x) - \psi_2(x), \tag{3}$$

where $\psi_1(x)$, $\psi_2(x)$ are monotone increasing. We then define the integral (1) by the equation

$$\int_a^b t(x)\,d\psi = \int_a^b t(x)\,d\psi_1 - \int_a^b t(x)\,d\psi_2.$$

The value of (1) is independent of the particular decomposition (3).

If $\{t_n(x)\}$ is a sequence of B-measurable functions which converges boundedly to $t(x)$, then

$$\lim_{n\to\infty} \int_a^b t_n(x)\,d\psi = \int_a^b t(x)\,d\psi.$$

If $\psi(x)$ is monotone increasing in (a, b) and $t_1(x)$, $t_2(x)$ are two bounded B-measurable functions which satisfy $t_1(x) \geqslant t_2(x)$, then

$$\int_a^b t_2(x)\,d\psi \geqslant \int_a^b t_1(x)\,d\psi.$$

It is important to notice that the value of (1) is, in general, altered by changing the value of $\psi(x)$ at a single point.

3. *Definition* 1. An increasing sequence S of positive integers is said to be dense if, given any positive integer ν, there are ν consecutive positive integers which belong to S.

If S is a dense sequence of positive integers, we can find a set of $2\nu+1$ consecutive numbers which belong to S. Let these numbers be $n_1, n_2, \ldots, n_{2\nu+1}$. Let $m > n_{2\nu+1}$; then, by definition, we can find a set of $2m$ consecutive integers which belong to S. Since $m \geqslant n_{2\nu+1}+1$, we have $2m \geqslant 2n_{2\nu+1}+2$. Hence there are at least

$$2n_{2\nu+1}+2-n_{2\nu+1} = n_{2\nu+1}+2 \geqslant 2\nu+3$$

consecutive integers of the second set which are greater than all the integers of the first set. Denote $2\nu+3$ of such consecutive integers by $n_1', n_2', \ldots, n_{2\nu+3}'$. In the same way we can find $2\nu+5$ consecutive integers $n_1'', n_2'', \ldots, n_{2\nu+5}''$, which belong to S and are greater than all the n's and all the n''s. This process can be continued indefinitely. The sequence $n_\nu, n_{\nu+1}', n_{\nu+2}'', \ldots$ possesses the property that, if k denotes any positive integer, then the sequence

$$(n_{\nu+r}^{(r)}+k) \quad (r > k),$$

and the sequence

$$(n_{\nu+r}^{(r)}-k) \quad (r > k)$$

both belong to S.

Definition 2. Let S be a dense sequence of positive integers. Let (n_p), $p = 1, 2, \ldots$, be a subsequence of S with the property that, given a positive integer k, there is a $p(k)$ such that, for $p \geqslant p(k)$, the sequences (n_p+k), (n_p-k) belong to S. Then (n_p) is said to be a central sequence of S.

By what precedes, every dense sequence of positive integers possesses a central sequence.

LEMMA 1. *Let S be a dense sequence of positive integers. Then S possesses a central sequence which is dense.*

By what precedes we can find a sequence $S_1, S_2, S_r, \ldots$ of sets of integers such that

(i) S_r consists of $3r$ consecutive integers;

(ii) every member of S_{r+1} is greater than every member of S_r;

(iii) every member of S_r is a member of S.

Let S_r consist of the integers $n_1^{(r)}, n_2^{(r)}, \ldots, n_{3r}^{(r)}$. Let S_r' denote the set $n_{r+1}^{(r)}, \ldots, n_{2r}^{(r)}$. Then $S' = \Sigma S_r'$ is a dense sequence. It is, however, a central sequence of S. For, if ν runs through the sequence S', then, from and after a certain value, it belongs to sets S_r' with $r > k$, where k is an assigned positive integer. From and after that value, $\nu \pm k$ belongs to S.

4. Let $f(x)$ be a function defined in the closed interval $(0, 2\pi)$ and of bounded variation in that interval. If there exists a dense sequence S of positive integers such that

$$\lim_{n\to\infty} \int_0^{2\pi} e^{inx}\, df = 0 \quad (n \subset S),$$

then we say that $f(x)$ is a member of class B. The sequence S is said to be associated with $f(x)$.

THEOREM I. *Let $f(x)$ be of class B and associated with the dense sequence S. Let (n_p) denote a central sequence of S. If $t(x)$, defined in the closed interval $(0, 2\pi)$, is bounded, B-measurable, and satisfies $t(0) = t(2\pi)$, then*

$$\lim_{p\to\infty} \int_0^{2\pi} e^{in_p x} t(x)\, df = 0.$$

Suppose in the first place that $t(x)$ is a trigonometric polynomial. Then we have an equation of the form

$$e^{in_p x}t(x) = \sum_{k=-r}^{r} A_k e^{i(n_p+k)x},$$

where the coefficients A_k are real or complex constants. Hence

$$\lim_{p\to\infty} \int_0^{2\pi} e^{in_p x}t(x)\,df = \sum_{k=-r}^{r} A_k \lim_{p\to\infty} \int_0^{2\pi} e^{i(n_p+k)x}\,df,$$

assuming that each of the $2r+1$ limits of the second member exists. This, however, is so, each of the limits being zero. For, since (n_p) is a central sequence of S, n_p+k belongs to S for a fixed k, from and after a certain value of p.

Suppose next that $t(x)$ is continuous with $t(0) = t(2\pi)$. Then, given $\epsilon > 0$, we can find a trigonometric polynomial $\tau(x)$ such that

$$|t(x)-\tau(x)| < \epsilon$$

for all x. We have

$$\int_0^{2\pi} e^{in_p x}t(x)\,df = \int_0^{2\pi} e^{in_p x}\{t(x)-\tau(x)\}\,df + \int_0^{2\pi} e^{in_p x}\tau(x)\,df.$$

Let V denote the total variation of $f(x)$ in $(0, 2\pi)$. It follows that

$$\overline{\lim_{p\to\infty}} \left| \int_0^{2\pi} e^{in_p x}t(x)\,df \right| \leqslant \epsilon V.$$

Since ϵ is arbitrary, Theorem I is true when $t(x)$ is continuous.

Suppose now that $t(x)$ is of Baire's first class. We can find a boundedly convergent sequence $\{t_\nu(x)\}$ of continuous functions such that $t_\nu(0) = t_\nu(2\pi)$ and such that $\lim t_\nu(x) = t(x)$. Then

$$\overline{\lim_{p\to\infty}} \left| \int_0^{2\pi} e^{in_p x}t(x)\,df \right| \leqslant \overline{\lim_{p\to\infty}} \left| \int_0^{2\pi} e^{in_p x}\{t(x)-t_\nu(x)\}\,df \right| + \lim_{p\to\infty} \left| \int_0^{2\pi} e^{in_p x}t_\nu(x)\,df \right|$$

$$\leqslant \int_0^{2\pi} |t(x)-t_\nu(x)|\,dv(x),$$

where $v(x)$ is the total variation of $f(x)$ in $(0, x)$. The first member is independent of ν, while the last member tends to zero as $\nu\to\infty$. Hence the value of the first member is zero.

Finally, since any bounded B-measurable function enters into Baire's classification, the preceding argument shows that the proof can be completed by transfinite induction.

5. The preceding theorem can be used to investigate the possible discontinuities of the function $f(x)$ of class B. Let S denote the associated sequence, and let (n_p) denote a central sequence of S. Let ξ be a point of discontinuity of $f(x)$ such that $0 < \xi < 2\pi$. Let $t(x)$ be defined by

$$t(\xi) = 1, \quad t(x) = 0 \quad (x \neq \xi).$$

This function satisfies the conditions of Theorem I. We therefore have

$$\lim_{p\to\infty} \int_0^{2\pi} e^{in_p x} t(x)\, df = 0. \tag{4}$$

On the other hand,

$$\int_0^{2\pi} e^{in_p x} t(x)\, df = e^{in_p \xi}\{f(\xi+0) - f(\xi-0)\}.$$

The modulus of the second member is $|f(\xi+0) - f(\xi-0)|$, so that, by (4), we must have

$$f(\xi+0) = f(\xi-0).$$

Thus $f(x)$ can only have removable discontinuities in the open interval $(0, 2\pi)$.

To examine what happens at the extremities of $(0, 2\pi)$, define $t(x)$ by the equations

$$t(0) = t(2\pi) = 1, \quad t(x) = 0 \quad (0 < x < 2\pi).$$

By Theorem I the relation (4) holds. Further,

$$\int_0^{2\pi} e^{in_p x} t(x)\, df = f(2\pi) - f(2\pi-0) + f(+0) - f(0).$$

We must therefore have

$$f(2\pi) - f(2\pi-0) + f(+0) - f(0) = 0. \tag{5}$$

We have shown that $f(x)$ is equivalent to a function $\psi(x)$ which is continuous in the closed interval $(0, 2\pi)$. The relation (5) will enable us to prove that, for any integer n,

$$\int_0^{2\pi} e^{inx}\, df = \int_0^{2\pi} e^{inx}\, d\psi.$$

To see this, let the discontinuities of $f(x)$ in the open interval $(0, 2\pi)$ be enumerated as $\xi_1, \xi_2, \ldots, \xi_\nu, \ldots$. Let $f_\nu(x)$ denote the function obtained from $f(x)$ by changing the values $f(\xi_1), \ldots, f(\xi_\nu)$ to $\psi(\xi_1), \ldots, \psi(\xi_\nu)$, respectively. Then we have

$$\int_0^{2\pi} e^{inx}\, df = \int_0^{2\pi} e^{inx}\, df_\nu. \tag{6}$$

Let $\phi(x)$ be defined by the equations

$$\phi(0)=f(0), \quad \phi(2\pi)=f(2\pi), \quad \phi(x)=\psi(x) \quad (0<x<\pi).$$

Then, if $\omega_\nu=|f(\xi_\nu)-\psi(\xi_\nu)|$, we have

$$\left|\int_0^{2\pi} e^{inx}\,d(\phi-f_\nu)\right|<2\sum_{r=\nu+1}^{\infty}\omega_\nu.$$

Hence, by (6),

$$\int_0^{2\pi} e^{inx}\,df=\int_0^{2\pi} e^{inx}\,d\phi. \tag{7}$$

Further,

$$\begin{aligned}\int_0^{2\pi} e^{inx}\,d(\phi-\psi) &= f(2\pi)-f(2\pi-0)-f(0)+f(+0)\\ &= 0\end{aligned}$$

by (5). By (7), our assertion follows. We thus have

THEOREM II. *If $f(x)$ is of class B and is associated with the dense sequence S, then $f(x)$ is equivalent to a continuous function $\psi(x)$ such that, for every integer n,*

$$\int_0^{2\pi} e^{inx}\,df=\int_0^{2\pi} e^{inx}\,d\psi.$$

Consequently, $\psi(x)$ is of class B and is associated with S.

Conversely, given a continuous function $\psi(x)$ of class B, associated with S, we can construct a discontinuous function $f(x)$ of class B and associated with S by writing

$$f(0)=\psi(0)+c, \quad f(2\pi)=\psi(2\pi)+c, \quad f(x)=\psi(x) \quad (0<x<2\pi).$$

We then have, for every integer n,

$$\int_0^{2\pi} e^{inx}\,d(f-\psi)=c-c=0.$$

The result of Theorem I is no longer true if the condition $t(0)=t(2\pi)$ is omitted. For, if we define $t(x)$ by

$$t(2\pi)=1, \quad t(x)=0 \quad (0\leqslant x<2\pi),$$

then, for any integer n,

$$\int_0^{2\pi} e^{inx}\,t(x)\,df=f(2\pi)-f(2\pi-0).$$

The following theorem, however, is true.

THEOREM III. *Let $f(x)$ be of class B and associated with S. Let $\psi(x)$ be the continuous function which is equivalent to $f(x)$. Let (n_p) denote a*

central sequence of S. Let $t(x)$, defined in $(0, 2\pi)$, be bounded and B-measurable. Then

$$\lim_{p\to\infty}\int_0^{2\pi} e^{in_p x}\,t(x)\,d\psi = 0.$$

Define a function $\tau(x)$ by the equations

$$\tau(2\pi) = t(0), \quad \tau(x) = t(x) \quad (0 \leqslant x < 2\pi).$$

The function $\psi(x)$ is, by Theorem II, of class B and associated with S. By Theorem I,

$$\lim_{p\to\infty}\int_0^{2\pi} e^{in_p x}\,\tau(x)\,d\psi = 0.$$

Further,

$$\int_0^{2\pi} e^{in_p x}\{t(x)-\tau(x)\}\,d\psi = \{t(2\pi)-t(0)\}\{\psi(2\pi)-\psi(2\pi-0)\}$$

$$= 0,$$

since $\psi(x)$ is continuous.

6. Let $\psi(x)$ be a continuous function of class B associated with the dense sequence S of natural numbers. By Lemma 1 there is a central sequence S' of S which is dense; and, by the proof of that lemma, S' may be taken to be of the form $S' = \Sigma S_r'$, where

(i) S_r' consists of r consecutive integers,

(ii) every member of S_{r+1}' is greater than every member of S_r'.

Let $0 \leqslant \theta \leqslant 1$. Then $\cos\theta x$, $\sin\theta x$ are bounded B-measurable functions. If (n_p), $p = 1, 2, \ldots$, denotes the central sequence S', we have, by Theorem III,

$$\lim_{p\to\infty}\int_0^{2\pi} e^{i(n_p+\theta)x}\,d\psi = 0. \tag{8}$$

Suppose now that (θ_p) is any sequence of real numbers which satisfy $0 \leqslant \theta_p \leqslant 1$. We can show that

$$\lim_{p\to\infty}\int_0^{2\pi} e^{i(n_p+\theta_p)x}\,d\psi = 0. \tag{9}$$

For, if this relation was false, we could select a subsequence $p_1, p_2, \ldots, p_s, \ldots$ such that

$$\lim_{s\to\infty}\left|\int_0^{2\pi} e^{i(n_{p_s}+\theta_{p_s})x}\,d\psi\right| = \eta,$$

where $\eta > 0$. From the sequence (θ_{p_s}) we can select a subsequence (θ_ν) such that $\lim \theta_\nu = \theta$. Then

$$\lim_{\nu\to\infty} \left| \int_0^{2\pi} e^{i(n_\nu+\theta_\nu)x}\, d\psi \right| = \eta. \tag{10}$$

But

$$\left| \int_0^{2\pi} e^{i(n_\nu+\theta_\nu)x}\, d\psi - \int_0^{2\pi} e^{i(n_\nu+\theta)x}\, d\psi \right| \leqslant \int_0^{2\pi} |e^{i\theta_\nu x} - e^{i\theta x}|\, dv(x),$$

where $v(x)$ denotes the total variation of $\psi(x)$ in $(0, x)$. Hence

$$\lim_{\nu\to\infty} \int_0^{2\pi} e^{i(n_\nu+\theta_\nu)x}\, d\psi = \lim_{\nu\to\infty} \int_0^{2\pi} e^{i(n_\nu+\theta)x}\, d\psi$$

$$= 0$$

by (8). By (10), we have $\eta = 0$. This contradiction establishes (9).

The first and last members of S_r' form the end points of an interval Δ_r of length $r-1$. The relation (9) is the same as

$$\lim_{\mu\to\infty} \int_0^{2\pi} e^{i\mu x}\, d\psi = 0 \quad (\mu \subset \Sigma\Delta_r). \tag{11}$$

We introduce the following definition.

Definition 3. A set R of positive real numbers is said to be compact if, given any integer ν, R contains an interval of length ν.

The relation (11) establishes the following theorem.

THEOREM IV. *Let $\psi(x)$ be a continuous function of class B, associated with the dense sequence S. Then there is a compact set R of real numbers such that*

$$\lim_{\mu\to\infty} \int_0^{2\pi} e^{i\mu x}\, d\psi = 0 \quad (\mu \subset R), \tag{12}$$

and R contains a dense central sequence of S.

We shall say that the continuous function $\psi(x)$ of class B is associated with the compact set R of real numbers.

7. *Definition* 4. Let R be a compact set of real numbers. A set $T \subset S$, which is not bounded above, is said to be a central set of R if, given any natural number k, there is a real number $\nu(k)$ such that, for all $\mu \geqslant \nu(k)$ which belong to T, $\mu \pm k$ belongs to S.

Every compact set R contains a central set T. For, by an argument similar to that used in § 3, we can select from R a sequence $R_1, R_2, \ldots$ of

non-overlapping intervals such that

(i) R_ν is of length 3ν,

(ii) $R_{\nu+1}$ lies to the right of R_ν.

If R_ν' denotes the central third of R_ν, then $T = \Sigma R_\nu'$ is the required central set. We have proved incidentally

LEMMA 2. *If R is a compact set of real numbers, then R contains a central set T which is compact.*

By an argument similar to that used for proving Theorem I, we can prove

LEMMA 3. *Let $\psi(x)$ be a continuous function of class B associated with the compact set R. Let T be a central set of R. If $t(x)$, defined in $(0, 2\pi)$, is bounded and B-measurable, and $t(0) = t(2\pi)$, then*

$$\lim_{\mu\to\infty} \int_0^{2\pi} e^{i\mu x} t(x)\, d\psi = 0 \quad (\mu \subset T).$$

Just as in the proof of Theorem III we can show that, in virtue of the continuity of $\psi(x)$, the condition $t(0) = t(2\pi)$ is superfluous. We therefore have

THEOREM V. *Let $\psi(x)$ be a continuous function of class B associated with the compact set R. Let T be a central set of R. If $t(x)$, defined in $(0, 2\pi)$, is bounded and B-measurable, then*

$$\lim_{\mu\to\infty} \int_0^{2\pi} e^{i\mu x} t(x)\, d\psi = 0 \quad (\mu \subset T). \tag{13}$$

From this theorem we can infer

THEOREM VI. *Let $\psi(x)$ be a continuous function of class B associated with the compact set R. Let T be a central set of R, and let $\mu \subset T$. Let $\{t_\mu(x)\}$ be a uniformly bounded family of B-measurable functions with the property that any sequence $\{t_{\mu_p}(x)\}$, $\mu_p \to \infty$, contains a convergent subsequence. Then*

$$\lim_{\mu\to\infty} \int_0^{2\pi} e^{i\mu x} t_\mu(x)\, d\psi = 0 \quad (\mu \subset T).$$

If the theorem was false, we could find a sequence $\mu_p \to \infty$ such that

$$\lim_{p\to\infty} \left| \int_0^{2\pi} e^{i\mu_p x} t_{\mu_p}(x)\, d\psi \right| = \eta,$$

where $\eta > 0$. By hypothesis, there is a subsequence (μ_ν) of (μ_p) such that $t_{\mu_\nu}(x) \to t(x)$ as $\nu \to \infty$. We have

$$\lim_{\nu\to\infty} \left| \int_0^{2\pi} e^{i\mu_\nu x} t_{\mu_\nu}(x)\, d\psi \right| = \eta.$$

Further,

$$\left| \int_0^{2\pi} e^{i\mu_\nu x} t_{\mu_\nu}(x)\, d\psi - \int_0^{2\pi} e^{i\mu_\nu x} t(x)\, d\psi \right| \leqslant \int_0^{2\pi} |t_{\mu_\nu}(x) - t(x)|\, dv(x),$$

where $v(x)$ denotes the total variation of $\psi(x)$ in $(0, x)$. The last expression tends to zero as $\nu \to \infty$. Hence

$$\lim_{\nu\to\infty} \left| \int_0^{2\pi} e^{i\mu_\nu x} t(x)\, d\psi \right| = \eta,$$

which contradicts (13).

From Theorem VI we infer

LEMMA 4. *Let $\psi(x)$ be a continuous function of class B associated with the compact set R. Let T be a central set of R, and let $\mu \subset T$. Let $\{\theta_\mu(x)\}$ be a family of B-measurable functions which converges as $\mu \to \infty$. Let $\tau(x)$ be a bounded B-measurable function. Then*

$$\lim_{\mu\to\infty} \int_0^{2\pi} e^{i[\mu x + \theta_\mu(x)]} \tau(x)\, d\psi = 0.$$

The functions $\cos\{\theta_\mu(x)\}$, $\sin\{\theta_\mu(x)\}$ are continuous functions of B-measurable functions, and are therefore B-measurable. For varying μ they are uniformly bounded, and they converge as $\mu \to \infty$. Hence $[\tau(x)\cos\{\theta_\mu(x)\}]$, $[\tau(x)\sin\{\theta_\mu(x)\}]$ satisfy the conditions of Theorem VI.

8. The enunciation of Theorem V can be transformed by using the following lemma.

LEMMA 5. *Let $\phi(x)$ be continuous and of bounded variation in $(0, 2\pi)$. Let $t(x)$ be bounded and B-measurable in $(0, 2\pi)$, and let*

$$\Phi(x) = \int_0^x t(x)\, d\phi, \quad \Phi(0) = 0.$$

Then, if $l(x)$ is bounded and B-measurable in $(0, 2\pi)$,

$$\int_0^{2\pi} l(x)\, d\Phi = \int_0^{2\pi} l(x)\, t(x)\, d\phi. \tag{14}$$

The function $\Phi(x)$ is continuous and of bounded variation. If we can express $l(x)$ as the limit of a uniformly bounded sequence $\{l_\nu(x)\}$ of functions

for which the lemma holds, then, in virtue of the results

$$\lim_{\nu\to\infty}\int_0^{2\pi} l_\nu(x)\,d\Phi = \int_0^{2\pi} l(x)\,d\Phi,$$

$$\lim_{\nu\to\infty}\int_0^{2\pi} l_\nu(x)\,t(x)\,d\phi = \int_0^{2\pi} l(x)\,t(x)\,d\phi,$$

the lemma will hold for $l(x)$. Since $l(x)$ enters into Baire's classification, we need only prove the lemma when $l(x)$ is continuous. If we divide $(0, 2\pi)$ into sub-intervals δ_i, in each of which the oscillation of $l(x)$ is less than an arbitrarily assigned $\epsilon > 0$, then the difference between the two members of (14) is

$$\sum_{(i)}\int_{\delta_i} l(x)\,d\Phi - \int_{\delta_i} l(x)\,t(x)\,d\phi. \tag{15}$$

Let V denote the total variation of $\phi(x)$ in $(0, 2\pi)$. Then

$$\Sigma\left|\int_{\delta_i}\{l(x)-l(\alpha_i)\}\,t(x)\,d\phi\right| < \epsilon MV,$$

where α_i is the left-hand end point of δ_i and $|t(x)| < M$. Also

$$\Sigma\left|\int_{\delta_i}\{l(x)-l(\alpha_i)\}\,d\Phi\right| < \epsilon MV,$$

since the total variation of $\Phi(x)$ in $(0, 2\pi)$ is less than MV. Finally,

$$\Sigma\left|l(\alpha_i)\int_{\delta_i}\{d\Phi - t(x)\,d\phi\}\right| = 0.$$

The last three relations show that the expression (15) is less in absolute value than $2\epsilon MV$.

In Theorem V we can write

$$\Psi(x) = \int_0^x t(x)\,d\psi, \quad \Psi(0) = 0,$$

in which case, by Lemma 5,

$$\int_0^{2\pi} e^{i\mu x}\,d\Psi = \int_0^{2\pi} e^{i\mu x}\,t(x)\,d\psi.$$

The variable μ can take values belonging to the central set T; and, by Lemma 2, T can be so chosen as to be a compact set. Any compact set contains a dense sequence of natural numbers. The set T depends only on $\psi(x)$, not on the particular function $t(x)$. It follows that $\Psi(x)$ is a continuous function of class B and is associated with a compact set which is independent of $t(x)$. We enunciate this result as a lemma.

LEMMA 6. *If $\psi(x)$ is a continuous function of class B associated with the compact set R, then there is a central set T of R which is compact, and such that, if $t(x)$ is any bounded B-measurable function, then*

$$\int_0^x t(x)\,d\psi$$

is a continuous function of class B associated with the compact set T.

9. Let $\psi(x)$ be a continuous function of bounded variation in $(0, 2\pi)$. Let E denote the set of points at which $\psi'(x) = \pm\infty$. Then, if $t(x)$ is the function which is zero at the points of E and is equal to 1 at the points of CE, we have

$$\int_0^x t(x)\,d\psi = \int_0^x \psi'(x)\,dx.$$

We shall, in what follows, take for granted results of this nature. As a specimen of the arguments by which such results may be established, we prove the following lemma.

LEMMA 7. *Let $\psi(x)$ be continuous and of bounded variation in $(0, 2\pi)$. Let $t(x) = 1$ at those points at which*

$$\overline{\lim_{h\to 0}} \frac{\psi(x+h)-\psi(x)}{h} \geqslant 0,$$

and let $t(x) = 0$ at the other points of the interval. If $P(x)$ denotes the positive variation of $\psi(x)$ in $(0, x)$, then

$$P(x) = \int_0^x t(x)\,d\psi.$$

We can write

$$\psi(x)-\psi(0) = \lambda(x)+\mu(x),$$

where $\mu(x)$ is absolutely continuous and $\lambda(x)$ is the function of singularities of $\psi(x)$. Then

$$\int_0^x t(x)\,d\psi = \int_0^x t(x)\,d\lambda + \int_0^x t(x)\,d\mu. \tag{16}$$

It is known that, if $\theta(x)$ is a monotone absolutely continuous function, then

$$\int_0^x t(x)\,d\theta = \int_0^x t(x)\,\theta'(x)\,dx,$$

this equation simply being the theorem concerning the change of variable in a Lebesgue integral. Thus (16) becomes

$$\int_0^x t(x)\,d\psi = \int_0^x t(x)\,d\lambda + \int_0^x t(x)\,\mu'(x)\,dx. \tag{17}$$

We put $$\lambda(x) = \lambda_1(x) - \lambda_2(x),$$

where $\lambda_1(x)$, $\lambda_2(x)$ denote respectively the positive and negative variations of $\lambda(x)$. It is known that

$$P(x) = \lambda_1(x) + \int_0^x t(x)\,\mu'(x)\,dx. \tag{18}$$

It is further known that, if E denotes the set of points at which $\psi'(x) = +\infty$, and $E(0, x)$ denotes that part of E which lies in $(0, x)$, then, if

$$\{d_\nu\} \equiv \{(\alpha_\nu, \beta_\nu)\}$$

is a set of non-overlapping open intervals in $(0, x)$ containing $E(0, x)$, we have

$$\sum_{(\nu)} \lambda_1(\beta_\nu) - \lambda_1(\alpha_\nu) = \lambda_1(x). \tag{19}$$

Let $\tau(x) = 1$ for $x \subset E$ and $\tau(x) = 0$ for $x \subset CE$. The set E is an inner limiting set of zero measure. Let $\{O_n\}$ with $O_n \supset O_{n+1}$ be a sequence of open sets such that $\lim O_n = E$. Let $\tau_n(x) = 1$ for $x \subset O_n$ and $\tau_n(x) = 0$ for $x \subset CO_n$. Then $\tau_n(x) \to \tau(x)$, and

$$\int_0^x \tau(x)\,d\lambda_1 = \lim_{n\to\infty} \int_0^x \tau_n(x)\,d\lambda_1. \tag{20}$$

If $\{(\alpha_r^{(n)}, \beta_r^{(n)})\}$ denote the intervals which constitute $O_n(0, x)$, we have, by (19),

$$\begin{aligned}\int_0^x \tau_n(x)\,d\lambda_1 &= \sum_{(r)} \lambda_1(\beta_r^{(n)}) - \lambda_1(\alpha_r^{(n)}) \\ &= \lambda_1(x),\end{aligned}$$

so that, by (20), $$\int_0^x \tau(x)\,d\lambda_1 = \lambda_1(x). \tag{21}$$

But $$\lambda_1(x) \geqslant \int_0^x t(x)\,d\lambda_1 \geqslant \int_0^x \tau(x)\,d\lambda_1 = \lambda_1(x).$$

Hence $$\int_0^x t(x)\,d\lambda_1 = \lambda_1(x). \tag{22}$$

It follows from (21) that, if $\sigma(x) = 1$ for $x \subset CE$, $\sigma(x) = 0$ for $x \subset E$, then

$$\int_0^x \sigma(x)\,d\lambda_1 = 0.$$

In a similar manner we can show that, if E' is the set at which $\psi'(x) = -\infty$, and if $\sigma'(x) = 1$ for $x \subset CE'$, $\sigma'(x) = 0$ for $x \subset E'$, then

$$\int_0^x \sigma'(x)\,d\lambda_2 = 0.$$

Since $EE' = 0$, it follows that $t(x) \leqslant \sigma'(x)$, so that

$$\int_0^x t(x)\, d\lambda_2 = 0. \tag{23}$$

From (22) and (23),

$$\lambda_1(x) = \int_0^x t(x)\, d\lambda,$$

and the lemma then follows from (17) and (18).

Applying Lemma 7 to Lemma 6 we obtain

LEMMA 8. *If $\psi(x)$ is a continuous function of class B associated with the compact set R, and if T denotes a central set of R which is compact, then $P(x)$, $N(x)$, the positive and negative variations of $\psi(x)$, are continuous functions of class B, associated with T.*

10. *Definition* 5. A function $f(x)$ defined in the closed interval (a, b) is said to be J-measurable if there is an enumerable set (α_n) of real numbers, everywhere dense in $(-\infty, \infty)$, such that the sets $(0, 2\pi)$ $[f(x) > \alpha_n]$ are measurable in the sense of Jordan.

LEMMA 8. *A function $f(x)$ which is integrable R (i.e. in the sense of Riemann) in (a, b) is J-measurable.*

We may suppose that $f(x) > 0$. Then the two-dimensional set

$$E\colon\ 0 \leqslant y \leqslant f(x), \quad 0 \leqslant x \leqslant 2\pi$$

is J-measurable, *i.e.* the frontier of E has the superficial measure zero. Hence, for almost all real ξ, the closed set $E(\xi)$, in which $y = \xi$ meets the frontier of E, is of linear measure zero. The projection of $E(\xi)$ on the x-axis contains the frontier of the set $(0, 2\pi)$ $[f(x) > \xi]$. Hence, for almost all ξ, the latter set is J-measurable.

It follows from the properties of J-measurable sets that the set $(0, 2\pi)$ $[\alpha_m < f(x) \leqslant \alpha_n]$ is J-measurable, where $\alpha_m < \alpha_n$. Further, a bounded J-measurable function is the limit as $\nu \to \infty$ of a uniformly convergent sequence of J-measurable functions of the form

$$\sum_{i=1}^{\nu} \alpha_i^{(\nu)} \phi_i^{(\nu)},$$

where the α's belong to the set (α_n) and the ϕ's are characteristic functions of J-measurable sets.

Definition 6. Let R be a compact set of real numbers. An increasing sequence $\{\mu_p\}$, $p = 1, 2, \ldots$, of real numbers, with $\lim \mu_p = \infty$, is said to be

a regular sequence of R if, given any positive integer k, there is a $p(k)$ such that, for $p \geqslant p(k)$, we have $k\mu_p \subset R$.

Let (λ_p) be a sequence of positive real numbers such that $\lambda_{p+1}/\lambda_p \to \infty$. Let (δ_p) be a sequence of positive numbers such that $\delta_p \lambda_p < \lambda_{p+1}$, and $\lim \delta_p = \infty$. Consider the aggregate of intervals $\{(\lambda_p, \delta_p \lambda_p)\}$. This aggregate forms a compact set R, since the length of the p-th interval tends to ∞ with p. The sequence (λ_p) is a regular sequence of R. For, if k is any positive integer, we have $\lambda_p \leqslant k\lambda_p < \delta_p \lambda_p$, provided that $k < \delta_p$, *i.e.* for $p \geqslant p(k)$.

A sequence $\{t_p(x)\}$ of B-measurable functions is said to be semi-convergent if any subsequence contains a subsequence which is convergent.

THEOREM VII. *Let $\psi(x)$ be a continuous function of class B associated with the compact set R. Let T be a central set of R which is compact. Let (μ_p), $p = 1, 2, \ldots$, be a regular sequence of T. Let $\{\theta_p(x)\}$ be a semi-convergent sequence of B-measurable functions defined in $(0, 2\pi)$. Then, if $t(x)$ is bounded, J-measurable, and periodic, we have*

$$\lim_{p\to\infty} \int_0^{2\pi} t[\mu_p x + \theta_p(x)]\, d\psi = \frac{\psi(2\pi) - \psi(0)}{2\pi} \int_0^{2\pi} t(x)\, dx.$$

Let $P(x)$, $N(x)$ denote respectively the positive and negative variations of $\psi(x)$ in $(0, x)$. If we can prove the relation

$$\lim_{p\to\infty} \int_0^{2\pi} t[\mu_p x + \theta_p(x)]\, dP = \frac{P(2\pi) - P(0)}{2\pi} \int_0^{2\pi} t(x)\, dx, \tag{24}$$

and the similar relation in which $P(x)$ is replaced by $N(x)$, then the theorem will follow.

Let $\tau(x)$ denote the B-measurable function which is equal to 1 at the points at which

$$\overline{\lim_{h\to 0}} \frac{\psi(x+h) - \psi(x)}{h} \geqslant 0$$

and which is zero elsewhere. Then, by Lemma 7,

$$P(x) = \int_0^x \tau(x)\, d\psi,$$

and, by Lemma 5, (24) is the same as

$$\lim_{p\to\infty} \int_0^{2\pi} t[\mu_p x + \theta_p(x)]\, \tau(x)\, d\psi = \frac{P(2\pi) - P(0)}{2\pi} \int_0^{2\pi} t(x)\, dx.$$

We observe in the first place that (24) is true when $t(x)$ is a constant. Suppose next that $t(x) = e^{ikx}$, where k is an integer. We may suppose that

k is a positive integer. We must show that

$$\lim_{p\to\infty}\int_0^{2\pi} e^{i[kx\mu_p+k\theta_\nu(x)]}\,dP=0. \tag{25}$$

If now (25) was false, we could select a sequence $p_1, p_2, \ldots, p_s, \ldots$ such that

$$\lim_{s\to\infty}\left|\int_0^{2\pi} e^{i[k\mu_{p_s}x+k\theta_{p_s}(x)]}\,dP\right|=\eta,$$

where $\eta>0$. Since $\{\theta_p(x)\}$ is semi-convergent, we can select from $\{\theta_{p_s}(x)\}$ a convergent subsequence $\{\theta_\nu(x)\}$. We then have

$$\lim_{\nu\to\infty}\left|\int_0^{2\pi} e^{i[k\mu_\nu x+k\theta_\nu(x)]}\,dP\right|=\eta. \tag{26}$$

The sequence (μ_ν) as a subsequence of (μ_p) is a regular sequence of T, so that $k\mu_\nu$ belongs to T for $\nu\geqslant\nu(k)$. It follows from Lemma 4 that

$$\lim_{\nu\to\infty}\int_0^{2\pi} e^{i[k\mu_\nu x+k\theta_\nu(x)]}\tau(x)\,d\psi=0,$$

which contradicts (26). This proves (25). Thus (24) is true when $t(x)$ is a trigonometric polynomial.

Suppose now that $t(x)$ is continuous. Given $\epsilon>0$, we can find a trigonometric polynomial $\tau(x)$ such that, for all x, $|t(x)-\tau(x)|<\epsilon$. Writing

$$t_p(x)=t\,[\mu_p x+\theta_p(x)],\quad \tau_p(x)=\tau\,[\mu_p x+\theta_p(x)],$$

we have $|t_p(x)-\tau_p(x)|<\epsilon$ for all x and p. Then

$$\left|\int_0^{2\pi} t_p(x)\,dP-\int_0^{2\pi}\tau_p(x)\,dP\right|<\epsilon\,[P(2\pi)-P(0)].$$

Since (24) is true for $\tau(x)$, we have

$$\lim_{p\to\infty}\int_0^{2\pi}\tau_p(x)\,dP=\frac{P(2\pi)-P(0)}{2\pi}\int_0^{2\pi}\tau(x)\,dx.$$

Hence

$$\overline{\lim_{p\to\infty}}\left|\int_0^{2\pi} t_p(x)\,dP-\frac{P(2\pi)-P(0)}{2\pi}\int_0^{2\pi} t(x)\,dx\right|<2\epsilon\,[P(2\pi)-P(0)].$$

Since ϵ is arbitrary, it follows that (24) is true for any continuous $t(x)$.

The preceding argument also shows that, if (24) is true for each function of a uniformly convergent sequence, then it is true for the limiting function. Any bounded J-measurable function is the limit of a uniformly convergent

sequence of functions of the form

$$\sum_{i=1}^{\nu} a_i^{(\nu)} \phi_i^{(\nu)}.$$

It is thus sufficient to prove (24) when $t(x)$ is the characteristic function of a J-measurable set E. We can find two sets E_1, E_2 such that $E_1 \supset E \supset E_2$ where E_1, E_2 each consist of a finite number of intervals, and $mE_1 - mE_2 < \epsilon$, where $\epsilon > 0$ denotes an arbitrarily assigned number. Let $t^{(1)}(x)$, $t^{(2)}(x)$ denote the characteristic functions of E_1, E_2 respectively, and let them be continued by the condition of periodicity. Then $t^{(1)}(x) \geqslant t(x) \geqslant t^{(2)}(x)$ for all x, and $t_p^{(1)}(x) \geqslant t_p(x) \geqslant t_p^{(2)}(x)$ for all p and x. Then, since $P(x)$ is monotone increasing, we have

$$\int_0^{2\pi} t_p^{(1)}(x)\, dP \geqslant \int_0^{2\pi} t_p(x)\, dP \geqslant \int_0^{2\pi} t_p^{(2)}(x)\, dP.$$

If we can prove (24) for the characteristic function of a finite set of intervals, then (24) will follow for $t(x)$ since

$$\overline{\lim_{p\to\infty}} \left| \int_0^{2\pi} t_p^{(1)}(x)\, dP - \int_0^{2\pi} t_p(x)\, dP \right| \leqslant \lim_{p\to\infty} \left| \int_0^{2\pi} t_p^{(1)}\, dP - \int_0^{2\pi} t_p^{(2)}\, dP \right|$$

$$\leqslant \frac{P(2\pi)-P(0)}{2\pi} \left[\int_0^{2\pi} t^{(1)}(x)\, dx - \int_0^{2\pi} t^{(2)}(x)\, dx \right]$$

$$\leqslant \frac{P(2\pi)-P(0)}{2\pi}\, \epsilon$$

and of $$\left| \int_0^{2\pi} t^{(1)}(x)\, dx - \int_0^{2\pi} t(x)\, dx \right| < \epsilon.$$

It remains then to prove (24) for the characteristic function of a single interval. Consider the case when the interval is (a, b) and $0 < a < b < 2\pi$. The other possibilities are treated in a similar way. For sufficiently large integers ν define $t^{(\nu)}(x)$ by

$$t^{(\nu)}(x) = 1 \quad (a \leqslant x \leqslant b),$$

$$t^{(\nu)}(x) = 0 \quad (0 \leqslant x \leqslant a - 1/\nu, \quad b + 1/\nu \leqslant x \leqslant 2\pi),$$

and let $t^{(\nu)}(x)$ be linear in the remaining two intervals and continuous in $(0, 2\pi)$. Then $t(x) \leqslant t^{(\nu)}(x)$ for all x and ν, and $t_p(x) \leqslant t_p^{(\nu)}(x)$ for all p, x, and ν. We have

$$\int_0^{2\pi} t_p(x)\, dP \leqslant \int_0^{2\pi} t_p^{(\nu)}(x)\, dP.$$

The relation (24) holds for the continuous function $t^{(\nu)}(x)$. Hence

$$\overline{\lim_{p\to\infty}} \int_0^{2\pi} t_p(x)\,dP \leqslant \overline{\lim_{p\to\infty}} \int_0^{2\pi} t_p^{(\nu)}(x)\,dP$$
$$\leqslant \frac{P(2\pi)-P(0)}{2\pi} \int_0^{2\pi} t^{(\nu)}(x)\,dx.$$

Letting $\nu\to\infty$, we obtain

$$\overline{\lim_{p\to\infty}} \int_0^{2\pi} t_p(x)\,dP \leqslant \frac{P(2\pi)-P(0)}{2\pi} \int_0^{2\pi} t(x)\,dx.$$

By constructing the functions $\tau^{(\nu)}(x)$ such that

$$\tau^{(\nu)}(x) = 1 \quad (a+1/\nu \leqslant x \leqslant b-1/\nu),$$
$$\tau^{(\nu)}(x) = 0 \quad (0 \leqslant x \leqslant a, \quad b \leqslant x \leqslant 2\pi),$$

and $\tau^{(\nu)}(x)$ is linear in the remaining two intervals and continuous in $(0, 2\pi)$, we can show that

$$\underline{\lim_{p\to\infty}} \int_0^{2\pi} t_p(x)\,dP \geqslant \frac{P(2\pi)-P(0)}{2\pi} \int_0^{2\pi} t(x)\,dx.$$

11. When $\psi(x)$ is absolutely continuous, in which case

$$\int_0^{2\pi} e^{i\mu x}\,d\psi = \int_0^{2\pi} e^{i\mu x}\psi'(x)\,dx$$

tends to zero as $\mu\to\infty$, Theorem VII becomes a special case of a well-known proposition. To see this, we require the following lemma.

LEMMA 9. *Let $t(x)$ be bounded, J-measurable, and periodic, and let $t_p(x) = t[\mu_p x+\theta_p(x)]$. Then, if $0 \leqslant a < b \leqslant 2\pi$,*

$$\lim_{p\to\infty} \int_a^b t_p(x)\,dx = \frac{b-a}{2\pi} \int_0^{2\pi} t(x)\,dx.$$

The lemma is true when $t(x)$ is a constant. We shall show that it is true when $t(x) = e^{ikx}$, where k is a positive integer. From this it will follow that the lemma is true when $t(x)$ is a trigonometric polynomial. The remainder of the proof is precisely similar to that of Theorem VII; we simply write $P(x) = x$.

We must show that

$$\lim_{p\to\infty} \int_a^b e^{i[k\mu_p x+k\theta_p(x)]}\,dx = 0 \tag{27}$$

If (27) is false, we can find a sequence (ν) such that

$$\lim_{\nu\to\infty}\left|\int_a^b e^{i[k\mu_\nu x+k\theta_\nu(x)]}\,dx\right|=\eta, \tag{28}$$

where $\eta>0$ and $\{\theta_\nu(x)\}$ is a convergent sequence.

Now $\psi(x)=x$ is a continuous function of class B since, with no restriction on λ,

$$\lim_{\lambda\to\infty}\int_0^{2\pi} e^{i\lambda x}\,dx=0.$$

The associated compact set R is the interval $(0,\ \infty)$, Then R is its own central set T and (μ_ν) is a regular sequence of T. Define the B-measurable function $\tau(x)$ by

$$\tau(x)=1\quad(a\leqslant x\leqslant b),\qquad \tau(x)=0\quad(0\leqslant x<a,\ b<x\leqslant 2\pi).$$

By Lemma 4,
$$\lim_{\nu\to\infty}\int_a^b e^{i[k\mu_\nu x+k\theta_\nu(x)]}\,dx=0,$$

which contradicts (28).

We can now show that, when $\psi(x)$ is absolutely continuous, Theorem VII is a consequence of the following proposition.

If $\{h_p(x)\}$ is a uniformly bounded sequence of functions integrable in $(0,\ 2\pi)$, with the property that, if $0\leqslant a<b\leqslant 2\pi$, then

$$\lim_{p\to\infty}\int_a^b h_p(x)\,dx=0,$$

then, if $g(x)$ is integrable in $(0,\ 2\pi)$, we have

$$\lim_{p\to\infty}\int_0^{2\pi} h_p(x)\,g(x)\,dx=0.$$

The result of Lemma 9 can be written in the form

$$\lim_{p\to\infty}\int_a^b\left[t_p(x)-\frac{1}{2\pi}\int_0^{2\pi}t(x)\,dx\right]dx=0.$$

If then we take

$$h_p(x)=t_p(x)-\frac{1}{2\pi}\int_0^{2\pi}t(x)\,dx,$$

we have, since $\psi(x)$ is absolutely continuous,

$$\lim_{p\to\infty}\int_0^{2\pi}h_p(x)\,d\psi=\lim_{p\to\infty}\int_0^{2\pi}h_p(x)\,\psi'(x)\,dx$$

$$=0.$$

Thus
$$\lim_{p\to\infty}\int_0^{2\pi} t_p(x)\,d\psi = \left[\frac{1}{2\pi}\int_0^{2\pi} t(x)\,dx\right]\int_0^{2\pi} d\psi$$
$$= \frac{\psi(2\pi)-\psi(0)}{2\pi}\int_0^{2\pi} t(x)\,dx.$$

If $\psi(x)$ is not absolutely continuous, Theorem VII is no longer true when the condition that $t(x)$ should be J-measurable is replaced by the wider condition that $t(x)$ should be B-measurable. To see this, let $\psi(x)$ be a continuous monotone increasing function of class B which satisfies the conditions of Theorem VII. We shall construct a B-measurable function $t(x)$ such that, for this function, the special case of Theorem VII obtained by taking $\theta_p(x)\equiv 0$ is false.

Let σ denote the set of points in $(0, 2\pi)$ at which $\psi'(x) = +\infty$. Let $E(x)$ denote the excess of x over the greatest integral multiple of 2π which does not exceed x. As x describes σ, $E(\mu_p x)$ describes a set σ_p. Let $G = \Sigma\sigma_p$. Then G is B-measurable and $mG = 0$. Let $t(x)$ be defined in $(0, 2\pi)$ by $t(x) = 1$ for $x \subset G$ and $t(x) = 0$ for $x \subset CG$. Let $t(x)$ be altered if necessary so that $t(0) = t(2\pi) = 1$, and let $t(x)$ be defined outside $(0, 2\pi)$ by the condition of periodicity. Then, if $x \subset \sigma$, $E(\mu_p x) \subset G$, and $t(\mu_p x) = t[E(\mu_p x)] = 1$. Hence, if $\tau(x) = 1$ for $x \subset \sigma$ and $\tau(x) = 0$ at the other points of $(0, 2\pi)$, we have $t(\mu_p x) \geqslant \tau(x)$ for all p and x. Therefore

$$\int_0^{2\pi} t(\mu_p x)\,d\psi \geqslant \int_0^{2\pi} \tau(x)\,d\psi.$$

If the theorem was true for $t(x)$, the limit of the first member would be zero, since

$$\int_0^{2\pi} t(x)\,dx = 0.$$

But the second member is equal to

$$\psi(2\pi)-\psi(0)-\int_0^{2\pi} \psi'(x)\,dx,$$

which is positive since $\psi(x)$ is not absolutely continuous.

If we examine the proof of (24) we see that it is true, not essentially in virtue of the fact that $P(x)$ is the positive variation of $\psi(x)$, but in virtue of the fact that $P(x)$ possesses the properties

(i) $P(x)$ is monotone increasing,

(ii) $P(x)$ is of the form $P(x) = \int_0^x \tau(x)\,d\psi$,

where $\tau(x)$ is bounded and B-measurable. Any function $Q(x)$ which possesses these properties will give rise to an equation similar to (24).

Thus, if $f(x)$ is the characteristic function of a B-measurable set, then

$$Q(x) = \int_0^x f(x)\, dP$$

$$= \int_0^x f(x)\, \tau(x)\, d\psi$$

possesses the required properties. We enunciate the result as a lemma.

LEMMA 10. *With the notation of Theorem VII, if $f(x)$ is the characteristic function of a B-measurable set and*

$$Q(x) = \int_0^x f(x)\, dP,$$

then

$$\lim_{p\to\infty} \int_0^{2\pi} t[\mu_p x + \theta_p(x)]\, dQ = \frac{Q(2\pi) - Q(0)}{2\pi} \int_0^{2\pi} t(x)\, dx. \tag{29}$$

12. Let σ be a B-measurable set in $(0, 2\pi)$. Let (μ_p) be a sequence of positive real numbers such that $\lim \mu_p = \infty$. As x describes σ, $E(\mu_p x)$ describes a set σ_p. We denote by δ_p the upper bound of the lengths of intervals in $(0, 2\pi)$ which are free from points of σ_p. Let $\phi(x)$ denote a function which is continuous and monotone increasing in $(0, 2\pi)$. Let (λ_p) denote a sequence of positive real numbers which tends to infinity.

THEOREM VIII. *If $f(x)$ is the characteristic function of σ and*

$$\operatorname*{Max}_{(k)} \left| \int_0^{2\pi} e^{ik\mu_p x} f(x)\, d\phi \right| \Big/ \delta_p{}^2 \quad \left(1 \leqslant k \leqslant \left[\frac{\lambda_p}{\delta_p{}^2}\right]\right),$$

where k is a natural number, tends to zero as $p \to \infty$, then

$$\int_0^{2\pi} f(x)\, d\phi = 0.$$

By the definition of δ_p, there is an interval (α_p, β_p) within $(0, 2\pi)$, and of length $d_p = \delta_p/2$, which is free from points of σ_p. Let $t_p(x)$ be the function of period 2π which is defined in $(0, 2\pi)$ by

$$t_p(x) = 1 \qquad (\alpha_p + \tfrac{1}{3}d_p \leqslant x \leqslant \beta_p - \tfrac{1}{3}d_p),$$

$$t_p(x) = \sin\left[(x-\alpha_p)\frac{3\pi}{2d_p}\right] \qquad (\alpha_p \leqslant x \leqslant \alpha_p + \tfrac{1}{3}d_p),$$

$$t_p(x) = -\cos\left[(x-\alpha_p)\frac{3\pi}{2d_p}\right] \qquad (\alpha_p + \tfrac{2}{3}d_p \leqslant x \leqslant \beta_p),$$

$$t_p(x) = 0 \qquad (0 \leqslant x \leqslant \alpha_p,\ \beta_p \leqslant x \leqslant 2\pi).$$

Then the total variation of $t_p'(x)$ in $(0, 2\pi)$ is $3\pi/d_p+2$; and there is an absolute constant C such that $3\pi/d_p+2 < C/d_p$ for all p. Let

$$t_p(x) = \tfrac{1}{2}a_0^{(p)} + \sum_{n=1}^{\infty} a_n^{(p)} \cos nx + b_n^{(p)} \sin nx.$$

Then, writing $a_{-n}^{(p)} = a_n^{(p)}$, $b_{-n}^{(p)} = -b_n^{(p)}$, we have

$$t_p(x) = \tfrac{1}{2} \sum_{n=-\infty}^{\infty} \{a_n^{(p)} - ib_n^{(p)}\} e^{inx}.$$

Further,
$$a_n^{(p)} - ib_n^{(p)} = \frac{1}{\pi}\int_0^{2\pi} e^{-inx} t_p(x)\, dx$$

$$= -\frac{1}{i\pi n}\int_0^{2\pi} e^{-inx} t_p'(x)\, dx \quad (n \neq 0).$$

Since the total variation of $t_p'(x)$ in $(0, 2\pi)$ is less than C/d_p, we have

$$|a_n^{(p)} - ib_n^{(p)}| < 4C/(\pi n^2 d_p) \quad (n \neq 0).$$

Now

$$\int_0^{2\pi} t_p(\mu_p x) f(x)\, d\phi = \tfrac{1}{2} \sum_{n=-\infty}^{\infty} \int_0^{2\pi} \{a_n^{(p)} - ib_n^{(p)}\} e^{i\mu_p nx} f(x)\, d\phi$$

$$= \tfrac{1}{2}a_0^{(p)} \int_0^{2\pi} f(x)\, d\phi + \tfrac{1}{2} \sum_{n=-\infty}^{\infty}{}' \int_0^{2\pi} \{a_n^{(p)} - ib_n^{(p)}\} e^{i\mu_p nx} f(x)\, d\phi,$$

where the dash denotes that the value $n = 0$ is omitted. Hence

$$\left| \int_0^{2\pi} t_p(\mu_p x) f(x)\, d\phi - \tfrac{1}{2}a_0^{(p)} \int_0^{2\pi} f(x)\, d\phi \right| < \sum_{n=1}^{\infty} \frac{4C}{\pi n^2 d_p} \left| \int_0^{2\pi} e^{i\mu_p nx} f(x)\, d\phi \right|.$$

If $x \subset \sigma$, then $E(\mu_p x)$ lies outside d_p, and $t_p(\mu_p x) = t_p[E(\mu_p x)] = 0$. Hence, if $\tau(x) = 0$ for $x \subset \sigma$ and $\tau(x) = 1$ at the other points of $(0, 2\pi)$, we have $t_p(\mu_p x) \leqslant \tau(x)$ for all x in $(0, 2\pi)$. Since $f(x) \geqslant 0$ and $\phi(x)$ is monotone increasing, we have

$$\int_0^{2\pi} t_p(\mu_p x) f(x)\, d\phi \leqslant \int_0^{2\pi} \tau(x) f(x)\, d\phi$$

$$\leqslant 0,$$

since $\tau(x) f(x) \equiv 0$. Hence

$$0 \leqslant \tfrac{1}{2}a_0^{(p)} \int_0^{2\pi} f(x)\, d\phi < \sum_{n=1}^{\infty} \frac{4C}{\pi n^2 d_p} \left| \int_0^{2\pi} e^{i\mu_p nx} f(x)\, d\phi \right|$$

$$\leqslant \sum_{n=1}^{[\lambda_p/\delta_p^2]} \frac{4C}{\pi n^2 d_p} \left| \int_0^{2\pi} e^{i\mu_p nx} f(x)\, d\phi \right| + 8CV \frac{\delta_p}{\lambda_p},$$

where V denotes the total variation of $\phi(x)$ in $(0, 2\pi)$. But

$$\tfrac{1}{2}a_n^{(p)} > d_p/(6\pi).$$

Hence

$$0 \leqslant \int_0^{2\pi} f(x)\,d\phi \leqslant \left[\operatorname*{Max}_{(k)} \left| \int_0^{2\pi} e^{i\mu_p kx} f(x)\,d\phi \right| \Big/ d_p{}^2\right] \sum_{n=1}^{\infty} \frac{24C}{n^2} + \frac{8CV}{\lambda_p},$$

where k is restricted as in the enunciation of the theorem. The last expression tends to zero as $p \to \infty$.

If σ is a B-measurable set in $(0, 2\pi)$ and $\psi(x)$ is continuous and of bounded variation in $(0, 2\pi)$, then the total variation of $\psi(x)$ on σ is defined to be

$$\int_0^{2\pi} t(x)\,dP + \int_0^{2\pi} t(x)\,dN,$$

where $t(x)$ is the characteristic function of σ and $P(x)$, $N(x)$ are respectively the positive and negative variations of $\psi(x)$ in $(0, x)$.

If η be a set which is not B-measurable and which is contained in a B-measurable set on which the total variation of $\psi(x)$ is zero, we say that the total variation of $\psi(x)$ on η is zero.

In § 1 we defined a set of type H. The sequence (n_p) which entered into the definition was a sequence of positive integers. It is convenient to remove this restriction according to the following definition.

Definition 7. Let (μ_p), $p = 1, 2, \ldots$, be a sequence of positive real numbers such that $\lim \mu_p = \infty$. Let σ be a set in $(0, 2\pi)$. As x describes σ, $E(\mu_p x)$ describes a set σ_p. If $\Sigma\sigma_p$ is not everywhere dense in $(0, 2\pi)$, then σ is said to be of type H with respect to (μ_p).

Let δ_p have the meaning assigned at the beginning of this section. If σ is of type H with respect to (μ_p), there is a $\delta > 0$ such that $\delta_p > \delta$ for all p. Suppose conversely that $\delta_p > \delta > 0$. Then there is an interval of length greater than $\frac{1}{2}\delta$ which is free from points of σ_p. Let ξ_p be the mid-point of that interval. Then we can find a subsequence $p_1, p_2, \ldots, p_s, \ldots$ such that $\xi_{p_s} \to \xi$. If $0 < \xi < 2\pi$, there is a complete neighbourhood of ξ which is free from points of σ_{p_s} for $s \geqslant s_0$; and if $\xi = 0$ or $\xi = 2\pi$ there is a one-sided neighbourhood of ξ with the same property. Hence the set σ is of type H with respect to a subsequence of (μ_p), namely with respect to (μ_{p_s}) for $s \geqslant s_0$.

Definition 8. Let T be a compact set of real numbers. Let σ be a set in $(0, 2\pi)$. If there is a sequence (μ_p) of positive real numbers such that $\lim \mu_p = \infty$, and a sequence (λ_p) with $\lim \lambda_p = \infty$, such that the intervals $(\mu_p, \lambda_p \mu_p/\delta_p{}^2)$ belong to T, then (μ_p) will be said to be a special sequence of T with respect to σ.

Suppose, for example, that $\mu_p = p!$. It is easy to construct a set σ for which $\delta_p = p^{-\frac{1}{2}}$. Then, taking $\lambda_p = p^{\frac{3}{2}}$, we have

$$\frac{\lambda_p}{\delta_p^2}\mu_p = p \cdot p!$$

$$< (p+1)!.$$

Let T be the compact set which consists of the intervals $(p!, p \cdot p!)$. Then $\{p!\}$ is a special sequence of T with respect to σ.

In the enunciation of the following theorem we confine ourselves to the case $\delta_p > \delta > 0$.

THEOREM IX. *Let $\psi(x)$ be a continuous function of class B associated with the compact set R. Let T be a compact central set of R. If there is a set σ and a sequence (μ_p) such that σ is of type H with respect to (μ_p), and (μ_p) is a special sequence of T with respect to σ, then the total variation of $\psi(x)$ on σ is zero.*

It is clear from Definition 7 that, if σ is of type H with respect to (μ_p), then so is $\sigma+\sigma'$, where σ' is the derived set of σ. The numbers δ_p are the same for σ and for $\sigma+\sigma'$. Thus for the purpose of the theorem we may suppose that σ is B-measurable. Let $f(x)$ be the characteristic function of σ. We must show that

$$\int_0^{2\pi} f(x)\,dP = 0,$$

and that a similar relation holds in $N(x)$. By Lemma 8,

$$\lim_{\mu\to\infty} \int_0^{2\pi} e^{i\mu x} f(x)\,dP = 0 \quad (\mu \subset T).$$

Since (μ_p) is a special sequence of T with respect to σ, the numbers $\mu_p, 2\mu_p, \ldots, [\lambda_p/\delta_p^2]\mu_p$ belong to T. Hence

$$\operatorname*{Max}_{(k)} \left| \int_0^{2\pi} e^{ik\mu_p x} f(x)\,dP \right| \quad \left(1 \leqslant k \leqslant \left[\frac{\lambda_p}{\delta_p^2}\right]\right),$$

where k is a natural number, tends to zero as $p \to \infty$. Since σ is of type H with respect to (μ_p), we have $\delta_p > \delta > 0$. By Theorem VIII,

$$\int_0^{2\pi} f(x)\,dP = 0.$$

By using that particular case of Lemma 10 which is obtained by taking $\theta_p(x) \equiv 0$, we can prove an analogous result when (μ_p) is simply taken to be a regular sequence of the compact set T.

THEOREM X. *Let $\psi(x)$ be a continuous function of class B associated with the compact set R. Let T be a central set of R which is compact. Let (μ_p), $p = 1, 2, \ldots$, be a regular sequence of T. Let σ be a B-measurable set which is of type H with respect to (μ_p). Then the total variation of $\psi(x)$ on σ is zero.*

Let $f(x)$ be the characteristic function of σ. We must show that

$$\int_0^{2\pi} f(x)\,dP = 0,$$

or, writing

$$Q(x) = \int_0^x f(x)\,dP,$$

that $Q(2\pi) - Q(0) = 0$. The set $G = \Sigma\sigma_p$ of Definition 7 is, by hypothesis, not everywhere dense in $(0, 2\pi)$. We can therefore find an interval within $(0, 2\pi)$ of length $\delta > 0$ which is free from points of G. We denote that interval by δ. We define the function $t(x)$ in $(0, 2\pi)$ by

$$t(x) = 1 \quad (x \subset \delta), \qquad t(x) = 0 \quad (x \subset C\delta),$$

and outside $(0, 2\pi)$ by the condition of periodicity. By Lemma 10,

$$\lim_{p\to\infty} \int_0^{2\pi} t(\mu_p x)\,dQ = \frac{Q(2\pi) - Q(0)}{2\pi} \int_0^{2\pi} t(x)\,dx. \tag{30}$$

If $x \subset \sigma$, then $E(\mu_p x)$ lies outside δ, and $t(\mu_p x) = 0$. Hence, if $\tau(x)$ denotes the function which is equal to 1 for $x \subset C\sigma$, and is equal to 0 for $x \subset \sigma$, we have $t(\mu_p x) \leqslant \tau(x)$ for all p and all x in $(0, 2\pi)$. Since $Q(x)$ is monotone increasing,

$$\begin{aligned} \int_0^{2\pi} t(\mu_p x)\,dQ &\leqslant \int_0^{2\pi} \tau(x)\,dQ \\ &\leqslant \int_0^{2\pi} \tau(x) f(x)\,dP \\ &\leqslant 0, \end{aligned} \tag{31}$$

since $\tau(x)f(x) \equiv 0$. Thus (30) and (31) give

$$\frac{Q(2\pi) - Q(0)}{2\pi}\,\delta \leqslant 0.$$

It might be thought that, by using Lemma 10 in its full generality, we

could infer more than the result of Theorem IX. Suppose, for example, we introduce the following definition.

Definition 9. Let σ be a set in $(0, 2\pi)$. Let $\{\theta_p(x)\}$ be a semi-convergent sequence of B-measurable functions. Let (μ_p) be a sequence of real numbers such that $\lim \mu_p = \infty$. As x describes σ, $E[\mu_p x + \theta_p(x)]$ describes a set σ_p. If $\Sigma \sigma_p$ is not everywhere dense in $(0, 2\pi)$, then σ is said to be of type η with respect to (μ_p), $\{\theta_p(x)\}$.

By the argument of Theorem IX we can show that, if (μ_p) is a regular sequence of T and σ is a B-measurable set of type η with respect to (μ_p), $\{\theta_p(x)\}$, then the total variation of $\psi(x)$ on σ is zero. By the definition of semi-convergence we can select a subsequence (p_s) such that $\{\theta_{p_s}(x)\}$ is a convergent sequence. Then σ is of type η with respect to $(\mu_{p_s})\{\theta_{p_s}(x)\}$. We may therefore confine our attention to the case in which $\{\theta_p(x)\}$ is a convergent sequence. Now any set S which is of type η with respect to (μ_p), $\{\theta_p(x)\}$, where $\{\theta_p(x)\}$ is convergent, is contained in a set of the form ΣH_r, where H_r is of type H with respect to (μ_p). Consequently, Theorem IX tells us that the total variation of $\psi(x)$ on S is zero. To see that $S \subset \Sigma H_r$ we observe that, if $\xi \subset S$, then the enumerable set $\{E(\mu_p \xi)\}$ is not everywhere dense in $(0, 2\pi)$. For otherwise, since $\theta_p(\xi) \to \theta(\xi)$ say, the enumerable set $\{E[\mu_p \xi + \theta_p(\xi)]\}$ would be everywhere dense in $(0, 2\pi)$. Therefore S is contained in the set M of all points x for which $\{E(\mu_p x)\}$ is not everywhere dense in $(0, 2\pi)$. In that case there is an interval ω, whose end points are rational multiples of 2π, which is contained in $(0, 2\pi)$ and which is free from points of the enumerable set $\{E(\mu_p x)\}$. The intervals such as ω can be enumerated as $\omega_1, \omega_2, \ldots, \omega_r, \ldots$; and to ω_r corresponds a set of type H with respect to (μ_p), namely the set H_r of all points x for which the enumerable set $\{E(\mu_p x)\}$ has no point in ω_r. Hence $S \subset M \subset \Sigma H_r$.

13. Let (n_p) be a sequence of positive integers which tends to infinity so that

$$n_{p+1} - n_p \to \infty.$$

Let λ be a positive integer. Let (λ_p) be a sequence of positive integers which tends to infinity so that

$$n_p - \lambda_p > n_{p-1} + \lambda + \lambda_{p-1}.$$

Let S denote the aggregate

$$\{n_p - \lambda_p,\ n_p - \lambda_p + 1,\ \ldots,\ n_p - 2,\ n_p - 1,\ n_p + \lambda,\ n_p + \lambda + 1,\ \ldots,\ n_p + \lambda + \lambda_p\}$$

$$(p = 2, 3, \ldots),$$

and let A denote the aggregate

$$\{n_p,\ n_p+1,\ \ldots,\ n_p+\lambda-1\} \quad (p=2,\ 3,\ \ldots).$$

Consider the class $D(\lambda)$ of functions $f(x)$ which are of bounded variation in $(0,\ 2\pi)$ and which satisfy the condition

$$\lim_{n\to\infty}\int_0^{2\pi} e^{inx}\,df=0 \quad (n\subset S).$$

Since S is a dense sequence, the class $D(\lambda)$ is a sub-class of the class B and so $f(x)$ is equivalent to a continuous function $\psi(x)$ of class $D(\lambda)$. We shall show that, for a continuous function of class $D(\lambda)$, we necessarily have

$$\lim_{n\to\infty}\int_0^{2\pi} e^{inx}\,d\psi=0 \quad (n\subset A).$$

THEOREM X. *Let $t(x)$, defined in $(0,\ 2\pi)$, be bounded and B-measurable. Let*

$$t(x)\sim a_0+\Sigma\,(a_n\cos nx+b_n\sin nx).$$

If $\psi(x)$ is a continuous function of class $D(\lambda)$, then

$$\overline{\lim_{p\to\infty}}\left|\int_0^{2\pi} e^{i(n_p+\mu)x}\,t(x)\,d\psi-\left[\frac{1}{2\pi}\int_0^{2\pi} t(x)\,dx\right]\int_0^{2\pi} e^{i(n_p+\mu)x}\,d\psi\right|$$

$$\leqslant \sum_{m=0}^{\lambda-1}(|a_m|+|b_m|)\,V-|a_0|\,V, \quad (32)$$

where μ is a fixed one of the numbers $0,\ 1,\ \ldots,\ \lambda-1$, and V is the total variation of $\psi(x)$ in $(0,\ 2\pi)$.

It is convenient to write a_0 instead of $\frac{1}{2}a_0$. When $\lambda=1$, the second member is zero.

Since $\psi(x)$ is continuous, we can alter the value of $t(x)$ at a point, and we may therefore suppose that $t(0)=t(2\pi)$. Suppose in the first place that $t(x)$ is a trigonometric polynomial. Thus

$$t(x)=a_0+\sum_{m=1}^{\nu}(a_m\cos mx+b_m\sin mx)$$

$$=a_0+\sum_{m=1}^{\nu}\{\tfrac{1}{2}(a_m-ib_m)\,e^{imx}+\tfrac{1}{2}(a_m+ib_m)\,e^{-imx}\}.$$

We may suppose that $\nu\geqslant\lambda$. Then, for $m\geqslant\lambda$, $(n_p+\mu\pm m)$ belongs to S from and after a certain value of p. Hence

$$\lim_{p\to\infty}\int_0^{2\pi} e^{i(n_p+\mu)x}\,e^{\pm imx}\,d\psi=0 \quad (m=\lambda,\ \lambda+1,\ \ldots,\ \nu),$$

and the first member of (32) is

$$\overline{\lim_{p\to\infty}}\left|\int_0^{2\pi} e^{i(n_p+\mu)x}\left\{\sum_{m=0}^{\lambda-1}(a_m\cos mx+b_m\sin mx)-a_0\right\}d\psi\right|$$

$$\leqslant\sum_{m=0}^{\lambda-1}(|a_m|+|b_m|)V-|a_0|V.$$

Suppose next that $t(x)$ is continuous. Let $t_r(x)$ be the r-th Cesàro mean of order 1 of the Fourier series of $t(x)$. Then $t_r(x)$ tends to $t(x)$ uniformly. Since the theorem is true for $t_r(x)$, we have

$$\overline{\lim_{p\to\infty}}\left|\int_0^{2\pi} e^{i(n_p+\mu)x}t_r(x)\,d\psi-\left[\frac{1}{2\pi}\int_0^{2\pi}t_r(x)\,dx\right]\int_0^{2\pi}e^{i(n_p+\mu)x}\,d\psi\right|$$

$$\leqslant\sum_{m=0}^{\lambda-1}(|a_m|+|b_m|)\left(1-\frac{m}{r}\right)V-|a_0|V.$$

Letting $r\to\infty$, we obtain the required inequality.

Suppose now that $t(x)$ is of Baire's first class. Then $t(x)$ is the limit of a sequence $\{t_s(x)\}$ of continuous functions which converge boundedly. We have

$$\left|\int_0^{2\pi}e^{i(n_p+\mu)x}t(x)\,d\psi-\int_0^{2\pi}e^{i(n_p+\mu)x}t_s(x)\,d\psi\right|\leqslant\int_0^{2\pi}|t(x)-t_s(x)|\,dv(x),$$

where $v(x)$ is the total variation of $\psi(x)$ in $(0, x)$. The last expression tends to zero as $s\to 0$. Further,

$$\lim_{s\to\infty}\int_0^{2\pi}t_s(x)\,dx=\int_0^{2\pi}t(x)\,dx,$$

and if $a_{m,s}$, $b_{m,s}$ denote the Fourier constants of $t_s(x)$, we have

$$\lim_{s\to\infty}a_{m,s}=a_m,\quad\lim_{s\to\infty}b_{m,s}=b_m.$$

By writing down the inequality which is true for $t_s(x)$ and letting $s\to\infty$, we obtain the required result. The proof is completed by transfinite induction.

THEOREM XI. *If $\psi(x)$ is a continuous function of class $D(\lambda)$, then*

$$\lim_{p\to\infty}\int_0^{2\pi}e^{i(n_p+\mu)x}\,d\psi=0,$$

where μ is a fixed one of the numbers $0, 1, \ldots, \lambda-1$.

We can write $\psi(x)=\theta(x)+\mu(x)$, where $\mu(x)$ is absolutely continuous and $\theta(x)$ is the function of singularities of $\psi(x)$. We further write

$\theta(x) = \theta_1(x) - \theta_2(x)$, where $\theta_1(x)$, $\theta_2(x)$ are respectively the positive and negative variations of $\theta(x)$. Then $\theta_1'(x) = \theta_2'(x) = 0$ for almost all x. Let E be the set of measure 2π at which these differential coefficients vanish. Then we can find $2\lambda+1$ points $0 < \xi_1 < \xi_2 < \ldots \xi_j \ldots < \xi_{2\lambda+1} < 2\pi$ which belong to E, and such that the determinant

$$\begin{vmatrix} 1 & 1 & \ldots & 1 \\ \cos\xi_1 & \cos\xi_2 & \ldots & \cos\xi_{2\lambda+1} \\ \cos 2\xi_1 & \ldots & \ldots & \cos 2\xi_{2\lambda+1} \\ \ldots & \ldots & \ldots & \ldots \\ \cos\lambda\xi_1 & \ldots & \ldots & \cos\lambda\xi_{2\lambda+1} \\ \sin\xi_1 & \ldots & \ldots & \sin\xi_{2\lambda+1} \\ \ldots & \ldots & \ldots & \ldots \\ \sin\lambda\xi_1 & \ldots & \ldots & \sin\lambda\xi_{2\lambda+1} \end{vmatrix}$$

differs from zero. For, if the determinant is multiplied by

$$\prod_{j=1}^{2\lambda+1} e^{i2\lambda\xi_j},$$

it becomes a polynomial in $e^{i\xi_j}$ $(j = 1, 2, \ldots, 2\lambda+1)$ which does not vanish identically in the ξ_j, and since there is an infinity of choices for each of the points ξ_j, the required condition can be satisfied. In that case the system of linear equations in the unknowns γ_j $(j = 1, 2, \ldots, 2\lambda+1)$,

$$\sum_{(j)} \gamma_j = -2\pi, \tag{33}$$

$$\sum_{(j)} \gamma_j \cos r\xi_j = 0 \quad (r = 1, 2, \ldots, \lambda), \tag{34}$$

$$\sum_{(j)} \gamma_j \sin r\xi_j = 0 \quad (r = 1, 2, \ldots, \lambda), \tag{35}$$

has a single solution.

Let δ_j denote an interval with centre ξ_j whose length tends to zero as the parameter k tends to ∞ through integral values. Let c_j denote a constant depending on j and k. Let d_m $(m = 1, 2, \ldots, 2\lambda+2)$ denote the intervals complementary to $\{\delta_j\}$. We define a function $t(x)$ which is equal to 1 in d_m $(m = 1, 2, \ldots, 2\lambda+2)$ and is equal to c_j in δ_j. We make $t(x)$ satisfy the condition

$$\int_0^{2\pi} t(x)\,dx = 0.$$

This condition is

$$\sum_{(j)} c_j\delta_j + \sum_{(m)} d_m = 0$$

that is

$$\sum_{(j)} (c_j - 1)\,\delta_j = -2\pi. \tag{36}$$

We further have

$$\int_0^{2\pi} t(x)\sin rx\,dx = \sum_{(j)} c_j \Delta(\delta_j, \sin rx) + \sum_{(m)} \Delta(d_m, \sin rx), \tag{37}$$

$$\int_0^{2\pi} t(x)\cos rx\,dx = \sum_{(j)} c_j \Delta(\delta_j, \cos rx) + \sum_{(m)} \Delta(d_m, \cos rx), \tag{38}$$

where we use the notation

$$\Delta\{d, f(x)\} = \int_\alpha^\beta f(x)\,dx \quad [d \equiv (\alpha, \beta)].$$

Since

$$\sum_{(j)} \Delta(\delta_j, \sin rx) + \sum_{(m)} \Delta(d_m, \sin rx) = 0,$$

$$\sum_{(j)} \Delta(\delta_j, \cos rx) + \sum_{(m)} \Delta(d_m, \cos rx) = 0,$$

we have, by (37) and (38),

$$\int_0^{2\pi} t(x)\sin rx\,dx = \sum_{(j)} (c_j - 1)\Delta(\delta_j, \sin rx), \tag{39}$$

$$\int_0^{2\pi} t(x)\cos rx\,dx = \sum_{(j)} (c_j - 1)\Delta(\delta_j, \cos rx). \tag{40}$$

We choose c_j so that $(c_j - 1)\delta_j = \gamma_j$, where (γ_j) is the solution of the equations (33), (34), and (35). Then, by (33), the equation (36) is satisfied. Further, since

$$\Delta(\delta_j, \sin rx) = \delta_j(\sin r\xi_j + \epsilon_{j,r}),$$

$$\Delta(\delta_j, \cos rx) = \delta_j(\cos r\xi_i + \epsilon'_{j,r}),$$

where $\epsilon_{j,r}$, $\epsilon'_{j,r}$ tend to zero as $k \to \infty$, we have, from (39) and (40),

$$\int_0^{2\pi} t(x)\sin rx\,dx = \sum_{(j)} \gamma_j \sin r\xi_j + o(1),$$

$$\int_0^{2\pi} t(x)\cos rx\,dx = \sum_{(j)} \gamma_j \cos r\xi_j + o(1).$$

By (34) and (35) we have

$$\left|\frac{1}{\pi}\int_0^{2\pi} t(x)\sin rx\,dx\right| < \frac{\epsilon}{2\lambda V}$$

$$\left|\frac{1}{\pi}\int_0^{2\pi} t(x)\cos rx\,dx\right| < \frac{\epsilon}{2\lambda V} \qquad (r = 1, 2, \ldots, \lambda)$$

for $k \geqq k(\epsilon)$, where $\epsilon > 0$ is an assigned number, and V denotes the total variation of $\psi(x)$ in $(0, 2\pi)$. We choose a suitable k and apply Theorem X

to $t(x)$. Since

$$\int_0^{2\pi} t(x)\,dx = 0,$$

we obtain

$$\overline{\lim_{p\to\infty}} \left| \int_0^{2\pi} e^{i(n_p+\mu)x} t(x)\,d\psi \right| < \epsilon. \tag{41}$$

We now have

$$\left| \int_0^{2\pi} e^{i(n_p+\mu)x} t(x)\,d\psi - \int_0^{2\pi} e^{i(n_p+\mu)x}\,d\psi \right| = \left| \sum_{(j)} \int_{\delta_j} (c_j-1)\, e^{i(n_p+\mu)x}\,d\psi \right|$$

$$= \left| \sum_{(j)} \int_{\delta_j} (c_j-1)\, e^{i(n_p+\mu)x} \{d\theta_1 - d\theta_2 + d\mu\} \right|.$$

Further,

$$\overline{\lim_{p\to\infty}} \left| \int_{\delta_j} (c_j-1)\, e^{i(n_p+\mu)x}\,d\theta_1 \right| \leqslant |(c_j-1)\{\theta_1(\beta_j)-\theta_1(\alpha_j)\}|$$

$$\leqslant \frac{|\gamma_j|}{\delta_j} \{\theta_1(\beta_j)-\theta_1(\alpha_j)\},$$

which tends to zero as $k\to 0$, since $\theta_1'(\xi_j)=0$ and γ_j is independent of k. Similarly

$$\overline{\lim_{p\to\infty}} \left| \int_{\delta_j} (c_j-1)\, e^{i(n_p+\mu)x}\,d\theta_2 \right| = o(1)$$

as $k\to\infty$. Finally, since $\mu(x)$ is absolutely continuous,

$$\lim_{p\to\infty} \int_{\delta_j} (c_j-1)\, e^{i(n_p+\mu)x}\,d\mu = 0.$$

It follows that, if we choose k sufficiently large, we have

$$\overline{\lim_{p\to\infty}} \left| \int_0^{2\pi} e^{i(n_p+\mu)x}\,d\psi \right| < 2\epsilon.$$

14. We showed in § 5 that, if $f(x)$ is a function of class B, then it can have only removable discontinuities within $(0, 2\pi)$ and that it satisfies the relation

$$f(2\pi)-f(2\pi-0)+f(+0)-f(0) = 0.$$

We now wish to establish this result for a wider class of function. Let $\lambda > 1$ denote a positive integer. We have the following elaboration of Definitions 1 and 2.

Definition 10. An increasing sequence S of positive integers is said to be λ-dense if, given any positive integer ν, there are ν positive integers of the form α, $\alpha+\lambda$, ..., $\alpha+(\nu-1)\lambda$, which belong to S.

Definition 11. Let S be a λ-dense sequence. Let (n_p) be a subsequence of S such that, given any positive integer k, there is a $p(k)$ such that, for $p \geqslant p(k)$, the sequences $(n_p+\lambda k)$, $(n_p-\lambda k)$ belong to S. Then (n_p) is said to be a central λ-sequence of S.

Every λ-dense sequence contains a central λ-sequence which is itself a λ-dense sequence. This is analogous to the result of Lemma 1 and is proved in a similar way.

Let $f(x)$ be defined in the closed interval $(0, 2\pi)$ and let it be of bounded variation in that interval. Let there be a λ-dense sequence S such that

$$\lim_{n\to\infty} \int_0^{2\pi} e^{inx}\, df = 0 \quad (n \subset S).$$

Then $f(x)$ is said to be a member of class $B(\lambda)$, associated with S.

We have the following analogue of Theorem I.

LEMMA 11. *Let $f(x)$ be of class $B(\lambda)$ and associated with S. Let (n_p) be a central λ-sequence of S. If $t(x)$ be bounded, B-measurable, and of period $2\pi/\lambda$, then*

$$\lim_{p\to\infty} \int_0^{2\pi} e^{in_p x}\, t(x)\, df = 0. \tag{42}$$

Suppose in the first place that $t(x)$ is a trigonometric polynomial. Then it can be expressed as

$$t(x) = \sum_{m=-r}^{r} A_m e^{im\lambda x},$$

and

$$e^{in_p x}\, t(x) = \sum_{m=-r}^{r} A_m e^{i(n_p+m\lambda)x}.$$

For a fixed m, $n_p+m\lambda$ belongs to S from and after a certain p. Hence

$$\lim_{p\to\infty} \int_0^{2\pi} e^{i(n_p+m\lambda)x}\, df = 0.$$

The lemma is therefore true when $t(x)$ is a trigonometric polynomial. The remainder of the proof follows the method used in the proof of Theorem I.

Not every function of class $B(\lambda)$ is equivalent to a continuous function. The simplest discontinuous function of that class is the function $f(x)$ defined for $0 \leqslant x \leqslant 2\pi$ as the sum of

$$\sum_{n=1}^{\infty} \frac{\sin nx}{n} - \sum_{n=1}^{\infty} \frac{\sin \lambda nx}{\lambda n}.$$

This function has non-removable discontinuities at

$$2\pi/\lambda, \quad 4\pi/\lambda, \quad \ldots, \quad (\lambda-1)\,2\pi/\lambda,$$

and

$$\int_0^{2\pi} e^{inx}\,df = 0 \quad [n \equiv 0 \pmod{\lambda}].$$

The peculiarity of this example is that the λ-dense sequence S for which

$$\lim_{n\to\infty} \int_0^{2\pi} e^{inx}\,df = 0 \quad (n \subset S)$$

consists of multiples of λ. The case in which we can assert that $f(x)$ is equivalent to a continuous function is given by the following theorem.

THEOREM XII. *Let $f(x)$ be a function of class $B(\lambda)$ associated with the λ-dense sequence S. Let (n_p) denote a central λ-sequence of S such that the residue of n_p to modulus λ takes each of the values $0, 1, \ldots, \lambda-1$ an infinite number of times as $p \to \infty$. Then $f(x)$ can have only removable discontinuities within $(0, 2\pi)$, and*

$$f(2\pi)-f(2\pi-0)+f(+0)-f(0) = 0.$$

Let ξ be a point such that $0 < \xi < 2\pi/\lambda$. Let $t(x)$ be the function of period $2\pi/\lambda$, defined in $(0, 2\pi/\lambda)$ by

$$t(\xi) = 1, \quad t(x) = 0 \quad (0 \leqslant x < \xi, \quad \xi < x \leqslant 2\pi/\lambda).$$

By Lemma 11,

$$\lim_{p\to\infty} \int_0^{2\pi} e^{in_p x}\,t(x)\,df = 0. \tag{43}$$

If we write

$$\Delta(\xi) = f(\xi+0)-f(\xi-0),$$

then (43) gives

$$\lim_{p\to\infty} \sum_{r=0}^{\lambda-1} e^{in_p(\xi+2\pi r/\lambda)}\,\Delta(\xi+2\pi r/\lambda) = 0.$$

In this relation we may remove the factor $e^{in_p\xi}$, and we obtain

$$\lim_{p\to\infty} \sum_{r=0}^{\lambda-1} e^{in_p 2\pi r/\lambda}\,\Delta(\xi+2\pi r/\lambda) = 0. \tag{44}$$

If we write $n_p = \nu_p\lambda + s_p$, where ν_p, s_p are positive integers and $0 \leqslant s_p < \lambda$, then s_p takes each of the values $0, 1, \ldots, \lambda-1$ an infinite number of times. We therefore have

$$\sum_{r=0}^{\lambda-1} e^{is2\pi r/\lambda}\,\Delta(\xi+2\pi r/\lambda) = 0 \tag{45}$$

for $s = 0, 1, \ldots, \lambda-1$, or, what is the same, for $s = 1, 2, \ldots, \lambda$. We thus have a system of linear homogeneous equations in the λ unknowns

$\Delta(\xi+2\pi r/\lambda)$. If we write $\alpha = e^{2\pi i/\lambda}$, then the determinant of the system is

$$\begin{vmatrix} 1 & \alpha & \alpha^2 & \dots & \alpha^{\lambda-1} \\ 1 & \alpha^2 & \alpha^4 & \dots & \alpha^{2(\lambda-1)} \\ \dots & \dots & \dots & \dots & \dots \\ 1 & \alpha^2 & \alpha^{2\lambda} & \dots & \alpha^{\lambda(\lambda-1)} \end{vmatrix} = \prod_{\sigma > s} (\alpha^\sigma - \alpha^s) \quad (s, \sigma = 1, 2, \dots, \lambda).$$

This expression differs from zero. It follows that

$$\Delta(\xi+2\pi r/\lambda) = 0 \quad (r = 0, 1, \dots, \lambda-1).$$

Thus, at points $x \not\equiv 0 \pmod{2\pi/\lambda}$, $f(x)$ can have only removable discontinuities.

Suppose now that $\xi = 0$. Let $t(x)$ be the function of period $2\pi/\lambda$ defined in $(0, 2\pi/\lambda)$ by

$$t(0) = 1, \quad t(x) = 0 \quad (0 < x < 2\pi/\lambda).$$

By applying Lemma 11 we obtain

$$\lim_{p \to \infty} \sum_{r=0}^{\lambda} e^{in_p 2\pi r/\lambda} \Delta(2\pi r/\lambda),$$

where

$$\Delta(0) = f(+0) - f(0), \quad \Delta(2\pi) = f(2\pi) - f(2\pi - 0),$$

while, for $r \neq 0, \lambda$, the expression $\Delta(2\pi r/\lambda)$ has the meaning previously assigned. Our system of equations is

$$\sum_{r=0}^{\lambda-1} e^{is 2\pi r/\lambda} \Delta'(2\pi r/\lambda) = 0 \quad (s = 1, 2, \dots, \lambda),$$

where $\Delta'(0) = \Delta(0) + \Delta(2\pi)$, and $\Delta'(2\pi r/\lambda) = \Delta(2\pi r/\lambda)$ for $r \neq 0$. Hence

$$\Delta'(2\pi r/\lambda) = 0 \quad (r = 0, 1, \dots, \lambda-1).$$

Thus, at the points $2\pi/\lambda, 4\pi/\lambda, \dots, (\lambda-1)\,2\pi/\lambda$, there can be only removable discontinuities, and $\Delta'(0) = 0$ gives

$$f(2\pi) - f(2\pi - 0) + f(+0) - f(0) = 0.$$

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CORRIGENDUM.

DEFORMATIONS OF A CURVE, IN A RIEMANNIAN n-SPACE, WHICH DISPLACE CERTAIN VECTORS PARALLELLY AT EACH POINT

[*Proc. London Math. Soc.* (2), 32 (1931), 321–326.]

H. A. HAYDEN.

With reference to a foot-note on p. 322, Prof. Hlavatý has explained to me that he regards his vector-field $\xi^i_{(1)}$ (or i' in his notation) as defined at every point of the space, and consequently his curve C as being one of a congruence, namely the congruence determined by the field $\xi^i_{(1)}$. He has omitted to state that fact explicitly in his paper, however, and, by reason of that omission, the operation mentioned in my foot-note appeared to have no meaning, as I remark there. I should like to state, therefore, that the operations and the results of Prof. Hlavaty's paper are valid, provided that the vector-field is interpreted in the manner which he suggests.

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